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Modular Edge-Gracefulness of Graphs without Stars

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Abstract: We investigate the modular edge-gracefulness $k(G)$ of a graph, i.e., the least integer k such that taking a cyclic group $\mathbb{Z}_k$ of order k, there exists a function $f : E(G) \to \mathbb{Z}_k$ so that the sums of edge labels incident with every vertex are distinct. So far the best upper bound on $k(G)$ for a general graph G is $2n$, where n is the order of G. In this note we prove that if G is a graph of order n without star as a component then $k(G) = n$ for $n \not\equiv 2 \pmod 4$ and $k(G) = n + 1$ otherwise. Moreover we show that for such G for every integer $t \geq k(G)$ there exists a $\mathbb{Z}_t$-irregular labeling.

Keywords: graph labeling; modular edge-gracefulness; cyclic abelian group; group irregularity strength

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1. Introduction

Let $G = (V, E)$ be a simple finite graph of order $|V| = n$. The neighborhood $N(v)$ of a vertex v is the set of vertices adjacent to x, and the degree $d(v)$ of v is $N(v)$, the size of the neighborhood of v. The minimal vertex degree is denoted by δ.

The irregularity strength is the smallest integer $s(G) = s$ for which there exists an edge labeling $f : E(G) \to \{1, \ldots, s\}$ such that $\sum_{e:u\in e} f(e) \neq \sum_{e:v\in e} f(e)$ for all pairs of different vertices $u, v \in V(G)$. The problem of finding $s(G)$ was introduced by Chartrand et al. in [1]. An upper bound $s(G) \leqslant n - 1$ was proved for all graphs containing no isolated edges and at most one isolated vertex, except for the graph K_3 [2,3]. A better upper bound can be improved for graphs with sufficiently large minimum degree δ. Best published result is due to Kalkowski, Karoński and Pfender (see [4]) is $s(G) \leqslant 6n/\delta$ for $\delta > 6$.

On the other hand Lo in [5] defined edge graceful labelings. A graph G of order n and size m is edge-graceful if there exists a bijective mapping $f : E(G) \to \{1, 2, \ldots, m\}$ such $\sum_{e:u\in e} f(e) \not\equiv \sum_{e:v\in e} f(e) \pmod n$ for all pairs of different vertices $u, v \in V(G)$. Jones combined the concepts of graceful labeling and irregular labeling into $\mathbb{Z}_k$-irregular labeling ([6–8]). He defined the modular edge-gracefulness of graphs as the smallest integer $k(G) = k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $w : V(G) \to \mathbb{Z}_k$ defined by

$$w(u) = \sum_{v\in N(u)} f(uv) \pmod k$$

is one-to-one. For a vertex u we call $w(u)$ the weighted degree of u.

Verifying a conjecture by Gnanajothi on trees [9], Fujie-Okamoto, Jones, Kolasinski and Zhang [6] showed that every tree of order $n \geqslant 3$ has the modular edge-gracefulness equal to n if and only if $n \not\equiv 2 \pmod 4$ and moreover proved that it is true for any connected graph of order $n \geqslant 3$. Formally:

Theorem 1 ([6]). *Let G be a connected graph of order $n \geq 3$, then*

$$k(G) = \begin{cases} n, & \text{if} \quad n \not\equiv 2 \pmod 4, \\ n+1, & \text{if} \quad n \equiv 2 \pmod 4. \end{cases}$$

For a non-connected graph G with no K_2 components, it is known that $k(G) \leq 2|V(G)|$ [10]. Moreover Anholcer and Cichacz proved the following.

Theorem 2 ([11]). *Let G be a graph of order n with no components of order less than 3 and no $K_{1,2u+1}$ components for any integer $u \geq 1$. Then:*

$$\begin{array}{lll} k(G) = n, & \text{if} \quad n \equiv 1 & \pmod 2, \\ k(G) = n+1, & \text{if} \quad n \equiv 2 & \pmod 4, \\ k(G) \leq n+1, & \text{if} \quad n \equiv 0 & \pmod 4. \end{array}$$

Lemma 1 ([11]). *Let G be a graph of order n with no components of order less than 3 and no $K_{1,2u+1}$ components for any integer $u \geq 1$. Then for every odd integer $t \geq n$ there exists a $\mathbb{Z}_t$-irregular labeling.*

Corollary 1 ([11]). *Let G be a disconnected graph of order n with all the bipartite components having both color classes of even order and with no component of order less than 3. Let $s = n+1$ if $n \equiv 2 \pmod 4$ and $s = n$ otherwise. Then for every integer $t \geq s$ there exists a $\mathbb{Z}_t$-irregular labeling.*

In this paper we would like to develop the method of augmented walks from [11] in order to solve the following conjecture:

Conjecture 1 ([12]). *Let G be a graph of order n with no $K_{1,u}$ components for any integer $u \geq 0$. Then $k(G) = n$ if $n \not\equiv 2 \pmod 4$ and $k(G) = n+1$ otherwise.*

2. $\mathbb{Z}_n$-Irregular Labeling

Given any two vertices x_1 and x_2 belonging to the same connected component of G, there exist walks from x_1 to x_2. Some of them may consist of an even number of vertices (some of them being repetitions). We are going to call them even walks. Analogously, the walks with an odd number of vertices will be called odd walks. We will always choose a shortest even or a shortest odd walk from x_1 to x_2 (note that sometimes it is not a path).

As in [11] we start with 0 on all the edges of G. Then, in every step we will choose x_1 and x_2 and add some labels to all the edges of the chosen walk from x_1 to x_2. To be more specific, we will add some element a of the group to the labels of all the edges having an odd position on the walk (starting from x_1) and $-a$ to the labels of all the edges having even position. It is possible that some labels will be modified more than once, as the walk does not need to be a path. We will denote such situation with $\phi_e(x_1, x_2) = a$ if we label a shortest even walk and $\phi_o(x_1, x_2) = a$ if we label the shortest odd walk. Observe that putting $\phi_e(x_1, x_2) = a$ results in adding a to the weighted degrees of both x_1 and x_2, while $\phi_o(x_1, x_2) = a$ means adding a to the weighted degree of x_1 and $-a$ to the weighted degree of x_2. In both cases, the operation does not change the weighted degree of any other vertex of the walk. Note that if some component G_1 of G is not bipartite, then for any vertices $x_1, x_2 \in V(G_1)$ there exist both even and odd walks.

Differently than in [11], where the zero-sum cyclic group was decomposed into zero-sum subsets, we will find some zero-sum subsets in the cyclic group that is not zero-sum itself. Therefore, in the proof we shall apply the following corollary of Lemma 3.2. from [13].

Corollary 2. *Let $n \equiv 0 \pmod 4$ and m, l be natural numbers such that $3m + 2l = n - 2$. Then the set $S = \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$ can be partitioned into m triples $A_1, A_2, \ldots, A_m$ and l pairs $B_1, B_2, \ldots, B_l$ such that*

$\sum_{x \in A_i} x = 0$ for $i = 1, 2, \ldots, m$ and $\sum_{x \in B_j} x = 0$ for $j = 1, 2, \ldots, l$. Moreover there exists such partition of S fulfilling the above conditions that if $m \geq \frac{n}{4}$, then $\frac{n}{4} \in A_i$ for some i, whereas for $m < \frac{n}{4}$ we have $\frac{n}{4} \in B_j$ for some j.

Lemma 2. *Let G be a graph of order n with no $K_{1,u}$ components for any integer $u \geq 0$. Let $s = n + 1$ if $n \equiv 2 \pmod 4$ and $s = n$ otherwise. Then for every integer $t \geq s$ there exists a $\mathbb{Z}_t$-irregular labeling of G.*

Proof. By Theorems 1 and 2 we can assume that G is disconnected. We are going to divide the vertices of G into triples and pairs. Namely, we will take one triple from each component (or part of a bipartite component) which has odd order and all the remaining vertices we will join into pairs, then using the partition of the group $\mathbb{Z}_t$ from Corollary 2 we will arrive distinct weighted degrees of vertices. For this process we need some notation. Let p_1 be the number of bipartite components of G with both color classes odd, p_2 with both classes even and p_3 with one class odd and one even. Let p_4 be the number of remaining components of odd order and p_5—the number of remaining components of even order. The number of triples equals to $2p_1 + p_3 + p_4$. The remaining vertices form the pairs. Observe that by Corollary 1 we can assume that $p_1 + p_3 > 0$. Moreover by Lemma 1 we assume that t is even.

We start with the case $t = n \equiv 0 \pmod 4$ what implies $2p_1 + p_3 + p_4 - 2 \geq 0$. Let $m = p_3 + p_4$ if ($p_1 = 0$ and $p_3 + p_4 - 2 \geq \frac{n}{4}$) and $m = 2p_1 + p_3 + p_4 - 2$ otherwise. Let $l = (n - 2 - 3m)/2$. Note that $l \geq 0$, therefore the set $S = \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$ can be partitioned into m triples $A_1, A_2, \ldots, A_m$ and l pairs $B_1, B_2, \ldots, B_l$ such that $\sum_{x \in A_i} x = 0$ for $i = 1, 2, \ldots, m$ and $\sum_{x \in B_j} x = 0$ for $j = 1, 2, \ldots, l$ by Corollary 2. Let $A_i = \{a_i, b_i, c_i\}$ for $i = 1, 2, \ldots, m$ and let $B_j = \{d_j, -d_j\}$ for $j = 1, \ldots, l$. It is easy to observe that for a given element $g \in S$ not belonging to any triple, we have $(g, -g) = B_j$ for some j.

Let us start the labeling. For both vertices and labels, we are numbering the pairs and triples consecutively, in the same order as they appear in the labeling algorithm described below, every time using the lowest index that has not been used so far (independently for the lists of couples and triples).

Given any bipartite component G with both color classes even, we divide the vertices of every color class into pairs (x_j^1, x_j^2), putting

$$\phi_o(x_j^1, x_j^2) = d_j$$

for every such pair. This gives as weighted degrees $w(x_j^1) = d_j$ and $w(x_j^2) = -d_j$. We proceed in a similar way in the case of all the non-bipartite components of even order, coupling the vertices of every such component in any way.

Note, that since we use Corollary 2, we still need to find a way to include 0 and $\frac{n}{2}$ as weighted degrees, in order to do that, we will consider now two cases on p_1.

Case 1. $p_1 > 0$

Note that in this case $l \geq 2$. If both color classes of a bipartite component are of odd order, then they both have at least 3 vertices. We choose three of them, denoted with x_j, y_j and z_j, in one class and another three, x_{j+1}, y_{j+1} and z_{j+1}, in another one.

Suppose first that $\frac{n}{4} \in A_i$ for some i. Without loss of generality we can assume that $i = 1$ and $a_1 = \frac{n}{4}$. Note that $b_1 + c_1 = -\frac{n}{4}$. We apply

$$\begin{aligned}
\phi_e(x_1, z_2) &= b_1, \\
\phi_e(y_1, z_2) &= c_1, \\
\phi_e(z_1, z_2) &= \tfrac{n}{2}, \\
\phi_o(x_2, y_2) &= d_l.
\end{aligned}$$

Observe that $w(x_1) = b_1$, $w(y_1) = c_1$, $w(z_1) = \frac{n}{2}$, $w(x_2) = d_l$, $w(y_2) = -d_l$ and $w(z_2) = b_1 + c_1 + \frac{n}{2} = \frac{n}{4}$. Taking $A_{m+1} = B_{l-1} \cup \{0\}$ we proceed with the remaining vertices of these components as in the case when both color classes are even.

If now $\frac{n}{4} \in B_j$ for some j. Without loss of generality we can assume that $j = l$ and $d_l = \frac{n}{4}$. We apply

$$\begin{aligned} \phi_e(x_1, x_2) &= d_l, \\ \phi_e(y_1, x_2) &= \tfrac{n}{2}, \\ \phi_e(z_1, x_2) &= 0, \\ \phi_o(y_2, z_2) &= d_{l-1}. \end{aligned}$$

Observe that $w(x_1) = d_l = \frac{n}{4}$, $w(y_1) = \frac{n}{2}$, $w(z_1) = 0$, $w(x_2) = \frac{n}{4} + \frac{n}{2} = \frac{3n}{4} = -d_l$, $w(y_2) = d_{l-1}$ and $w(z_2) = -d_{l-1}$. We proceed with the remaining vertices of these components as in the case when both color classes are even.

In the rest of bipartite component with both odd color classes, we apply:

$$\begin{aligned} \phi_e(x_j, z_{j+1}) &= a_j, \\ \phi_e(y_j, z_{j+1}) &= b_j, \\ \phi_e(z_j, z_{j+1}) &= c_j, \\ \phi_e(x_{j+1}, z_j) &= a_{j+1}, \\ \phi_e(y_{j+1}, z_j) &= b_{j+1}, \\ \phi_e(z_{j+1}, z_j) &= c_{j+1}. \end{aligned}$$

In the case of non-bipartite components of odd order, we choose three vertices. We apply

$$\begin{aligned} \phi_e(x_j, z_j) &= a_j, \\ \phi_e(y_j, z_j) &= b_j, \\ \phi_e(z_j, z_j) &= c_j. \end{aligned}$$

Finally for bipartite components of odd order we choose four vertices x_j, y_j, z_j and v_2 (v_2 belongs to the even color class and three other vertices to the odd one). We apply

$$\begin{aligned} \phi_e(x_j, v_2) &= a_j, \\ \phi_e(y_j, v_2) &= b_j, \\ \phi_e(z_j, v_2) &= c_j. \end{aligned}$$

We proceed with the remaining vertices of these components as in the case when both color classes are even.

The labeling defined above is $\mathbb{Z}_n$-irregular. Indeed, for $j > 2$ in the j^{th} triple of vertices the weighted degrees are equal to $w(x_j) = a_j$, $w(y_j) = b_j$ and $w(z_j) = c_j$ and in the j^{th} pair we have $w(x_j^1) = d_j$ and $w(x_j^2) = -d_j$.

Case 2. $p_1 = 0$.

Note that by Corollary 1 $p_3 > 0$, what implies that $p_3 + p_4 - 2 \geq 0$.

Assume first that $p_3 + p_4 - 2 < \frac{n}{4}$. Then $m = p_3 + p_4 - 2$, $l \geq 3$ and moreover $\frac{n}{4} \in B_j$ for some j by Corollary 2. Without loss of generality we can assume that $j = l$ and it is still unused. Let $A_{m+1} = B_{l-2} \cup \{0\}$. For the case of the non-bipartite component of odd order we choose three vertices x_j, y_j, z_j from the odd color class. For of such a component let t_2, h_2 be two vertices from the even class. We apply

$$\begin{aligned} \phi_e(x_1, t_2) &= d_l, \\ \phi_e(x_1, h_2) &= \tfrac{n}{2}, \\ \phi_o(y_1, z_1) &= d_{l-1}. \end{aligned}$$

Observe that $w(x_1) = d_l + \frac{n}{2} = -d_l$, $w(y_1) = d_{l-1}$, $w(z_1) = -d_{l-1}$, $w(t_2) = d_l$, $w(h_2) = \frac{n}{2}$. We proceed with the remaining vertices of these components as in the case when both color classes are even.

If now $p_2 + p_4 - 2 \geq \frac{n}{4}$, then $m = p_3 + p_4 \geq \frac{n}{4}$ and $\frac{n}{4} \in A_i$ for some i by Corollary 2. Without loss of generality we can assume that $i = 1$ and $a_1 = \frac{n}{4}$. We apply

$$\phi_e(x_1, t_2) = b_1,$$
$$\phi_e(y_1, t_2) = c_1,$$
$$\phi_e(z_1, t_2) = \tfrac{n}{2}.$$

Observe that $w(x_1) = b_1$, $w(y_1) = c_1$, $w(z_1) = \frac{n}{2}$, $w(t_2) = -a_1 + \frac{n}{2} = \frac{n}{4} = a_1$, $w(h_2) = 0$. We proceed with the remaining vertices of these components as in the case when both color classes are even.

For all the remaining components we proceed the same as in Case 1.

For $t \neq n$ there is $t > n$. If $t = n+1$, then $m = 2p_1 + p_3 + p_4 - 1$ and $l = (n-2-3m)/2$ and $A_{m+1} = B_l \cup \{0\}$. For $t = n+2$, then $m = 2p_1 + p_3 + p_4$ and $l = (n-2-3m)/2$. If $t = n+3$, then $m = 2p_1 + p_3 + p_4 - 1$ and $l = (n-2-3m)/2$ and $A_{m+1} = B_l \cup \{0\}$. For $t \geq n+4$ let $m = 2p_1 + p_3 + p_4$ and $l = (n-2-3m)/2$ for n even and $m = 2p_1 + p_3 + p_4 + 1$ and $l = (n-2-3m)/2$ otherwise. We proceed in the same way as above (but not using the element $\frac{n}{2}$). □

By Theorem 2 and Lemma 2 we obtain immediately the following.

Theorem 3. *Let G be a graph of order n with no $K_{1,u}$ components for any integer $u \geq 0$. Then $k(G) = n$ if $n \not\equiv 2 \pmod 4$ and $k(G) = n+1$ otherwise.*

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