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Exact Inference for an Exponential Parameter under Generalized Adaptive Progressive Hybrid Censored Competing Risks Data

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Abstract: It is known that the lifetimes of items may not be recorded exactly. In addition, it is known that more than one risk factor (RisF) may be present at the same time. In this paper, we discuss exact likelihood inference for competing risk model (CoRiM) with generalized adaptive progressive hybrid censored exponential data. We derive the conditional moment generating function (ConMGF) of the maximum likelihood estimators of scale parameters of exponential distribution (ExpD) and the resulting lower confidence bound under generalized adaptive progressive hybrid censoring scheme (GeAdPHCS). From the example data, it can be seen that the PDF of MLE is almost symmetrical.

Keywords: competing risk; exact likelihood inference; exponential distribution; generalized adaptive progressive hybrid censoring

1. Introduction

Let us consider a lifetime test where items are kept under observation until failure. These items could be patients put under certain medical research or they could be parts or some system in reliability test. However, it is known that the lifetimes of items may not be recorded exactly. There are also conditions wherein the elimination of items prior to failure is prearranged in order to lower the cost or time associated with test. In addition, it is known that more than one RisF may be present at the same time. Following Cox [1], we refer to this model as CoRiM. In CoRiM, it is supposed that the RisFs are statistically independent. Moreover, it is supposed that an indicator denoting the RisF of failure and competing risks (CoR) data consist of observed failure times. Lately, we are interested in one special cause in the presence of other RisFs.

Based on CoRiM, Author1 [2] developed CoRiM under generalized Type I hybrid censoring scheme and constructed exact confidence intervals (CI) and approximate CIs by exact distributions, asymptotic distributions, the parametric bootstrap method and the Bayesian posterior distribution, respectively. Step stress partially accelerated life testing plan for competing risk using adaptive Type I progressive hybrid censoring (Ad1PHCS) was discussed by Author1 [3]. For Weibull distribution, Author1 [4] developed a CoRiM under progressive Type II censoring scheme (Pr2CS) with binomial removals. For Lindley distribution, Author1 [5] developed a CoRiM under Pr2CS with binomial removals. For Chen distribution, [6] developed a CoRiM under Pr2CS.

Conventional Type I and II censoring schemes cannot be used if the experimenter wants to remove the live experimental unit at a point other than the final end point of the experiment. Therefore, recently, Pr2CS and Ad1PHCS have become quite popular in a life-testing problem and reliability analysis [7–11]. Although Pr2CS and Ad1PHCS assure a pre-assigned number of failures, they have the drawback that it might take a long time to observe a pre-assigned number of failures and terminate

the test. For this reason, Author1 [12] suggested a GeAdPHCS in which the test is assured to end at a pre-assigned time. The survival test based on the GeAdPHCS can save both the total time and cost on tests. If the experimenter has prepaid for the use of the facility for $\mathcal{T}$ units of time, GeAdPHCS can be applied.

GeAdPHCS can be explained as follows. Consider a life-testing experiment in which n identical units are put on test. The times $\mathcal{T}_1$ and $\mathcal{T}_2$ and integer m are pre-assigned such that $m \leq n$ and $0 < \mathcal{T}_1 < \mathcal{T}_2 < \infty$. In addition, the pre-assigned PrCS $(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_m)$ satisfy $\sum_{i=1}^m \mathfrak{R}_i + m = n$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ represent the number of failures up to pre-assigned times $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Likewise, let d_1 and d_2 be the observed value of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. When the first failure ($X_{1:m:n}$) is observed, the $\mathfrak{R}_1$ survival units are removed randomly from the test. Furthermore, when the second failure ($X_{2:m:n}$) is observed, the $\mathfrak{R}_2$ survival units are removed randomly from the test, and so on. If the m th failure is observed before the pre-assigned time $\mathcal{T}_1$ ($X_{m:m:n} < \mathcal{T}_1$), terminate the test at $X_{m:m:n}$ (Case I). If $\mathcal{T}_1 < X_{m:m:n} < \mathcal{T}_2$, then, instead of terminating the test by removing all survival units at pre-assigned time $\mathcal{T}_1$, continue to observe failures, without any removals, up to the m th failure (Case II). Therefore, $\mathfrak{R}_{d_1+1} = \dots = \mathfrak{R}_{m-1} = 0$. If $\mathcal{T}_2 < X_{m:m:n}$, and terminate the test at pre-assigned time $\mathcal{T}_2$ (Case III). Here, the pre-assigned time $\mathcal{T}_2$ expresses the longest test time that the experimenter is willing to allow the test to continue. In the GeAdPHCS, therefore, there are Cases I–III as follows:

Case I: $\{X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}\}$, if $X_{m:m:n} < \mathcal{T}_1$.

Case II: $\{X_{1:m:n}, X_{2:m:n}, \dots, X_{d_1:m:n}, \dots, X_{m:m:n}\}$, if $\mathcal{T}_1 < X_{m:m:n} < \mathcal{T}_2$, $\mathfrak{R}_{d_1+1} = \dots = \mathfrak{R}_{m-1} = 0$.

Case III: $\{X_{1:m:n}, X_{2:m:n}, \dots, X_{d_1:m:n}, \dots, X_{d_2:m:n}\}$, if $X_{m:m:n} > \mathcal{T}_2$, $\mathfrak{R}_{d_1+1} = \dots = \mathfrak{R}_{d_2-1} = 0$.

Here, $X_{d_1:m:n} < \mathcal{T}_1 < X_{d_1+1:m:n}$, $X_{d_2:m:n} < \mathcal{T}_2 < X_{d_2+1:m:n}$ and $X_{d_2+1:m:n}, \dots, X_{m:m:n}$ are not observed for Case III. A schematic representation of the GeAdPHCS is presented in Figure 1.

Figure 1. Schematic representation of GeAdPHCS.

In this paper, we consider independent identically distributed (iid) exponential CoRiM under GeAdPHCS. In Section 2, we derive the distributions of the MLEs of parameters as well as CIs for MLEs of parameters. In Section 3, we present the results of a numerical study to investigate the MSEs, biases, confidence lengths (CL) and coverage percentages (CP) of the MLEs under various GeAdPHCS.

In addition, an illustrative example is presented. Finally, the summary and conclusion are presented in Section 4.

2. Model and Conditional MLEs

2.1. Model

Suppose that n randomly selected items with CoR data for an ExpD are placed on a life test. In addition, we suppose that $X_1, X_2, \dots, X_n$ are independent and identically distributed with an ExpD. Here, $X_i = \min\{X_{i1}, X_{i2}, \dots, X_{ij}\}$, X_{ij} denotes the lifetime of the i th item under the j th RisF with probability density function (PDF) and cumulative distribution function (CDF) such as $g_j(x) = \exp(-x/\theta_j)/\theta_j$ and $G_j(x) = 1 - \exp(-x/\theta_j)$, respectively. In addition, we suppose that there are two RisFs for the failure of items. Then, the PDF and CDF of lifetime can be obtained as

$$F(x; \theta) = 1 - \exp \left[- \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) x \right],$$

$$f(x; \theta) = \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \exp \left[- \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) x \right], x > 0, \theta_1 > 0, \theta_2 > 0,$$

where $\theta = (\theta_1, \theta_2)$.

It is well known that each failure observation is composed of failure lifetime and the cause of failure under the CoRiM. Let $\mathbf{X} = \{x_{1:m:n}, x_{2:m:n}, \dots, x_{m:m:n}\}$ denote ordered Pr2CS data of n items and $\mathbf{Z} = \{z_1, z_2, \dots, z_m\}$ denote the indicator of risk cause corresponding to the ordered Pr2CS data. Here, $z_i = 1, i = 1, 2, \dots, m$ denotes the failure of the i th unit caused by the first RisF. On the other hand, $z_i = 0$ denotes that the second RisF is responsible for the i th failure. Based on the above assumption, the joint PDF of lifetime and corresponding factor $(\mathbf{X}, \mathbf{Z})$ is given by

$$f_{\mathbf{X}, \mathbf{Z}}(x, j) = \left(\frac{1}{\theta_j} \right) \exp \left[- \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) x \right], j = 1, 2.$$

From GeAdPHCS data, therefore, we have the following data:

$$(x_{1:m:n}, z_1), (x_{2:m:n}, z_2), \dots, (x_{u:m:n}, z_u),$$

where $u = m$ for Cases I and II and $u = \mathcal{D}_2$ for Case III.

Based on the three scenarios as discussed above, the likelihood function ($\mathcal{L}$) is

$$\mathcal{L}(\theta | \mathbf{x}) = \begin{cases} \zeta'_m \prod_{j=1}^m f_{X,Z}(x_{j:m:n}, 1)^{z_j} f_{X,Z}(x_{j:m:n}, 2)^{1-z_j} [1 - \mathcal{F}(x_{j:m:n})]^{\Re_j}, & \mathcal{D}_1 = m, \\ \zeta'_m \prod_{j=1}^m f_{X,Z}(x_{j:m:n}, 1)^{z_j} f_{X,Z}(x_{j:m:n}, 2)^{1-z_j} [1 - \mathcal{F}(x_{j:m:n})]^{\Re_j}, & \mathcal{D}_1 = 0, 1, \dots, m-1, \mathcal{D}_2 = m, \\ \zeta'_{\mathcal{D}_2} \prod_{j=1}^{\mathcal{D}_2} f_{X,Z}(x_{j:m:n}, 1)^{z_j} f_{X,Z}(x_{j:m:n}, 2)^{1-z_j} [1 - \mathcal{F}(x_{j:m:n})]^{\Re_j} [1 - \mathcal{F}(\mathcal{T}_2)]^{\Re'_{d_2}}, & \mathcal{D}_2 = 1, 2, \dots, m-1, \end{cases} \quad (1)$$

where $\zeta'_d = \prod_{j=1}^d \sum_{k=j}^m (\Re_k + 1)$ and $\Re'_{d_2} = n - \sum_{i=1}^{d_1} \Re_i - d_2$. From (1), we obtain the MLE of θ_j as

$$\hat{\theta}_j = \begin{cases} \frac{1}{n_j} \left[\sum_{j=1}^m (1 + \Re_j) x_{j:m:n} \right], & \mathcal{D}_1 = m, \\ \frac{1}{n_j} \left[\sum_{j=1}^{d_1} (1 + \Re_j) x_{j:m:n} + \sum_{j=d_1+1}^m x_{j:m:n} \right], & \mathcal{D}_1 = 0, 1, \dots, m-1, \mathcal{D}_2 = m, \\ \frac{1}{n_j} \left[\sum_{j=1}^{d_1} (1 + \Re_j) x_{j:m:n} + \sum_{j=d_1+1}^{d_2} x_{j:m:n} + \Re'_{d_2} \mathcal{T}_2 \right], & \mathcal{D}_2 = 1, \dots, m-1. \end{cases} \quad (2)$$

Here, we denote the total failure number of units due to the RisF j by n_j , $j = 1, 2$, and then it is easy to obtain $n_1 = \sum_{i=1}^u z_i$ and $n_2 = \sum_{i=1}^u (1 - z_i) = u - n_1$.

Note that, from (2), the MLEs do not exist when $n_j = 0$, $j = 1, 2$. To estimate θ_j , we have to observe at least one failure caused by each RisF. That is,

$$\zeta^{(u)} = \{n_1 \geq 1, n_2 \geq 1 | n_1 + n_2 = u\}.$$

2.2. Exact Conditional Inference for MLE

Lemma 1 established in [13] is used to derive the explicit expression of the ConMGF of MLE.

Lemma 1. Let $v_j > 0$ where $j = 1, 2, \dots, m$, and let X denote the absolutely continuous random variable with PDF $f(x)$ and CDF $\mathcal{F}(x)$. Then, for $m \geq 1$, we have

$$\int_{-\infty}^{x_{m+1}} \cdots \int_{-\infty}^{x_3} \int_{-\infty}^{x_2} \prod_{j=1}^m f(x_j) \{1 - \mathcal{F}(x_j)\}^{v_j-1} dx_1 dx_2 \cdots dx_m = \sum_{i=0}^m \zeta_{i,m}(\mathbf{v}_m) \{1 - \mathcal{F}(x_{m+1})\}^{\varphi_{i,m}(\mathbf{v}_m)},$$

where $\mathbf{v}_m = (v_1, v_2, \dots, v_m)$; $\zeta_{i,m}(\mathbf{v}_m) = \frac{(-1)^i}{\{\prod_{j=1}^i \sum_{k=m-i+1}^{m-i+j} v_k\} \{\prod_{j=1}^{m-i} \sum_{k=j}^{m-i} v_k\}}$, $\varphi_{i,m}(\mathbf{v}_m) = \sum_{j=m-i+1}^m v_j$ with the usual conventions that $\prod_{j=1}^0 \zeta_j \equiv 1$ and $\sum_{j=i}^0 \zeta_j \equiv 0$.

Theorem 1. Conditional on $\zeta^{(u)}$, the ConMGF of $\hat{\theta}_1$ is given by

$$\begin{aligned} M_{\hat{\theta}_1}(t) &= E(e^{t\hat{\theta}_1} | \zeta^{(u)}) \\ &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)} | \mathcal{D}_1 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_m + 1) \\ &\quad \times q_1^{(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}) \mathfrak{R}_{m-j+1}^*} + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)} | \mathcal{D}_1 = d_1, \mathcal{D}_2 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-m} \\ &\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1, d_1}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1) \zeta_{i_2, d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m + 1) q_1^{(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}) \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1)} \\ &\quad \times q_2^{(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}) \sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1)} + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\zeta^{(d_2)} | \mathcal{D}_2 = d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^{d_2}} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-d_2} \\ &\quad \times \sum_{j=0}^{d_2} \zeta_{j, d_2}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2} + 1) q_2^{(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}) \mathfrak{R}'_{d_2-i+1}}. \end{aligned}$$

where $q_1 = \exp\left[-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) \mathcal{T}_1\right]$, $q_2 = \exp\left[-\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right) \mathcal{T}_2\right]$ and $\mathbf{1}_k$ means that all k elements equal to 1.

Proof. The proof of Theorem 1 is given in Appendix A. $\square$

Theorem 2. Conditional on $\zeta^{(u)}$, the ConMGF of $\hat{\theta}_2$ is given by

$$\begin{aligned}
 M_{\hat{\theta}_2}(t) &= E\left(e^{t\hat{\theta}_1} | \zeta^{(u)}\right) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)} | \mathcal{D}_1 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \left(1 - \frac{t}{m-i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_m + 1) \\
 &\quad \times q_1^{\left(1 - \frac{t}{m-i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \mathfrak{R}_{m-j+1}^*} + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)} | \mathcal{D}_1 = d_1, \mathcal{D}_2 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \left(1 - \frac{t}{m-i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-m} \\
 &\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1, d_1}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1) \zeta_{i_2, d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m + 1) q_1^{\left(1 - \frac{t}{m-i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1)} \\
 &\quad \times q_2^{\left(1 - \frac{t}{m-i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1)} + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\zeta^{(d_2)} | \mathcal{D}_2 = d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^{d_2}} \left(1 - \frac{t}{d_2-i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-d_2} \\
 &\quad \times \sum_{j=0}^{d_2} \zeta_{j, d_2}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2} + 1) q_2^{\left(1 - \frac{t}{d_2-i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \mathfrak{R}'_{d_2-i+1}}.
 \end{aligned}$$

Corollary 1. The first and second moments of $\hat{\theta}_1$ are given by

$$\begin{aligned}
 E_{\theta_1}(\hat{\theta}_1) &= M'_{\hat{\theta}_1}(0) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)} | \mathcal{D}_1 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_m + 1) q_1^{\mathfrak{R}_{m-j+1}^*} \\
 &\quad \times \left[\frac{m}{i} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) + \frac{\mathcal{T}_1}{i} \mathfrak{R}_{m-j+1}^* \right] + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)} | \mathcal{D}_1 = d_1, \mathcal{D}_2 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \\
 &\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1, d_1}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1) \zeta_{i_2, d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m + 1) q_1^{\sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1)} q_2^{\sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1)} \\
 &\quad \times \left[\frac{m}{i} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) + \frac{\mathcal{T}_1}{i} \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1) + \frac{\mathcal{T}_2}{i} \sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1) \right] \\
 &\quad + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\zeta^{(d_2)} | \mathcal{D}_2 = d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^{d_2}} \sum_{j=0}^{d_2} \zeta_{j, d_2}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2} + 1) \\
 &\quad \times q_2^{\mathfrak{R}'_{d_2-i+1}} \left[\frac{d_2}{i} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) + \frac{\mathcal{T}_2}{i} \mathfrak{R}'_{d_2-i+1} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 E_{\theta_1}(\hat{\theta}_1^2) &= M''_{\theta_1}(0) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)}|\mathcal{D}_1=m)} \binom{m}{i} \frac{\theta_1^{m-i}\theta_2^i}{(\theta_1+\theta_2)^m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_m+1) q_1^{\mathfrak{R}_{m-j+1}^*} \\
 &\quad \times \left[\left\{ \frac{m}{i} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right) + \frac{\mathcal{T}_1}{i} \mathfrak{R}_{m-j+1}^* \right\}^2 + \frac{m}{i^2} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right)^2 \right] + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)}|\mathcal{D}_1=d_1, \mathcal{D}_2=m)} \\
 &\quad \times \binom{m}{i} \frac{\theta_1^{m-i}\theta_2^i}{(\theta_1+\theta_2)^m} \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1,d_1}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_{d_1}+1) \zeta_{i_2,d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m+1) \\
 &\quad \times q_1^{\sum_{j=d_1-i_1+1}^{m-i_2}(\mathfrak{R}_j+1)} q_2^{\sum_{j=m-i_2+1}^m(\mathfrak{R}_j+1)} \\
 &\quad \times \left[\left\{ \frac{m}{i} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right) + \frac{\mathcal{T}_1}{i} \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j+1) + \frac{\mathcal{T}_2}{i} \sum_{j=m-i_2+1}^m (\mathfrak{R}_j+1) \right\}^2 + \frac{m}{i^2} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right)^2 \right] \\
 &\quad + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\zeta^{(d_2)}|\mathcal{D}_2=d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i}\theta_2^i}{(\theta_1+\theta_2)^{d_2}} \sum_{j=0}^{d_2} \zeta_{j,d_2}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_{d_1}+1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2}+1) \\
 &\quad \times q_2^{\mathfrak{R}'_{d_2-i+1}} \left[\left\{ \frac{d_2}{i} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right) + \frac{\mathcal{T}_2}{i} \mathfrak{R}'_{d_2-i+1} \right\}^2 + \frac{d_2}{i^2} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right)^2 \right].
 \end{aligned}$$

Corollary 2. The first and second moments of $\hat{\theta}_2$ are given by

$$\begin{aligned}
 E_{\theta_2}(\hat{\theta}_2) &= M'_{\theta_2}(0) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)}|\mathcal{D}_1=m)} \binom{m}{i} \frac{\theta_1^{m-i}\theta_2^i}{(\theta_1+\theta_2)^m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_m+1) q_1^{\mathfrak{R}_{m-j+1}^*} \\
 &\quad \times \left[\frac{m}{m-i} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right) + \frac{\mathcal{T}_1}{m-i} \mathfrak{R}_{m-j+1}^* \right] + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)}|\mathcal{D}_1=d_1, \mathcal{D}_2=m)} \binom{m}{i} \frac{\theta_1^{m-i}\theta_2^i}{(\theta_1+\theta_2)^m} \\
 &\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1,d_1}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_{d_1}+1) \zeta_{i_2,d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m+1) q_1^{\sum_{j=d_1-i_1+1}^{m-i_2}(\mathfrak{R}_j+1)} q_2^{\sum_{j=m-i_2+1}^m(\mathfrak{R}_j+1)} \\
 &\quad \times \left[\frac{m}{m-i} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right) + \frac{\mathcal{T}_1}{m-i} \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j+1) + \frac{\mathcal{T}_2}{m-i} \sum_{j=m-i_2+1}^m (\mathfrak{R}_j+1) \right] \\
 &\quad + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\zeta^{(d_2)}|\mathcal{D}_2=d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i}\theta_2^i}{(\theta_1+\theta_2)^{d_2}} \sum_{j=0}^{d_2} \zeta_{j,d_2}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_{d_1}+1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2}+1) \\
 &\quad \times q_2^{\mathfrak{R}'_{d_2-i+1}} \left[\frac{d_2}{d_2-i} \left(\frac{\theta_1\theta_2}{\theta_1+\theta_2} \right) + \frac{\mathcal{T}_2}{d_2-i} \mathfrak{R}'_{d_2-i+1} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 E_{\theta_2}(\hat{\theta}_2^2) &= M''_{\hat{\theta}_2}(0) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_m + 1) q_1^{\mathfrak{R}_{m-j+1}^*} \\
 &\quad \times \left[\left\{ \frac{m}{m-i} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) + \frac{\mathcal{T}_1}{m-i} \mathfrak{R}_{m-j+1}^* \right\}^2 + \frac{m}{(m-i)^2} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right)^2 \right] \\
 &\quad + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1 = d_1, \mathcal{D}_2 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1, d_1}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1) \\
 &\quad \times \zeta_{i_2, d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m + 1) q_1^{\sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1)} q_2^{\sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1)} \\
 &\quad \times \left[\left\{ \frac{m}{m-i} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) + \frac{\mathcal{T}_1}{m-i} \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1) + \frac{\mathcal{T}_2}{m-i} \sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1) \right\}^2 \right. \\
 &\quad \left. + \frac{m}{(m-i)^2} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right)^2 \right] + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\xi^{(d_2)}|\mathcal{D}_2 = d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^{d_2}} \\
 &\quad \times \sum_{j=0}^{d_2} \zeta_{j, d_2}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2} + 1) q_2^{\mathfrak{R}'_{d_2-i+1}} \\
 &\quad \times \left[\left\{ \frac{d_2}{d_2-i} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right) + \frac{\mathcal{T}_2}{d_2-i} \mathfrak{R}'_{d_2-i+1} \right\}^2 + \frac{d_2}{(d_2-i)^2} \left(\frac{\theta_1 \theta_2}{\theta_1 + \theta_2} \right)^2 \right].
 \end{aligned}$$

Theorem 3. Conditional on $\xi^{(u)}$, the conditional PDF of $\hat{\theta}_1$ is given by

$$\begin{aligned}
 f_{\hat{\theta}_1}(x) &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_m + 1) q_1^{\mathfrak{R}_{m-j+1}^*} \gamma \left(x - \frac{\mathcal{T}_1 \mathfrak{R}_{m-j+1}^*}{i}; m, \frac{\theta_1 \theta_2}{i(\theta_1 + \theta_2)} \right) \\
 &\quad + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1 = d_1, \mathcal{D}_2 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1, d_1}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1) \\
 &\quad \times \zeta_{i_2, d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m + 1) q_1^{\sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1)} q_2^{\sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1)} \\
 &\quad \times \gamma \left(x - \frac{1}{i} \left[\mathcal{T}_1 \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1) + \mathcal{T}_2 \sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1) \right]; m, \frac{\theta_1 \theta_2}{i(\theta_1 + \theta_2)} \right) \\
 &\quad + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\xi^{(d_2)}|\mathcal{D}_2 = d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^{d_2}} \sum_{j=0}^{d_2} \zeta_{j, d_2}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2} + 1) \\
 &\quad \times q_2^{\mathfrak{R}'_{d_2-i+1}} \gamma \left(x - \frac{\mathcal{T}_2 \mathfrak{R}'_{d_2-i+1}}{i}; d_2, \frac{\theta_1 \theta_2}{i(\theta_1 + \theta_2)} \right),
 \end{aligned}$$

where $\gamma(x - c; a, b)$ denote gamma distribution with shape parameter a , rate parameter b and shift parameter c .

Proof. The proof of Theorem 3 is given in Appendix B. $\square$

Corollary 3. Conditional on $\xi^{(u)}$, the tail probability of $\hat{\theta}_1$ can be expressed as

$$\begin{aligned}
 & P_{\theta_1}(\hat{\theta}_1 > k) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1=m)} \binom{m}{i} \frac{\theta_1^{m-i}\theta_2^i}{(\theta_1+\theta_2)^m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_m+1) q_1^{\mathfrak{R}_{m-j+1}^*} \Gamma\left(m, \frac{\theta_1\theta_2}{i(\theta_1+\theta_2)} \left\langle k - \frac{\mathcal{T}_1 R_{m-j+1}^*}{i} \right\rangle\right) \\
 &+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1=d_1, \mathcal{D}_2=m)} \binom{m}{i} \frac{\theta_1^{m-i}\theta_2^i}{(\theta_1+\theta_2)^m} \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1,d_1}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_{d_1}+1) \\
 &\times \zeta_{i_2,d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m+1) q_1^{\sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j+1)} q_2^{\sum_{j=m-i_2+1}^m (\mathfrak{R}_j+1)} \\
 &\times \Gamma\left(m, \frac{\theta_1\theta_2}{i(\theta_1+\theta_2)} \left\langle k - \frac{1}{i} \left[\mathcal{T}_1 \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j+1) + \mathcal{T}_2 \sum_{j=m-i_2+1}^m (\mathfrak{R}_j+1) \right] \right\rangle\right) \\
 &+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\xi^{(d_2)}|\mathcal{D}_2=d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i}\theta_2^i}{(\theta_1+\theta_2)^{d_2}} \sum_{j=0}^{d_2} \zeta_{j,d_2}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_{d_1}+1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2}+1) \\
 &\times q_2^{\mathfrak{R}'_{d_2-i+1}} \Gamma\left(d_2, \frac{\theta_1\theta_2}{i(\theta_1+\theta_2)} \left\langle k - \frac{\mathcal{T}_2 R'_{d_2-i+1}}{i} \right\rangle\right),
 \end{aligned}$$

where k is an arbitrary constant, $\langle x \rangle = \max\{x, 0\}$ and $\Gamma(a, b) = \int_b^\infty (1/(a-1)!) x^{a-1} e^{-x} dx$.

Theorem 4. Conditional on $\xi^{(u)}$, the conditional PDF of $\hat{\theta}_2$ is given by

$$\begin{aligned}
 & f_{\hat{\theta}_2}(x) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1=m)} \binom{m}{i} \frac{\theta_1^{m-i}\theta_2^i}{(\theta_1+\theta_2)^m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_m+1) q_1^{\mathfrak{R}_{m-j+1}^*} \\
 &\times \gamma\left(x - \frac{\mathcal{T}_1 R_{m-j+1}^*}{m-i}; m, \frac{\theta_1\theta_2}{(m-i)(\theta_1+\theta_2)}\right) + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1=d_1, \mathcal{D}_2=m)} \binom{m}{i} \frac{\theta_1^{m-i}\theta_2^i}{(\theta_1+\theta_2)^m} \\
 &\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1,d_1}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_{d_1}+1) \zeta_{i_2,d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m+1) q_1^{\sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j+1)} q_2^{\sum_{j=m-i_2+1}^m (\mathfrak{R}_j+1)} \\
 &\times \gamma\left(x - \frac{1}{m-i} \left[\mathcal{T}_1 \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j+1) + \mathcal{T}_2 \sum_{j=m-i_2+1}^m (\mathfrak{R}_j+1) \right]; m, \frac{\theta_1\theta_2}{(m-i)(\theta_1+\theta_2)}\right) \\
 &+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\xi^{(d_2)}|\mathcal{D}_2=d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i}\theta_2^i}{(\theta_1+\theta_2)^{d_2}} \sum_{j=0}^{d_2} \zeta_{j,d_2}(\mathfrak{R}_1+1, \dots, \mathfrak{R}_{d_1}+1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2}+1) \\
 &\times q_2^{\mathfrak{R}'_{d_2-i+1}} \gamma\left(x - \frac{\mathcal{T}_2 R'_{d_2-i+1}}{d_2-i}; d_2, \frac{\theta_1\theta_2}{(d_2-i)(\theta_1+\theta_2)}\right).
 \end{aligned}$$

Corollary 4. Conditional on $\xi^{(u)}$, the tail probability of $\hat{\theta}_2$ can be expressed as

$$\begin{aligned}
 & P_{\theta_2}(\hat{\theta}_2 > k) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \sum_{j=0}^m \zeta_{j,m}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_m + 1) q_1^{\mathfrak{R}_{m-j+1}^*} \\
 &\quad \times \Gamma\left(m, \frac{\theta_1 \theta_2}{(m-i)(\theta_1 + \theta_2)} \left\langle k - \frac{\mathcal{T}_1 R_{m-j+1}^*}{m-i} \right\rangle\right) + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\xi^{(m)}|\mathcal{D}_1 = d_1, \mathcal{D}_2 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \\
 &\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1, d_1}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1) \zeta_{i_2, d_2}(\mathbf{1}_{m-d_1-1}, \mathfrak{R}_m + 1) q_1^{\sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1)} q_2^{\sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1)} \\
 &\quad \times \Gamma\left(m, \frac{\theta_1 \theta_2}{(m-i)(\theta_1 + \theta_2)} \left\langle k - \frac{1}{m-i} \left[\mathcal{T}_1 \sum_{j=d_1-i_1+1}^{m-i_2} (\mathfrak{R}_j + 1) + \mathcal{T}_2 \sum_{j=m-i_2+1}^m (\mathfrak{R}_j + 1) \right] \right\rangle\right) \\
 &\quad + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\xi^{(d_2)}|\mathcal{D}_2 = d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^{d_2}} \sum_{j=0}^{d_2} \zeta_{j, d_2}(\mathfrak{R}_1 + 1, \dots, \mathfrak{R}_{d_1} + 1, \mathbf{1}_{d_2-d_1}, \mathfrak{R}'_{d_2} + 1) \\
 &\quad \times q_2^{\mathfrak{R}'_{d_2-i+1}} \Gamma\left(d_2, \frac{\theta_1 \theta_2}{(d_2-i)(\theta_1 + \theta_2)} \left\langle k - \frac{\mathcal{T}_2 R'_{d_2-i+1}}{d_2-i} \right\rangle\right).
 \end{aligned}$$

Based on Corollaries 3 and 4, we construct $100(1 - \alpha)\%$ CI of $\hat{\theta}_j$ under the assumption that $P(\hat{\theta}_j > k)$ is an increasing function of θ_j , when the other parameter is fixed. Then, we can easily obtain the CI for θ_j , denoted by $(\theta_j^{(L)}, \theta_j^{(U)})$, satisfying the following equation with $\hat{\theta}_j^{(obs)}$ being the observed value of $\hat{\theta}_j$:

$$P_{\theta_j^{(L)}}(\hat{\theta}_j^{(L)} > \hat{\theta}_j^{(obs)}) = \frac{\alpha}{2}, P_{\theta_j^{(U)}}(\hat{\theta}_j^{(U)} > \hat{\theta}_j^{(obs)}) = 1 - \frac{\alpha}{2}, j = 1, 2.$$

3. Simulation Results and Data Analysis

3.1. Simulation Results

In this subsection, we consider various $n, m, \mathcal{T}_1$ and $\mathcal{T}_2$. We used three different Pr2CS: Scheme I, $\mathfrak{R}_m = n - m$ and $\mathfrak{R}_i = 0$ for $i = 1, \dots, m - 1$. Scheme II, $\mathfrak{R}_1 = \dots = \mathfrak{R}_{n-m} = 1$ and $\mathfrak{R}_i = 0$ for $i = n - m + 1, \dots, m$; and Scheme III, $\mathfrak{R}_1 = \dots = \mathfrak{R}_{(n-m)/2} = 1$, $\mathfrak{R}_m = (n - m)/2$ and $\mathfrak{R}_i = 0$ for $i = (n - m)/2 + 1, \dots, m - 1$.

For three Pr2CS, we generated Pr2CS data. In addition, we generated new random variable $\mathbf{U} = (u_1, u_2, \dots, u_m)$. Now, if $u_i < \theta_1/(\theta_1 + \theta_2)$, then assign $z_i = 1$, otherwise $z_i = 0$. Then, the corresponding GeAdPHCS CoR data are $\{(x_{1:m:n}, z_1), (x_{2:m:n}, z_2), \dots, (x_{m:m:n}, z_m)\}$. Without loss of generality, we take $\theta_1 = 0.4$ and $\theta_2 = 0.6$ in each case.

Furthermore, if $x_{m:m:n} < \mathcal{T}_1$, we have Case I and the corresponding GeAdPHCS data are $\{(x_{1:m:n}, z_1), (x_{2:m:n}, z_2), \dots, (x_{m:m:n}, z_m)\}$. If $\mathcal{T}_1 < x_{m:m:n} < \mathcal{T}_2$, we have Case II and the corresponding GeAdPHCS data are $\{(x_{1:m:n}, z_1), (x_{2:m:n}, z_2), \dots, (x_{m:m:n}, z_m)\}$, and $\mathfrak{R}_{d_1} = \dots = \mathfrak{R}_{m-1} = 0$. If $\mathcal{T}_2 < x_{m:m:n}$, we have Case III and we find $\mathcal{D}_2$ such that $x_{\mathcal{D}_2:m:n} < \mathcal{T}_2 < x_{\mathcal{D}_2+1:m:n}$. The corresponding GeAdPHCS data are $\{(x_{1:m:n}, z_1), (x_{2:m:n}, z_2), \dots, (x_{\mathcal{D}_2:m:n}, z_{\mathcal{D}_2})\}$. We reiterated the procedure 1000 times in each GeAdPHCS. We calculated the RMSEs of the estimator, and the corresponding average biases. The simulation results are presented in Table 1. In addition, we calculated the average CL and the corresponding CP. The results are presented in Table 2. Note that we used Python for the simulation study.

Table 1. Relative RMSEs and biases for the MLEs of parameters.

<i>n</i>	<i>m</i>	$\mathcal{F}_1$	RMSE (Bias)				
			$\mathcal{F}_2 = 0.5$		$\mathcal{F}_2 = 0.8$		
			$\mathfrak{R}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$
20	18	0.2	(0*17, 2)	0.1458 (0.0219)	0.3657 (0.0715)	0.1393 (0.0176)	0.2966 (0.0541)
			(1*2 [†] , 0*16 [†])	0.1499 (0.0243)	0.4263 (0.0897)	0.1428 (0.0203)	0.3251 (0.0617)
			(1, 0*16, 1)	0.1482 (0.0230)	0.3963 (0.0830)	0.1441 (0.0202)	0.3076 (0.0564)
	16		(0*15, 4)	0.1540 (0.0231)	0.3888 (0.0738)	0.1502 (0.0212)	0.3473 (0.0659)
			(1*4, 0*12)	0.1822 (0.0370)	0.4841 (0.1075)	0.1609 (0.0255)	0.4072 (0.0813)
			(1*2, 0*13, 2)	0.1633 (0.0287)	0.4048 (0.0843)	0.1520 (0.0219)	0.3523 (0.0672)
	14		(0*13, 6)	0.1768 (0.0292)	0.3977 (0.0974)	0.1722 (0.0282)	0.3975 (0.0969)
			(1*6, 0*8)	0.2068 (0.0430)	0.4947 (0.1349)	0.1926 (0.0365)	0.4130 (0.1084)
			(1*3, 0*10, 3)	0.1928 (0.0337)	0.4047 (0.1028)	0.1723 (0.0283)	0.3970 (0.0967)
	12		(0*11, 8)	0.2058 (0.0353)	0.5057 (0.1154)	0.2054 (0.0352)	0.5058 (0.1154)
			(1*8, 0*4)	0.2635 (0.0657)	0.5636 (0.1638)	0.2573 (0.0705)	0.5752 (0.1644)
			(1*4, 0*7, 4)	0.2165 (0.0385)	0.5095 (0.1167)	0.2054 (0.0352)	0.5058 (0.1154)
30	28	0.2	(0*27, 2)	0.1162 (0.0192)	0.3160 (0.0582)	0.1097 (0.0138)	0.2325 (0.0459)
			(1*2, 0*26)	0.1208 (0.0193)	0.3124 (0.0605)	0.1123 (0.0160)	0.3123 (0.0517)
			(1, 0*26, 1)	0.1169 (0.0184)	0.3172 (0.0610)	0.1114 (0.0148)	0.2461 (0.0474)
	26		(0*25, 4)	0.1146 (0.0159)	0.2197 (0.0343)	0.1109 (0.0129)	0.2138 (0.0300)
			(1*4, 0*22)	0.1252 (0.0205)	0.2472 (0.0453)	0.1127 (0.0150)	0.2259 (0.0358)
			(1*2, 0*23, 2)	0.1180 (0.0186)	0.2363 (0.0404)	0.1096 (0.0131)	0.2152 (0.0311)
	24		(0*23, 6)	0.1149 (0.0123)	0.2550 (0.0469)	0.1141 (0.0117)	0.2457 (0.0434)
			(1*6, 0*18)	0.1315 (0.0182)	0.3223 (0.0677)	0.1185 (0.0139)	0.2674 (0.0512)
			(1*3, 0*20, 3)	0.1219 (0.0158)	0.2721 (0.0554)	0.1143 (0.0118)	0.2468 (0.0437)
	22		(0*21, 8)	0.1246 (0.0124)	0.2547 (0.0380)	0.1240 (0.0122)	0.2550 (0.0380)
			(1*8, 0*14)	0.1429 (0.0228)	0.2884 (0.0525)	0.1278 (0.0147)	0.2604 (0.0430)
			(1*4, 0*17, 4)	0.1293 (0.0151)	0.2587 (0.0416)	0.1240 (0.0122)	0.2549 (0.0380)
	20		(0*19, 10)	0.1321 (0.0185)	0.2730 (0.0551)	0.1316 (0.0184)	0.2729 (0.0550)
			(1*10, 0*10)	0.1592 (0.0305)	0.3456 (0.0759)	0.1497 (0.0257)	0.2853 (0.0631)
			(1*5, 0*14, 5)	0.1349 (0.0200)	0.2733 (0.0565)	0.1316 (0.0184)	0.2729 (0.0550)
	18		(0*17, 12)	0.1378 (0.0169)	0.2942 (0.0534)	0.1378 (0.0169)	0.2942 (0.0534)
			(1*12, 0*6)	0.1646 (0.0365)	0.4243 (0.1003)	0.1639 (0.0433)	0.3895 (0.1008)
			(1*6, 0*11, 6)	0.1384 (0.0173)	0.2963 (0.0541)	0.1378 (0.0169)	0.2942 (0.0534)
40	38	0.2	(0*37, 2)	0.0963 (0.0109)	0.1836 (0.0197)	0.0889 (0.0067)	0.1715 (0.0138)
			(1*2, 0*36)	0.1001 (0.0120)	0.1909 (0.0213)	0.0919 (0.0081)	0.1782 (0.0161)
			(1, 0*36, 1)	0.0981 (0.0118)	0.1853 (0.0190)	0.0899 (0.0074)	0.1751 (0.0150)
	36		(0*35, 4)	0.0944 (0.0097)	0.2006 (0.0283)	0.0916 (0.0088)	0.1834 (0.0195)
			(1*4, 0*32)	0.0975 (0.0098)	0.2149 (0.0335)	0.0943 (0.0093)	0.1970 (0.0252)
			(1*2, 0*33, 2)	0.0962 (0.0104)	0.2111 (0.0323)	0.0931 (0.0096)	0.1881 (0.0215)
	34		(0*33, 6)	0.0953 (0.0087)	0.2008 (0.0289)	0.0923 (0.0069)	0.1915 (0.0260)
			(1*6, 0*28)	0.1000 (0.0101)	0.2294 (0.0383)	0.0954 (0.0083)	0.2028 (0.0295)
			(1*3, 0*30, 3)	0.0973 (0.0094)	0.2179 (0.0350)	0.0930 (0.0073)	0.1940 (0.0264)
	32		(0*31, 8)	0.1017 (0.0100)	0.1900 (0.0205)	0.1001 (0.0092)	0.1890 (0.0195)
			(1*8, 0*24)	0.1133 (0.0148)	0.2161 (0.0274)	0.1045 (0.0105)	0.1956 (0.0210)
			(1*4, 0*27, 4)	0.1074 (0.0129)	0.2037 (0.0239)	0.1002 (0.0093)	0.1889 (0.0195)
	30		(0*29, 10)	0.1042 (0.0108)	0.1987 (0.0268)	0.1040 (0.0108)	0.1981 (0.0264)
			(1*10, 0*20)	0.1156 (0.0155)	0.2361 (0.0413)	0.1079 (0.0121)	0.2186 (0.0321)
			(1*5, 0*24, 5)	0.1075 (0.0123)	0.2194 (0.0321)	0.1040 (0.0108)	0.1982 (0.0265)
	28		(0*16, 1*12)	0.1080 (0.0131)	0.2273 (0.0446)	0.1080 (0.0131)	0.2273 (0.0446)
			(1*12, 0*16)	0.1171 (0.0182)	0.3150 (0.0605)	0.1128 (0.0160)	0.3133 (0.0520)
			(1*6, 0*16, 1*6)	0.1110 (0.0144)	0.2317 (0.0462)	0.1080 (0.0131)	0.2273 (0.0446)

1*2[†]: (1, 1), 0*16[†]: (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0).

Table 2. Relative CL and CP for the MLEs of parameters.

<i>n</i>	<i>m</i>	$\mathcal{T}_1$	Confidence Length (Coverage Probability)				
			$\mathfrak{R}$	$\mathcal{T}_2 = 0.5$		$\mathcal{T}_2 = 0.8$	
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$
20	18	0.2	(0*17, 2)	0.5807 (94.7)	1.3001 (95.3)	0.5540 (94.6)	1.1730 (95.3)
			(1*2, 0*16)	0.6163 (95.2)	1.4882 (94.9)	0.5723 (94.8)	1.2355 (95.1)
			(1, 0*16, 1)	0.5962 (95.0)	1.3901 (95.0)	0.5631 (94.6)	1.1896 (95.3)
	16		(0*15, 4)	0.6104 (94.8)	1.4075 (95.5)	0.6027 (94.8)	1.3495 (95.7)
			(1*4, 0*12)	0.6973 (95.0)	1.7466 (95.3)	0.6266 (94.5)	1.4713 (95.3)
			(1*2, 0*13, 2)	0.6372 (94.7)	1.4858 (95.7)	0.6048 (94.7)	1.3567 (95.7)
	14		(0*13, 6)	0.6704 (95.2)	1.5825 (94.9)	0.6667 (95.1)	1.5794 (94.9)
			(1*6, 0*8)	0.7688 (94.2)	1.9668 (95.3)	0.7029 (94.2)	1.6595 (95.0)
			(1*3, 0*10, 3)	0.6935 (95.0)	1.6221 (95.1)	0.6669 (95.2)	1.5790 (94.9)
	12		(0*11, 8)	0.7755 (94.7)	2.0278 (94.5)	0.7751 (94.7)	2.0279 (94.5)
			(1*8, 0*4)	0.9108 (93.8)	2.3713 (94.4)	0.8669 (93.1)	2.2177 (93.5)
			(1*4, 0*7, 4)	0.7900 (94.7)	2.0405 (94.6)	0.7752 (94.7)	2.0280 (94.5)
30	28	0.2	(0*27, 2)	0.4496 (94.7)	0.9645 (95.4)	0.4240 (94.4)	0.8656 (95.4)
			(1*2, 0*26)	0.4667 (95.0)	1.0045 (95.7)	0.4355 (94.2)	0.9287 (95.2)
			(1, 0*26, 1)	0.4561 (94.6)	0.9884 (95.7)	0.4281 (94.8)	0.8784 (95.4)
	26		(0*25, 4)	0.4517 (95.3)	0.8881 (95.7)	0.4409 (95.4)	0.8676 (95.8)
			(1*4, 0*22)	0.4892 (95.7)	0.9813 (95.8)	0.4533 (95.5)	0.9006 (95.6)
			(1*2, 0*23, 2)	0.4673 (95.6)	0.9300 (95.8)	0.4421 (95.2)	0.8715 (95.7)
	24		(0*23, 6)	0.4595 (94.8)	0.9557 (94.9)	0.4571 (94.8)	0.9431 (94.8)
			(1*6, 0*18)	0.5067 (94.4)	1.1165 (94.9)	0.4709 (94.8)	0.9891 (94.8)
			(1*3, 0*20, 3)	0.4752 (94.3)	1.0009 (94.5)	0.4575 (94.7)	0.9444 (94.9)
	22		(0*21, 8)	0.4862 (93.8)	0.9956 (94.3)	0.4857 (93.9)	0.9956 (94.3)
			(1*8, 0*14)	0.5444 (94.8)	1.1201 (94.3)	0.4995 (93.9)	1.0275 (94.5)
			(1*4, 0*17, 4)	0.4955 (93.7)	1.0129 (94.5)	0.4858 (93.9)	0.9956 (94.3)
	20		(0*19, 10)	0.5198 (94.1)	1.0913 (95.6)	0.5195 (94.1)	1.0911 (95.6)
			(1*10, 0*10)	0.5832 (93.8)	1.2706 (95.7)	0.5425 (93.2)	1.1311 (95.1)
			(1*5, 0*14, 5)	0.5245 (94.1)	1.0975 (95.6)	0.5195 (94.1)	1.0911 (95.6)
	18		(0*17, 12)	0.5518 (94.6)	1.1689 (95.3)	0.5518 (94.6)	1.1689 (95.3)
			(1*12, 0*6)	0.6181 (93.8)	1.4257 (94.5)	0.5991 (94.4)	1.3025 (94.6)
			(1*6, 0*11, 6)	0.5530 (94.5)	1.1722 (95.3)	0.5518 (94.6)	1.1689 (95.3)
40	38	0.2	(0*37, 2)	0.3736 (94.5)	0.7129 (94.1)	0.3527 (94.8)	0.6721 (94.3)
			(1*2, 0*36)	0.3857 (94.7)	0.7372 (93.5)	0.3603 (94.9)	0.6882 (94.2)
			(1, 0*36, 1)	0.3799 (94.5)	0.7217 (93.8)	0.3556 (94.9)	0.6787 (93.8)
	36		(0*35, 4)	0.3732 (94.9)	0.7335 (93.8)	0.3634 (94.8)	0.6987 (93.8)
			(1*4, 0*32)	0.3930 (95.6)	0.7858 (93.6)	0.3718 (94.5)	0.7262 (93.6)
			(1*2, 0*33, 2)	0.3829 (95.2)	0.7606 (93.9)	0.3657 (94.7)	0.7051 (93.7)
	34		(0*33, 6)	0.3777 (94.8)	0.7478 (94.8)	0.3727 (94.9)	0.7359 (95.0)
			(1*6, 0*28)	0.4055 (94.9)	0.8252 (94.6)	0.3823 (95.3)	0.7596 (95.1)
			(1*3, 0*30, 3)	0.3878 (94.7)	0.7807 (95.1)	0.3735 (94.8)	0.7377 (95.1)
	32		(0*31, 8)	0.3900 (94.2)	0.7506 (94.8)	0.3882 (94.1)	0.7481 (94.7)
			(1*8, 0*24)	0.4266 (94.0)	0.8271 (94.1)	0.3983 (94.2)	0.7674 (94.3)
			(1*4, 0*27, 4)	0.4014 (93.8)	0.7731 (94.6)	0.3884 (94.1)	0.7481 (94.7)
	30		(0*29, 10)	0.4040 (96.0)	0.7894 (94.8)	0.4039 (95.9)	0.7885 (94.9)
			(1*10, 0*20)	0.4416 (95.6)	0.8875 (95.4)	0.4141 (95.6)	0.8197 (95.0)
			(1*5, 0*24, 5)	0.4104 (95.6)	0.8116 (94.6)	0.4039 (95.9)	0.7887 (94.9)
	28		(0*16, 1*12)	0.4218 (94.5)	0.8598 (95.3)	0.4218 (94.5)	0.8598 (95.3)
			(1*12, 0*16)	0.4602 (94.9)	0.9968 (95.8)	0.4342 (94.5)	0.9271 (95.1)
			(1*6, 0*16, 1*6)	0.4257 (94.5)	0.8676 (95.3)	0.4218 (94.5)	0.8598 (95.3)

In Table 1, the following general observations can be made. The MSEs decrease as sample size n increases. For fixed sample size n , the MSEs decrease generally as the number of Pr2CS data size m increases. For Fixed sample size n and Pr2CS data size m , the RMSEs decreases generally as the time $\mathcal{T}_2$ increases. In addition, we can observed that the estimator for Pr2CS I has smaller RMSE and bias than the corresponding estimator for the other two Pr2CS.

In Table 2, the CL decrease as sample size n increases. For fixed sample size n , the CL decrease generally as the number of Pr2CS data size m increases. For fixed sample size n and Pr2CS data size m , the CL decreases generally as the time $\mathcal{T}_2$ increases. In addition, we can observed that the estimator for Pr2CS I has smaller CL than the corresponding estimator for the other two Pr2CS. It is observed that the CI works well for all GeAdPHCS.

$\hat{\theta}_1$ has smaller RMSE, bias and CL than the corresponding $\hat{\theta}_2$. This is because, when θ_1 is smaller than θ_2 , we may observe more failure number due to Factor 1 than those due to Factor 2, so that the $\hat{\theta}_1$ is more precise than $\hat{\theta}_2$.

3.2. Data Analysis

To analyze real data, we use the estimators in the above section. The real data were from some small electronic appliances exposed to the automatic test machine [14]. These data were analyzed by the authors of [2,15]. From these data, let us express the failure of appliance due to ninth failure RisF with $z_i = 1$, and $z_i = 0$ denotes failure caused by other failure RisFs. Here, we suppose that the underlying distribution of these data is the ExpD based on the Pr2CS (i.e., $m = 28$, $\mathfrak{R}_{21} = \dots = \mathfrak{R}_{28} = 1$ and $\mathfrak{R}_i = 0$ for $i = 1, 2, \dots, 20$). Then, Pr2CS data are presented in Table 3.

Table 3. Progressive censored data of the example.

x_i	11.0	35.0	49.0	170.0	329.0	381.0	708.0	958.0	1062.0	1167.0
z_i	0	0	0	0	0	0	0	0	0	1
$\mathfrak{R}_i$	0	0	0	0	0	0	0	0	0	0
x_i	1594.0	1925.0	1990.0	2223.0	2327.0	2400.0	2451.0	2471.0	2551.0	2565.0
z_i	0	1	1	1	0	1	0	1	1	0
$\mathfrak{R}_i$	0	0	0	0	0	0	0	0	0	0
x_i	2568.0	2702.0	2831.0	3059.0	3214.0	3504.0	4329.0	6976.0		
z_i	1	0	0	0	1	1	1	1		
$\mathfrak{R}_i$	1	1	1	1	1	1	1	1		

In addition, we set Case I ($\mathcal{T}_1 = 7000$ and $\mathcal{T}_2 = 8000$), Case II ($\mathcal{T}_1 = 3000$ and $\mathcal{T}_2 = 7000$) and Case III ($\mathcal{T}_1 = 3000$ and $\mathcal{T}_1 = 5000$). Table 4 presents the 95% CIs for $\hat{\theta}_1$ and $\hat{\theta}_2$, and we include the standard error (SE) and MSE calculated from Corollaries 1 and 2. In addition, the PDFs of $\hat{\theta}_1$ and $\hat{\theta}_2$ based on the example data are shown in Figure 2.

Table 4. Inference of parameters of the example.

$\mathcal{T}_1$	$\mathcal{T}_2$	n_1	n_2	$\hat{\theta}_1$	SE ($\hat{\theta}_1$)	95% CI	$\hat{\theta}_2$	SE ($\hat{\theta}_2$)	95% CI
7000	8000	12	16	7144.417	2062.415	(4057.341, 12,580.330)	5358.312	1339.578	(3282.644, 8746.460)
3000	7000	12	16	8294.250	2394.344	(4710.336, 14,605.030)	6220.688	1555.172	(3810.957, 10,154.130)
3000	5000	11	16	7970.455	2403.182	(4057.341, 14,392.450)	5479.688	1369.922	(3357.001, 8944.583)

Figure 2. The PDFs of $\hat{\theta}_1$ and $\hat{\theta}_2$ of the example.

4. Conclusions

It is known that the lifetimes of items may not be recorded exactly. Therefore, recently, Pr2CS and Ad1PHCS have become quite popular in a life-testing problem and reliability analysis. Although Pr2CS and Ad1PHCS assure a pre-assigned number of failures, it has the drawback that it might take a long time to observe a pre-assigned number of failures and terminate the test. For this reason, Lee [12] suggested a GeAdPHCS in which the test is assured to end at a pre-assigned time. In addition, it is known that more than one RisF may be present at the same time. That is, several RisFs compete for the immediate failure cause of items. Following Cox [1], we refer to this model as CoRiM. In this paper, we discuss exact likelihood inference for CoRiM with GeAdPHCS exponential data. We derive the ConMGF of the maximum likelihood estimators of scale parameters of ExpD and the resulting lower confidence bound under GeAdPHCS. Consequently, for fixed sample and Pr2CS sample size, the RMSEs decrease as the time $\mathcal{T}_2$ increases. In addition, for fixed sample and Pr2CS sample size, the CLs decrease as the time $\mathcal{T}_2$ increases. Although we focus on the inference for scale parameter of the ExpD, the suggested GeAdPHCS CoRiM can be extended to other distributions.

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Abbreviations

The following abbreviations are used in this manuscript:

RisF	risk factor
CoRiM	competing risk model
ConMGF	conditional moment generating function
ExpD	exponential distribution
GeAdPHCS	generalized adaptive progressive hybrid censoring scheme

CoR	competing risks
CI	confidence interval
Ad1PHCS	adaptive Type I progressive hybrid censoring
Pr2CS	progressive Type II censoring scheme
CL	confidence lengths
CP	coverage percentages
PDF	probability density function
CDF	cumulative distribution function

Appendix A. Proof of Theorem 1

Conditional on $\zeta^{(u)}$, the MGF of $\hat{\theta}_1$ is given by

$$\begin{aligned}
 M_{\hat{\theta}_1}(t) &= E\left(e^{t\hat{\theta}_1} | \zeta^{(u)}\right) \\
 &= E\left(e^{t\hat{\theta}_1} | \mathcal{D}_1 = m, \zeta^{(m)}\right) P(\mathcal{D}_1 = m) + \sum_{d_1=0}^{m-1} E\left(e^{t\hat{\theta}_1} | \mathcal{D}_1 = d_1, \mathcal{D}_2 = m, \zeta^{(m)}\right) P(\mathcal{D}_1 = d_1, \mathcal{D}_2 = m) \\
 &\quad + \sum_{d_2=1}^{m-1} E\left(e^{t\hat{\theta}_1} | \mathcal{D}_2 = d_2, \zeta^{(d_2)}\right) P(\mathcal{D}_2 = d_2).
 \end{aligned} \tag{A1}$$

For convenience, let us denote the subset of indicator of failure causes as Q_u^* , where

$$Q_u^* = \{Z = (z_1, \dots, z_u) : z_i = 0 \text{ or } 1; i = 1, \dots, u\}.$$

For Case I ($\mathcal{D}_1 = m$): Conditional on $\mathcal{D}_1 = m$ and $n_1 = i$, the joint distribution of order statistics $x_{1:m:n} < \dots < x_{m:m:n} < \mathcal{T}_1$ has the form

$$\begin{aligned}
 f(x_{1:m:n}, \dots, x_{m:m:n} | \mathcal{D}_1 = m, n_1 = i) &= \frac{1}{P(\mathcal{D}_1 = m, n_1 = i)} f(x_{1:m:n}, \dots, x_{m:m:n}; \mathcal{D}_1 = m, \sum_{j=1}^m z_j = i) \\
 &= \frac{1}{P(\mathcal{D}_1 = m, n_1 = i)} \sum_{Z_m \in Q_m^*} \dots \sum_{\sum_{j=1}^m z_j = i} f(x_{1:m:n}, \dots, x_{m:m:n}; \mathcal{D}_1 = m) \\
 &= \frac{1}{P(\mathcal{D}_1 = m, n_1 = i)} \sum_{Z_m \in Q_m^*} \dots \sum_{\sum_{j=1}^m z_j = i} \zeta'_m \prod_{j=1}^m f_{X,Z}(x_{j:m:n}, 1)^{z_j} f_{X,Z}(x_{j:m:n}, 2)^{1-z_j} [1 - \mathcal{F}(x_{j:m:n})]^{\Re_j} \\
 &= \frac{\zeta'_m}{P(\mathcal{D}_1 = m, n_1 = i)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \exp \left[- \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \sum_{j=1}^m (1 + \Re_j) x_{j:m:n} \right].
 \end{aligned}$$

Upon the conditional PDF obtained above, we can readily have

$$\begin{aligned}
 E\left(e^{t\hat{\theta}_1} | \mathcal{D}_1 = m, \zeta^{(m)}\right) &= \sum_{i=1}^{m-1} E\left(e^{t\hat{\theta}_1} | \mathcal{D}_1 = m, n_1 = i\right) P(n_1 = i | \zeta^{(m)}, \mathcal{D}_1 = m) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\mathcal{D}_1 = m, n_1 = i)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \\
 &\quad \times \int_0^{\mathcal{T}_1} \dots \int_0^{x_{2:m:n}} \prod_{j=1}^m f(x_{j:m:n}) [1 - F(x_{j:m:n})]^{(1+\Re_j)} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-1} dx_{1:m:n} \dots dx_{m:m:n}.
 \end{aligned}$$

From Lemma 1 with $v_j = (1 + \Re_j) \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)$ and then factor $\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)$ out of all of the v_j s, the above expression can be easily simplified as

$$\sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\mathcal{D}_1 = m, n_1 = i)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-m} \sum_{j=0}^m \zeta_{j,m}(\Re_1 + 1, \dots, \Re_m + 1) q_1^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \Re_{m-j+1}^*}. \tag{A2}$$

For Case II ($\mathcal{D}_1 = 1, \dots, m, \mathcal{D}_2 = m$): Conditional on $\mathcal{D}_1 = 1, \dots, m, \mathcal{D}_2 = m$ and $n_1 = i$, the joint distribution of order statistics $x_{1:m:n} < \dots < x_{d_1:m:n} < \mathcal{T}_1 < \dots < x_{m:m:n} < \mathcal{T}_2$ has the form

$$\begin{aligned}
& f(x_{1:m:n}, \dots, x_{m:m:n} | \mathcal{D}_1 = d_1, \mathcal{D}_2 = m, n_1 = i) \\
&= \frac{1}{P(\mathcal{D}_1 = m, \mathcal{D}_2 = m, n_1 = i)} f(x_{1:m:n}, \dots, x_{m:m:n}; \mathcal{D}_1 = d_1, \mathcal{D}_2 = m, \sum_{j=1}^m z_j = i) \\
&= \frac{1}{P(\mathcal{D}_1 = d_1, \mathcal{D}_2 = m, n_1 = i)} \sum_{Z_m \in Q_m^*, \sum_{j=1}^m z_j = i} f(x_{1:m:n}, \dots, x_{m:m:n}; \mathcal{D}_1 = d_1, \mathcal{D}_2 = m) \\
&= \frac{\zeta'_m}{P(\mathcal{D}_1 = d_1, \mathcal{D}_2 = m, n_1 = i)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \exp \left[- \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \sum_{j=1}^m (1 + \Re_j) x_{j:m:n} \right].
\end{aligned}$$

Then, we immediately have

$$\begin{aligned}
E \left(e^{t\hat{\theta}_1} | \mathcal{D}_1 = d_1, \mathcal{D}_2 = m, \zeta^{(m)} \right) &= \sum_{i=1}^{m-1} E \left(e^{t\hat{\theta}_1} | \mathcal{D}_1 = d_1, \mathcal{D}_2 = m, n_1 = i \right) P(n_1 = i | \zeta^{(m)}, \mathcal{D}_1 = d_1, \mathcal{D}_2 = m) \\
&= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\mathcal{D}_1 = d_1, \mathcal{D}_2 = m, n_1 = i)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \\
&\quad \times \int_{\mathcal{T}_1}^{\mathcal{T}_2} \dots \int_{x_{m-1:m:n}}^{\mathcal{T}_2} \int_0^{\mathcal{T}_1} \dots \int_0^{x_{2:m:n}} \prod_{j=1}^m f(x_{j:m:n}) [1 - F(x_{j:m:n})]^{(1+\Re_j) \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) - 1} \\
&\quad \times dx_{1:m:n} \dots dx_{d_1:m:n} dx_{m:m:n} dx_{d_1+1:m:n}.
\end{aligned}$$

From Lemma 1 with $v_j = (1 + \Re_j) \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)$ and then factor $\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)$ out of all of the v_j s, the above expression can be easily simplified as

$$\begin{aligned}
& \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\mathcal{D}_1 = d_1, \mathcal{D}_2 = m, n_1 = i)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1, d_1} (\Re_1 + 1, \dots, \Re_{d_1} + 1) \\
&\quad \times q_1^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \sum_{j=d_1-i_1+1}^{d_1} (\Re_j + 1)} \int_{\mathcal{T}_1}^{\mathcal{T}_2} \dots \int_{x_{m-1:m:n}}^{\mathcal{T}_2} \prod_{j=d_1+1}^m f(x_{j:m:n}) [1 - F(x_{j:m:n})]^{(1+\Re_j) \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) - 1} \\
&\quad \times dx_{m:m:n} dx_{d_1+1:m:n} \\
&= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\mathcal{D}_1 = d_1, \mathcal{D}_2 = m, n_1 = i)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-m} \\
&\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1, d_1} (\Re_1 + 1, \dots, \Re_{d_1} + 1) \zeta_{i_2, d_2} (\mathbf{1}_{m-d_1-1}, \Re_m + 1) q_1^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \sum_{j=d_1-i_1+1}^{m-i_2} (\Re_j + 1)} \\
&\quad \times q_2^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \sum_{j=m-i_2+1}^m (\Re_j + 1)}. \tag{A3}
\end{aligned}$$

For Case III ($\mathcal{D}_2 = 1, \dots, m$): Conditional on $\mathcal{D}_1 = 1, \dots, m$ and $n_1 = i$, the joint distribution of order statistics $x_{1:m:n} < \dots < x_{d_2:m:n} < \mathcal{T}_2$ has the form

$$\begin{aligned}
& f(x_{1:m:n}, \dots, x_{d_2:m:n} | \mathcal{D}_2 = d_2, n_1 = i) \\
&= \frac{1}{P(\mathcal{D}_2 = d_2, n_1 = i)} f(x_{1:m:n}, \dots, x_{d_2:m:n}; \mathcal{D}_2 = d_2, \sum_{j=1}^{d_2} z_j = i) \\
&= \frac{1}{P(\mathcal{D}_2 = d_2, n_1 = i)} \sum_{Z_{d_2} \in Q_{d_2}^*, \sum_{j=1}^{d_2} z_j = i} f(x_{1:m:n}, \dots, x_{m:m:n}; \mathcal{D}_2 = d_2) \\
&= \frac{\zeta'_{d_2}}{P(\mathcal{D}_2 = d_2, n_1 = i)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)_{d_2}^2} \exp \left[- \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \sum_{j=1}^{d_2} (1 + \Re_j) x_{j:m:n} + \mathcal{T}_2 R'_{d_2} \right].
\end{aligned}$$

Then, we have

$$\begin{aligned}
 E\left(e^{t\hat{\theta}_1}|\mathcal{D}_2 = d_2, \zeta^{(d_2)}\right) &= \sum_{i=1}^{d_2-1} E\left(e^{t\hat{\theta}_1}|\mathcal{D}_2 = d_2, n_1 = i\right) P(n_1 = i|\zeta^{(d_2)}, \mathcal{D}_2 = d_2) \\
 &= \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\mathcal{D}_2 = d_2, n_1 = i)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^d} \\
 &\quad \times q_2^{R'_{d_2} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)} \int_0^{\mathcal{T}_2} \dots \int_0^{x_{2:m:n}} \prod_{j=1}^{d_2} f(x_{j:m:n}) [1 - F(x_{j:m:n})]^{(1+\Re_j) \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) - 1} \\
 &\quad \times dx_{1:m:n} \dots dx_{d_2:m:n}.
 \end{aligned}$$

From Lemma 1 with $v_j = (1 + \Re_j) \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)$ and then factor $\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)$ out of all of the v_j s, the above expression can be easily simplified as

$$\begin{aligned}
 &\sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\mathcal{D}_2 = d_2, n_1 = i)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^d} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-d_2} \\
 &\quad \times \sum_{j=0}^{d_2} \zeta_{j,d_2}(\Re_1 + 1, \dots, \Re_{d_1} + 1, \mathbf{1}_{d_2-d_1}, R'_{d_2}) q_2^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \Re_{d_2-j+1}}. \quad (\text{A4})
 \end{aligned}$$

The theorem then follows readily upon substituting (A2)–(A4) into (A1).

Appendix B. Proof of Theorem 3

From Theorem 1, the ConMGF of $\hat{\theta}_1$ is given by

$$\begin{aligned}
 M_{\hat{\theta}_1}(t) &= E\left(e^{t\hat{\theta}_1}|\zeta^{(u)}\right) \\
 &= \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)}|\mathcal{D}_1 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-m} \sum_{j=0}^m \zeta_{j,m}(\Re_1 + 1, \dots, \Re_m + 1) \\
 &\quad \times q_1^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \Re_{m-j+1}} + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\zeta'_m}{P(\zeta^{(m)}|\mathcal{D}_1 = d_1, \mathcal{D}_2 = m)} \binom{m}{i} \frac{\theta_1^{m-i} \theta_2^i}{(\theta_1 + \theta_2)^m} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-m} \\
 &\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \zeta_{i_1,d_1}(\Re_1 + 1, \dots, \Re_{d_1} + 1) \zeta_{i_2,d_2}(\mathbf{1}_{m-d_1-1}, \Re_m + 1) q_1^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \sum_{j=d_1-i_1+1}^{m-i_2} (\Re_j + 1)} \\
 &\quad \times q_2^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \sum_{j=m-i_2+1}^m (\Re_j + 1)} + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\zeta'_{d_2}}{P(\zeta^{(d_2)}|\mathcal{D}_2 = d_2)} \binom{d_2}{i} \frac{\theta_1^{d_2-i} \theta_2^i}{(\theta_1 + \theta_2)^{d_2}} \left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right)^{-d_2} \\
 &\quad \times \sum_{j=0}^{d_2} \zeta_{j,d_2}(\Re_1 + 1, \dots, \Re_{d_1} + 1, \mathbf{1}_{d_2-d_1}, \Re'_{d_2} + 1) q_2^{\left(1 - \frac{t}{i} \frac{\theta_1 \theta_2}{\theta_1 + \theta_2}\right) \Re'_{d_2-i+1}}.
 \end{aligned}$$

Because $(1 - t\theta_1\theta_2/\{i(\theta_1 + \theta_2)\})^{-m} \exp(\mathcal{T}_1 R_{m-j+1}^*/i)$ is the MGF of random variable X at t , where X is a gamma random variable with shape parameter m , rate parameter $\theta_1\theta_2/\{i(\theta_1 + \theta_2)\}$ and shift parameter $\mathcal{T}_1 R_{m-j+1}^*/i$, the theorem readily follows.

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