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Generic Existence of Solutions of Symmetric Optimization Problems

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Abstract: In this paper we study a class of symmetric optimization problems which is identified with a space of objective functions, equipped with an appropriate complete metric. Using the Baire category approach, we show the existence of a subset of the space of functions, which is a countable intersection of open and everywhere dense sets, such that for every objective function from this intersection the corresponding symmetric optimization problem possesses a solution.

Keywords: complete metric space; generic element; lower semicontinuos function; uniformity

MSC: [2010]49J27; 90C31

1. Introduction

Assume that (X, ρ) is a complete metric space. For each function $f : X \to R^1$ define

$$\inf(f) = \inf\{f(x) : \ x \in X\}.$$

For each $x \in X$ and each nonempty set $E \subset X$ define

$$\rho(x, E) = \inf\{\rho(x, y) : \ y \in E\}.$$

Denote by $\mathcal{M}_l$ the set of all lower semicontinuous and bounded from below functions $f : X \to R^1$. We equip the set $\mathcal{M}_l$ with the uniformity determined by the following base

$$\mathcal{E}(\epsilon) = \{(f, g) \in \mathcal{M}_l \times \mathcal{M}_l : \ |f(x) - g(x)| \le \epsilon \text{ for all } x \in X\},$$

where $\epsilon > 0$. It is known that this uniformity is metrizable (by a metric d) and complete [1].

Denote by $\mathcal{M}_c$ the set of all continuous functions $f \in \mathcal{M}_l$. It is not difficult to see that $\mathcal{M}_c$ is a closed subset of $\mathcal{M}_l$.

Consider a minimization problem

$$f(x) \to \min, \ x \in X,$$

where $f \in \mathcal{M}_l$.

This problem has a solution when X is compact or when f satisfies a growth conditions and bounded subsets of X are compact. When X does not satisfy a compactness assumption the existence problem becomes more difficult and less understood. It is possible to overcome this difficulty applying the Baire category approach. It turns out that this approach is useful in various fields of mathematics (see, for example, [1–7]). According to the Baire category approach, a property is valid for a generic (typical) point of a complete metric space (or it is valid generically) if the collection of all points of the space, which have this property, contains a G_δ everywhere dense subset of the metric space. In particular, it is known that the optimization problem stated above (see [1,3,7,8] and the references mentioned therein) can be solved generically (for a generic objective function). Namely, there exists a set $\mathcal{F} \subset \mathcal{M}_l$ which is a countable intersection of open and everywhere dense sets such that for each $f \in \mathcal{F}$ the minimization problem has a unique solution which is a limit of any minimizing sequence. This result and its numerous extensions are collected in [1]. It should be mentioned that generic existence results in optimal control and the calculus of variations are discussed in [9] while generic results in nonlinear analysis are presented in [10–14]. In particular, Ref. [9] contains generic results on the existence of solutions for large classes of optimal optimal control problems without convexity assumptions, generic existence results for best approximation problems are presented in [2,10,12], generic existence of fixed points for nonlinear operators is shown in [6,11,12] and the generic existence of a unique zero of maximally monotone operators is shown in [14]. In the present paper our goal is to obtain a generic existence of minimization problems with symmetry. This result is important because has applications in crystallography [15].

Assume that a mapping $T : X \to X$ is continuous and the mapping $T^2 = T \circ T$ is an identity mapping in X:

$$T^2(x) = x \text{ for all } x \in X. \tag{1}$$

This implies that $T(X) = X$, if $x_1, x_2 \in X$ and $T(x_1) = T(x_2)$, then $x_1 = x_2$ and that there exists $T^{-1} = T$.

Denote by $\mathcal{M}_{T,l}$ the set of all $f \in \mathcal{M}_l$ such that

$$f(T(x)) = f(x) \text{ for all } x \in X. \tag{2}$$

Clearly, $\mathcal{M}_{T,l}$ is a closed subset of $\mathcal{M}_l$. Set

$$\mathcal{M}_{T,c} = \mathcal{M}_{T,l} \cap \mathcal{M}_c.$$

All the sets $\mathcal{M}_{T,l}$, $\mathcal{M}_{T,c}$ and $\mathcal{M}_c$ are equipped with the metric d. We consider a minimization problem

$$f(x) \to \min, \ x \in X,$$

where $f \in \mathcal{M}_{T,l}$. Such problems with symmetric objective function have real world applications [15]. Note that the generic existence result for the space $\mathcal{M}_l$ does not give any information for its subspace $\mathcal{M}_{T,l}$.

In this paper, we prove the following result.

Theorem 1. *Assume that $\mathcal{M}_T$ is either $\mathcal{M}_{T,l}$ or $\mathcal{M}_{T,c}$. Then there is a subset $\mathcal{F} \subset \mathcal{M}_T$ which is a countable intersection of open and everywhere dense subsets of the space $\mathcal{M}_T$ such that for every objective function $f \in \mathcal{F}$ the two properties below hold:*

(i) there is a point $x_f \in X$ for which

$$\inf(f) = f(x_f) = f(T(x_f))$$

and if a point $z \in X$ satisfies the equation $f(z) = \inf(f)$, then the inclusion $z \in \{x_f, T(x_f)\}$ is true;

(ii) for every positive number ϵ, there exist a positive number δ and a neighborhood $\mathcal{U}$ of $f \in \mathcal{M}_l$ such that for every function $g \in \mathcal{U}$ and every point $z \in X$ for which

$$g(z) \leq \inf(g) + \delta$$

the relation

$$\min\{\rho(z, \{x_g, T(x_g)\}), \ \rho(T(z), \{x_g, T(x_g)\})\} \leq \epsilon$$

is true.

This theorem is proved in Section 3 while Section 2 contains an auxiliary result. It should be mentioned that the result stated above also holds for some closed subspaces of the space $\mathcal{M}_{T,l}$ with the same proof. This extension implies the main result of [15], obtained for some class of optimization problems arising in crystallography with $X = R^1$ and $T(x) = -x,\ x \in R^1$, which gave us the motivation for the result presented in this paper.

2. An Auxiliary Result

Lemma 1. *Assume that* $f \in \mathcal{M}_{T,l}$, $\epsilon \in (0,1)$, $\gamma > 0$,

$$\delta \in (0, 8^{-1}\epsilon\gamma), \tag{3}$$

$\bar{x} \in X$ *satisfies*

$$f(\bar{x}) \le \inf(f) + \delta, \tag{4}$$

$$\bar{f}(x) = f(x) + \gamma \min\{1,\ \rho(x,\bar{x}),\ \rho(T(x),\bar{x})\},\ x \in X \tag{5}$$

and that

$$U = \{g \in \mathcal{M}_l :\ (\bar{f}, g) \in \mathcal{E}(\delta)\}. \tag{6}$$

Then $\bar{f} \in \mathcal{M}_{T,l}$, *if* $f \in \mathcal{M}_{T,c}$, *then* $\bar{f} \in \mathcal{M}_{T,c}$ *and for each* $g \in U$ *and each* $z \in X$ *satisfying*

$$g(z) \le \inf(g) + \delta$$

the following inequality holds:

$$\min\{\rho(z,\bar{x}),\ \rho(T(z),\bar{x})\} < \epsilon.$$

Proof. By (1), (2) and (5), for every $x \in X$,

$$\bar{f}(T(x)) = f(T(x)) + \gamma \min\{1, \rho(T(x),\bar{x}), \rho(T^2(x),\bar{x})\}$$

$$= f(x) + \gamma \min\{1, \rho(x,\bar{x}), \rho(T(x),\bar{x})\} = \bar{f}(x).$$

Thus $\bar{f} \in \mathcal{M}_{T,l}$. Clearly, if $f \in \mathcal{M}_{T,c}$, then $\bar{f} \in \mathcal{M}_{T,c}$

Assume that

$$g \in U \tag{7}$$

and that $z \in X$ satisfies

$$g(z) \le \inf(g) + \delta. \tag{8}$$

By (4)–(8),

$$f(z) + \gamma \min\{1,\ \rho(z,\bar{x}),\ \rho(T(z),\bar{x})\} = \bar{f}(z)$$

$$\le g(z) + \delta \le \inf(g) + 2\delta \le g(\bar{x}) + 2\delta$$

$$\le \bar{f}(\bar{x}) + 3\delta = f(\bar{x}) + 3\delta \le \inf(f) + 4\delta \le f(z) + 4\delta$$

and

$$\min\{1,\ \rho(z,\bar{x}),\ \rho(T(z),\bar{x})\} \le 4\delta\gamma^{-1}.$$

In view of the inequality above and (3),

$$\min\{\ \rho(z,\bar{x}),\ \rho(T(z),\bar{x})\} \le 4\delta\gamma^{-1} < \epsilon.$$

Lemma 1 is proved. □

3. Proof of Theorem 1

Let $f \in \mathcal{M}_T$ and n be a natural number. By Lemma 1, there exist a nonempty open set $\mathcal{U}(f,n)$ in $\mathcal{M}_l$, $x(f,n) \in X$ and $\delta(f,n) > 0$ such that

$$\mathcal{U}(f,n) \cap \{g \in \mathcal{M}_T :\ d(f,g) < n^{-1}\} \neq \emptyset \tag{9}$$

and that the following property holds:

(P1) for each $g \in \mathcal{U}(f,n)$ and each $z \in X$ which satisfies

$$g(z) \leq \inf(g) + \delta(f,n) \tag{10}$$

the inequality

$$\min\{\rho(z, x(f,n)),\ \rho(T(z), x(f,n))\} < n^{-1} \tag{11}$$

holds.

Define

$$\mathcal{F} = [\cap_{p=1}^{\infty} \cup \{\mathcal{U}(f,n) :\ f \in \mathcal{M}_T,\ n \geq p \text{ is an integer }\}] \cap \mathcal{M}_T. \tag{12}$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\mathcal{M}_T$. Let

$$g \in \mathcal{F}. \tag{13}$$

Assume that $\{z_i\}_{i=1}^{\infty} \subset X$ satisfies

$$\lim_{i \to \infty} g(z_i) = \inf(g). \tag{14}$$

Let p be a natural number. By (12) and (13), there exist $f \in \mathcal{M}_T$ and an integer $n \geq p$ such that

$$g \in \mathcal{U}(f,n). \tag{15}$$

Property (P1) and (15) imply that for all large enough natural numbers i,

$$\min\{\rho(z_i, x(f,n)),\ \rho(T(z_i), x(f,n))\} \leq n^{-1} \leq p^{-1}.$$

Since p is an arbitrary natural number there exists a subsequence $\{z_{i_p}\}_{p=1}^{\infty}$ such that at least one of the sequences $\{z_{i_p}\}_{p=1}^{\infty}$ and $\{T(z_{i_p})\}_{p=1}^{\infty}$ converges to some point $x_g \in X$. Clearly,

$$g(x_g) = g(T(x_g)) = \inf(g). \tag{16}$$

Let $\xi \in X$ satisfy

$$g(\xi) = \inf(g). \tag{17}$$

Property (P1) and (15)–(17) imply that

$$\min\{\rho(\xi, x(f,n)),\ \rho(T(\xi), x(f,n))\} \leq p^{-1},$$

$$\min\{\rho(x_g, x(f,n)),\ \rho(T(x_g), x(f,n))\} \leq p^{-1}. \tag{18}$$

The relations above (see (18)) imply that

$$\min\{\rho(\xi, x_g),\ \rho(\xi, T(x_g)),\ \rho(T(\xi), x_g),\ \rho(T(\xi), T(x_g))\} \leq 2p^{-1}.$$

Since p is an arbitrary natural number we conclude that

$$\min\{\rho(\xi, x_g),\ \rho(\xi, T(x_g)),\ \rho(T(\xi), x_g),\ \rho(T(\xi), T(x_g))\} = 0$$

and at least one of the following relations hold:

$$\xi = x_g,\ \xi = T(x_g).$$

Assume that

$$h \in \mathcal{U}(f, n) \tag{19}$$

and $z \in X$ satisfies

$$h(z) \le \inf(h) + \delta(f, n). \tag{20}$$

Property (P1), (19) and (20) imply that

$$\min\{\rho(z, x(f, n)),\ \rho(T(z), x(f, n)) \le n^{-1} \le p^{-1}.$$

Together with (18) this implies that

$$\min\{\rho(z, x_g),\ \rho(z, T(x_g)),\ \rho(T(z), x_g),\ \rho(T(z), T(x_g))\} \le 2p^{-1}.$$

Since p is an arbitrary natural number this completes the proof of Theorem 1.

4. Conclusions

In this paper, using the Baire category approach, we study the large class of symmetric optimization problems which is identified with a space of objective functions, equipped with an appropriate complete metric. Such classes of optimization problems arise in crystallography. We show the existence of subset of the space of functions, which is a countable intersection of open and everywhere dense sets, such that for every objective function from this intersection the corresponding symmetric optimization problem possesses a solution.

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