Simple New Proofs of the Characteristic Functions of the F and Skew-Normal Distributions

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Abstract: For a statistical distribution, the characteristic function (CF) is crucial because of the one-to-one correspondence between a distribution function and its CF and other properties. In order to avoid the calculation of contour integrals, the CFs of two popular distributions, the F and the skew-normal distributions, are derived by solving two ordinary differential equations (ODEs). The results suggest that the approach of deriving CFs by the ODEs is effective for asymmetric distributions. A much simpler approach is proposed to derive the CF of the multivariate F distribution in terms of a stochastic representation without using contour integration. For a special case of the multivariate F distribution where the variable dimension is one, its CF is consistent with that of the former (univariate) F distribution. This further confirms that the derivations are reasonable. The derivation is quite simple, and is suitable for presentation in statistics theory courses.

Keywords: characteristic function; ordinary differential equation; F distribution; skew-normal distribution; asymmetric distribution

1. Introduction

For any distribution function, its characteristic function (CF) always exists because of the integrability for every bounded and measurable function with respect to any distribution over the real line. The CF plays an important role in statistics due to the one-to-one correspondence between a distribution function and its CF. By the continuity theorem, the one-to-one correspondence relationship is also continuous. According to [1,2], CFs have many pleasant properties, e.g., every characteristic function is uniformly continuous on the whole real line. Owing to the properties of the CF, the calculation of the CF for each distribution has received much interest in statistical literature.

For example, the F distribution (also known as Snedecor's F distribution or the Fisher–Snedecor distribution), which was derived in the year 1924 in [3], has been frequently used in many studies, particularly as a null distribution of test statistics. Its CF has been derived in many studies, such as [4–7], but the CF of the F distribution was listed incorrectly. The author of [8] corrected this using contour integration as

$$\psi_X(t) = Ee^{itX} = \frac{\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)} U\left(\frac{\nu_1}{2}, 1 - \frac{\nu_2}{2}, -\frac{\nu_2}{\nu_1} it\right), \quad t \in \mathbb{R}, i = \sqrt{-1}, \quad (1)$$

where the random variable $X > 0$ follows the F distribution with degrees of freedom $\nu_1 > 0$ and $\nu_2 > 0$, and $U(\cdot)$ is Tricomi's (confluent hypergeometric) function proposed in [9] with its specific expression in Appendix A. The application of this CF can be found in many studies. For instance, ref. [10] utilized this CF to evaluate the distribution of a linear combination of independent F random variables, which can be applied to compute the exact confidence intervals for the common mean of several normal populations. In [11] the authors applied this CF to obtain the distribution of an average F -statistic for testing linear asset pricing models. In [12], this CF is applied to calculate the moment of the ratio between Euclidean distances in high dimensional spaces.

Another example is regarding a skew-normal distribution that was first introduced in [13]. The skew-normal distribution generalizes the normal distribution to allow for non-zero skewness. For instance, it is applied to measure various body levels for sports athletes in [14], to measure prices of bottles of Barolo wine in [15], and to measure different chemical measurements of Piedmont wines in [16]. For a random variable Y with values in $\mathbb{R}$ following the skew-normal distribution with the skewness (or slant) parameter λ , that is, $Y \sim SN(0, 1, \lambda)$, the CF given by [17–19] and [15] is

$$\psi_Y(t) = Ee^{itY} = e^{-\frac{t^2}{2}} \left(1 + \sqrt{\frac{2}{\pi}} i \int_{-\infty}^{\delta t} e^{\frac{v^2}{2}} dv \right), \quad t \in \mathbb{R}, \quad (2)$$

with $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$. In [19] the authors also obtained different forms of the CF of the skew-normal distribution using contour integration. This CF is employed to estimate the location and scale parameters as a special case of the skew-symmetric distribution in [20]. Taking this CF as a special case, the goodness-of-fit test is done based on CFs in [21].

Because statistics students are unfamiliar with contour integration, the rigorous derivation of CFs based on such integration is mostly avoided. Using ordinary differential equations (ODEs), ref. [22] presented a proof of the CF of a Student's t -distribution. The current study extends the ODE method to derive the above two CFs, which are for asymmetric distributions.

The further objective of this study is to derive the CFs of some multivariate distributions without contour integration. For the multivariate F distribution introduced in [23,24] proposed its CF through the transformation of the general inverted Dirichlet distribution, and the calculation is done via the contour integrals. In terms of a simple stochastic representation, the derivation of the CF of the multivariate F distribution proposed in this study is much simpler than the one employing the contour integrals.

In Section 2, by applying the ODEs, we first calculate the CF of the F distribution. The reason that this CF consists of a Tricomi's function can be explained by the computing process. The calculation for the ODEs of the skew-normal distribution follows. The Faddeeva function is involved in the derivation of the CF for the skew-normal distribution. Even though it is not always easy to obtain the solution for an ODE, which can further yield the CF, solution for the ODE of the F distribution is found in Kummer's equation (see Appendix A) and that for the ODE of the skew-normal distributions is found in a differential equation of the Faddeeva function (see Appendix B). In Section 3, the CF of the multivariate F distribution is also derived using simple stochastic representation without using contour integration. Finally, a short conclusion is provided in Section 4.

2. Deriving the CFs by ODEs

2.1. Derivation for the F Distribution

For the random variable $X \sim F(\nu_1, \nu_2)$, the probability density function (PDF) is

$$f_X(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}-1} \left(1 + \frac{\nu_1}{\nu_2}x\right)^{-\frac{\nu_1+\nu_2}{2}}, \quad x > 0,$$

with degrees of freedom $\nu_1 > 0$ and $\nu_2 > 0$, where $\Gamma(\cdot)$ is a Gamma function.

It is first proved for $\nu_2 > 4$ and then $\nu_2 > 2$. Finally, the result will be proved for $\nu_2 > 0$. The first derivative at x of $f_X(x)$ leads to

$$2(\nu_2 + \nu_1 x) x f'_X(x) + \nu_1(\nu_2 + 2)x f_X(x) - \nu_2(\nu_1 - 2)f_X(x) = 0.$$

When $\nu_2 > 4$, an integration by parts to this differential equation further results in

$$\begin{aligned} 0 &= \int_0^\infty [2(\nu_2 + \nu_1 x) x f'_X(x) + \nu_1(\nu_2 + 2)x f_X(x) - \nu_2(\nu_1 - 2)f_X(x)] e^{itx} dx \\ &= -\nu_1 \nu_2 E(e^{itX}) + (\nu_1 \nu_2 - 2\nu_1 - 2it\nu_2) E(X e^{itX}) - 2it\nu_1 E(X^2 e^{itX}), \end{aligned}$$

where all integrals exist because $\nu_2 > 4$. Combining with the properties of the CF,

$$\psi'_X(t) = iE(X e^{itX}) \text{ and } \psi''_X(t) = -E(X^2 e^{itX}), \quad (3)$$

the ODE of the F distribution for the CF $\psi_X(t)$ with $\nu_2 > 4$ is

$$2\nu_1 t \psi''_X(t) + [2i\nu_2 t - \nu_1(\nu_2 - 2)] \psi'_X(t) + i\nu_1 \nu_2 \psi_X(t) = 0. \quad (4)$$

Referring to Appendix A, if $w(z)$ satisfies Kummer's Equation (A1), that is,

$$inz w''(z) + (b - z)w'(z) - aw(z) = 0,$$

then $-\frac{\nu_2}{\nu_1} iz w''\left(-\frac{\nu_2}{\nu_1} iz\right) + \left(1 - \frac{\nu_2}{2} + \frac{\nu_2}{\nu_1} iz\right) w'\left(-\frac{\nu_2}{\nu_1} iz\right) - \frac{\nu_1}{2} w\left(-\frac{\nu_2}{\nu_1} iz\right) = 0$ follows. Furthermore, $g_*(z) = w\left(-\frac{\nu_2}{\nu_1} iz\right)$ satisfies $2\nu_1 z g''_*(z) + [2i\nu_2 z - \nu_1(\nu_2 - 2)] g'_*(z) + i\nu_1 \nu_2 g_*(z) = 0$. Therefore, the complete solution for (4) is

$$\psi_X(t) = C_1 M\left(\frac{\nu_1}{2}, 1 - \frac{\nu_2}{2}, -\frac{\nu_2}{\nu_1} it\right) + C_2 U\left(\frac{\nu_1}{2}, 1 - \frac{\nu_2}{2}, -\frac{\nu_2}{\nu_1} it\right),$$

where C_1 and C_2 are arbitrary constants except that $M(a, b, z)$ (Kummer's function of the first kind) does not exist when b is a nonpositive integer. $U(a, b, z)$ is Tricomi's function with specific expression given in Appendix A.

The standard properties of a CF $\psi_X(t)$, which include $\lim_{t \rightarrow 0} \psi_X(t) = 1$ and $|\psi_X(t)| \leq 1$ for all $t \in \mathbb{R}$, can be combined with (A2)–(A5) in Appendix A, that is, with

$$\begin{aligned}\lim_{|t| \rightarrow 0} \psi_X(t) &= C_1 + C_2 \frac{\Gamma\left(\frac{\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right)} + \mathcal{O}(z), \text{ as } z \rightarrow 0, \\ \lim_{|t| \rightarrow \infty} \psi_X(t) &\sim C_1 \frac{\Gamma\left(1 - \frac{\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)} e^z z^{\frac{\nu_1 + \nu_2}{2} - 1} + C_2 z^{-\frac{\nu_1}{2}}, \text{ as } z \rightarrow \infty,\end{aligned}$$

where $z = -\frac{\nu_2}{\nu_1} it$, in order to conclude that the values of the constant coefficients are given by $C_1 = 0$ and $C_2 = \frac{\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$. Here, $|\text{ph}(z)| = \frac{1}{2}\pi$ for all complex z since the real part of z is zero. $|\text{ph}(z)| < \frac{3}{2}\pi - \delta$ follows for many small values of positive constant δ . Consequently, the CF of the F distribution with $\nu_2 > 4$ is given by (1).

Suppose $\nu_2 > 2$, and let $S \sim F(\nu_1, \nu_2 + 2)$ with the CF $\psi_S(t)$. On using the standard formula $\Gamma(\nu + 1) = \nu\Gamma(\nu)$,

$$\begin{aligned}\psi'_S\left(\frac{\nu_2}{\nu_2 + 2}t\right) &= i \frac{\nu_2}{\nu_2 + 2} E\left(S e^{i \frac{\nu_2}{\nu_2 + 2} t S}\right) \\ &= i \frac{(\nu_1 + \nu_2)}{\nu_2} \int_0^\infty x e^{itx} f_X(x) \frac{\nu_2}{\nu_2 + \nu_1 x} dx \\ &= i\psi_X(t) + \frac{2}{\nu_1} it\psi'_X(t),\end{aligned}$$

where $X \sim F(\nu_1, \nu_2)$. Furthermore, suppose that $\nu_2 > 0$ and $\nu_2^* = \nu_2 + 2 > 2$, and let $S^* \sim F(\nu_1, \nu_2^* + 2)$. Through a similar argument as for S , the CF of S^* can be obtained as

$$\psi'_{S^*}\left(\frac{\nu_2^*}{\nu_2^* + 2}t\right) = i\psi_S(t) + \frac{2}{\nu_1} it\psi'_S(t).$$

By applying the differential property of the Tricomi's function $U(a, b, z)$ (A6) to $\psi'_S\left(\frac{\nu_2}{\nu_2 + 2}t\right)$ and $\psi'_{S^*}\left(\frac{\nu_2^*}{\nu_2^* + 2}t\right)$ and by using the recurrence relationship (A7), the CF of the F distribution for $\nu_2 > 0$ is obtained as in (1).

2.2. Derivation for the Skew-Normal Distribution

The random variable Y following a skew-normal distribution $SN(0, 1, \lambda)$ has the PDF given by

$$f_Y(y) = 2\phi(y)\Phi(\lambda y), \quad y \in \mathbb{R}, \lambda \in \mathbb{R},$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and distribution functions of the standard normal distribution, respectively.

When combined with the property of a density function in the standard normal distribution $\phi'(y) = -y\phi(y)$, the first derivative at y of $f_Y(y)$ yields the following equation:

$$f'_Y(y) + yf_Y(y) - 2\lambda\phi(y)\phi(\lambda y) = 0.$$

Using integration by parts and the properties of the CF in (3) again and

$$\int_{-\infty}^{\infty} e^{ity} \phi(y) \phi(\lambda y) dy = \frac{1}{\sqrt{2\pi(1+\lambda^2)}} \psi_V \left(\frac{t}{\sqrt{1+\lambda^2}} \right),$$

the ODE of the skew-normal random variable Y for the CF $\psi_Y(t)$ is

$$\psi_Y'(t) + t\psi_Y(t) - \sqrt{\frac{2}{\pi}} \delta i \psi_V \left(\frac{t}{\sqrt{1+\lambda^2}} \right) = 0,$$

where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$, V is a standard normal random variable and $\psi_V \left(\frac{t}{\sqrt{1+\lambda^2}} \right) = e^{-\frac{t^2}{2(1+\lambda^2)}}$.

Referring to Appendix B, if $w(z)$ satisfies a differential equation of the Faddeeva function (A10), then $h_*(z) = e^{-\frac{z^2}{2(1+\lambda^2)}} w \left(\frac{\delta z}{\sqrt{2}} \right)$ satisfies

$$h_*'(z) + zh_*(z) - \sqrt{\frac{2}{\pi}} \delta i \psi_V \left(\frac{z}{\sqrt{1+\lambda^2}} \right) = 0.$$

By Lemma A1 in Appendix B, the particular solution $w \left(\frac{\delta z}{\sqrt{2}} \right)$ is

$$\begin{aligned} w \left(\frac{\delta z}{\sqrt{2}} \right) &= e^{-\frac{\delta^2 z^2}{2}} \left(1 + \frac{2}{\sqrt{\pi}} i \int_0^{\frac{\delta z}{\sqrt{2}}} e^{u^2} du \right) \\ &= e^{-\frac{\delta^2 z^2}{2}} \left(1 + \sqrt{\frac{2}{\pi}} i \int_0^{\delta z} e^{\frac{v^2}{2}} dv \right). \end{aligned}$$

Hence

$$h_*(z) = e^{-\frac{z^2}{2}} \left(1 + \sqrt{\frac{2}{\pi}} i \int_0^{\delta z} e^{\frac{v^2}{2}} dv \right).$$

Therefore, the CF of the skew-normal variable Y is obtained as (2) since $\psi_Y(t) = h_*(t)$. For the CF of the multivariate skew-normal distribution introduced by [25], we refer to [19].

3. Deriving the CF of the Multivariate F Distribution

Based on the stochastic representation (5), we derive the CF of the multivariate F distribution in a simple manner without resorting to contour integration.

Let $\chi_j, j = 0, 1, \dots, p$ be an independent chi-squared variable with ν_j degrees of freedom, then

$$X_j = \frac{\chi_j / \nu_j}{\chi_0 / \nu_0} \quad (5)$$

follows an F distribution with degrees of freedom ν_j and ν_0 . The joint density of $\mathbf{X} = (X_1, \dots, X_p)^\top$, is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma(\nu/2) \prod_{j=1}^p \left(\frac{\nu_j}{\nu_0} \right)^{\nu_j/2}}{\prod_{j=0}^p \Gamma(\nu_j/2)} \frac{\prod_{j=1}^p x_j^{\nu_j/2-1}}{\left(1 + \sum_{j=1}^p \frac{\nu_j}{\nu_0} x_j \right)^{\nu/2}}, \quad x_j > 0, j = 1, \dots, p,$$

with $\nu = \sum_{j=1}^p \nu_j$. This is known as the multivariate F distribution and is denoted as $\mathbf{X} \sim F(\nu_1, \dots, \nu_p, \nu_0)$ ([23]). When $p = 1$, this density reduces to that of (univariate) F distribution, $F(\nu_1, \nu_0)$.

Denote the CF of the multivariate F random vector $\mathbf{X}$ as $\psi_{\mathbf{X}}(\mathbf{t}) = E(e^{i\mathbf{t}^\top \mathbf{X}}) = E\left[\exp\left(i \sum_{j=1}^p t_j \frac{\chi_j/\nu_j}{\chi_0/\nu_0}\right)\right]$ with $\mathbf{t} = (t_1, t_2, \dots, t_p)^\top$. The law of iterated expectation and the independence among χ_j s lead to

$$\psi_{\mathbf{X}}(\mathbf{t}) = E_{\chi_0} \left[E \left(\prod_{j=1}^p \exp \left(i \frac{\nu_0 t_j}{\nu_j \chi_0} \chi_j \right) \middle| \chi_0 \right) \right] = E_{\chi_0} \prod_{j=1}^p \psi_{\chi_j} \left(\frac{\nu_0 t_j}{\nu_j \chi_0} \right),$$

where $\psi_{\chi_j} \left(\frac{\nu_0 t_j}{\nu_j \chi_0} \right) = \left(1 - 2i \frac{\nu_0 t_j}{\nu_j \chi_0} \right)^{-\nu_j/2}$ and $\psi_{\chi_j}(t)$ is the CF of the chi-squared variable χ_j . By the definition of the moment for the chi-squared random variable χ_0 , $\psi_{\mathbf{X}}(\mathbf{t})$ changes to

$$\begin{aligned} \psi_{\mathbf{X}}(\mathbf{t}) &= \int_0^\infty \frac{1}{\Gamma(\nu_0/2) 2^{\nu_0/2}} \chi_0^{\nu_0/2-1} e^{-\chi_0/2} \prod_{j=1}^p \left(1 - 2i \frac{\nu_0 t_j}{\nu_j \chi_0} \right)^{-\nu_j/2} d\chi_0 \\ &= \frac{1}{\Gamma(\nu_0/2)} \int_0^\infty e^{-\frac{\chi_0}{2}} \left(\frac{\chi_0}{2} \right)^{\sum_{j=1}^p \nu_j/2} \frac{1}{\chi_0} \prod_{j=1}^p \left(\frac{\chi_0}{2} - \frac{\nu_0}{\nu_j} i t_j \right)^{-\nu_j/2} d\chi_0 \\ &= \frac{1}{\Gamma(\nu_0/2)} \int_0^\infty e^{-y} y^{\sum_{j=1}^p \nu_j/2-1} \prod_{j=1}^p \left(y - \frac{\nu_0}{\nu_j} i t_j \right)^{-\nu_j/2} dy. \end{aligned}$$

The last one-dimensional integration is equal to the CF of the multivariate F distribution in [24], who obtained it via contour integration. Some other forms of CF including the p -dimensional integral form are also available; we refer to [24] for more information. Here, we developed the CF in an easy manner using the stochastic representation given in (5) not depending on contour integration.

When $p = 1$, the CF of the (univariate) F distribution becomes

$$\psi_X(t) = \frac{1}{\Gamma(\nu_0/2)} \int_0^\infty e^{-y} y^{\frac{\nu_0+\nu_1}{2}-1} \left(y - \frac{\nu_0}{\nu_1} i t \right)^{-\nu_1/2} dy.$$

Let $z = -\frac{\nu_0}{\nu_1} i t$, we find that, by the transformation $x = y/z$,

$$\begin{aligned} \psi_X(t) &= \frac{z^{\nu_0/2}}{\Gamma(\nu_0/2)} \int_0^\infty e^{-xz} x^{\frac{\nu_0+\nu_1}{2}-1} (1+x)^{-\nu_1/2} dx \\ &= \frac{z^{\nu_0/2}}{\Gamma(\nu_0/2)} \Gamma\left(\frac{\nu_0+\nu_1}{2}\right) U\left(\frac{\nu_0+\nu_1}{2}, 1 + \frac{\nu_0}{2}, z\right), \end{aligned}$$

by (A8) in Appendix A. Furthermore, using (A9) in Appendix A, that is,

$$U\left(\frac{\nu_1}{2}, 1 - \frac{\nu_0}{2}, z\right) = z^{\nu_0/2} U\left(\frac{\nu_0+\nu_1}{2}, 1 + \frac{\nu_0}{2}, z\right),$$

we have the result (1) after replacing ν_0 with ν_2 .

In deriving the CF of the multivariate F distribution, we found a much shorter derivation of the CF of the (univariate) F distribution as a by-product compared to the ODE approach. However, to the best of our knowledge, the ODE approach is new in this area, so it is still useful in devising proof for deriving the CF

of symmetric (Student's t -distribution in [22]) and asymmetric distributions such as F and skew-normal distributions. This method could be used to obtain CFs for many other continuous random variables, which is an ongoing project.

4. Conclusions

For two asymmetric distributions, F and skew-normal, the CFs are derived by the ODEs, which avoid the computation of the contour integration. The main scenario is to first find an ODE with polynomial coefficients by applying derivatives to the PDF, and then use integration by parts to obtain an ODE satisfied by the CF. Fortunately, the ODEs of the former two distributions are tractable to deduce a formula for the CF. In the F distribution, the calculation process explains why the CF consists of Tricomi's function. The Faddeeva function is applied in the computation of the CF for the skew-normal distribution. Although this study focuses on the canonical univariate situation, the result can be easily generalized to the case having location and scale parameters and to the multivariate case using simple stochastic representations. The derivation of the CF for the multivariate F distribution using a stochastic representation herein is much simpler and easier to understand. When the dimensions are limited to one in the multivariate F distribution, the consistency of this CF with that of the former (univariate) F distribution further suggests that the derivation is reasonable.

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Appendix A

See [7,26] for the following facts. The complete solution of Kummer's equation

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0, \quad (\text{A1})$$

is given by $w(z) = C_1 M(a, b, z) + C_2 U(a, b, z)$ where C_1 and C_2 are arbitrary constants except that $M(a, b, z)$ does not exist when b is a nonpositive integer. Tricomi's function, $U(a, b, z)$, has the following relationship with Kummer's function ([27]) of the first kind M :

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z),$$

where

$$M(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s, \quad (a)_s = a(a+1) \dots (a+s-1), \text{ and } (a)_0 = 1.$$

Tricomi's function, $U(a, b, z)$, has limiting forms such as

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + O(z), \quad \Re(b) \leq 0, b \neq 0 \text{ as } z \rightarrow 0, \quad (\text{A2})$$

$$U(a, b, z) \sim z^{-a} \text{ as } z \rightarrow \infty, \quad (\text{A3})$$

where $|\text{ph}(z)| < \frac{3}{2}\pi - \delta$ with an arbitrary small positive constant δ . $\text{ph}(z)$ is the phase or argument of the complex number z , and $\Re(b)$ is the real part of the complex number b . Two limiting forms of $M(a, b, z)$ are

$$M(a, b, z) = 1 + O(z) \text{ as } z \rightarrow 0, \quad (\text{A4})$$

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \text{ as } z \rightarrow \infty, \quad (\text{A5})$$

where $|\text{ph}(z)| < \frac{1}{2}\pi - \delta$ with an arbitrary small positive constant δ , and $a \neq 0, -1, \dots$ (polynomial cases). Tricomi's function, $U(a, b, z)$, has the following properties:

$$\frac{d^n}{dz^n} U(a, b, z) = (-1)^n (a)_n U(a + n, b + n, z), \quad (\text{A6})$$

$$(b - a)U(a, b, z) = -U(a - 1, b, z) + zU(a, b + 1, z), \quad (\text{A7})$$

$$\Gamma(a)U(a, b, z) = \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt, \quad (\text{A8})$$

$$U(a, 1 - n, z) = z^n U(a + n, 1 + n, z). \quad (\text{A9})$$

Appendix B

Lemma A1. The Faddeeva (or Faddeyeva) function ([26] (Chapter 7) and [28]),

$$w(z) = e^{-z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{u^2} du \right),$$

is the particular solution of the differential equation

$$\frac{dw}{dz} + 2zw - \frac{2}{\sqrt{\pi}} i = 0. \quad (\text{A10})$$

This equation is referred as a differential equation of the Faddeeva function.

Proof of Lemma 1. Derive the first derivative of w at z , then the differential equation is obtained. $\square$

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