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PID Control Design for SISO Strictly Metzlerian Linear Systems

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Abstract: For linear time-invariant Metzlerian systems, this paper proposes an original approach reflecting specific structural system constraints and positiveness in solving the problem of PID control. Refining parameter constraints and introducing enhanced equivalent system descriptions, the reformulated design task is consistent with the control law representation and is formulated as a linear matrix inequality feasibility problem. Taking into account structural restriction of Metzlerian positive systems, a characterization of PID control law parameters is permitted, to highlight dynamical properties of the closed-loop system solutions and the significant structural influence of derivative gain value of the control law parameters in design.

Keywords: linear Metzlerian systems; positive linear systems; diagonal stabilization; linear matrix inequalities; PID control

1. Introduction

Evolved from different considerations and exploiting positive properties of the system state and variables [1], positive systems represent a specific class of technical processes. Since their positiveness is conditioned through nonnegative parameters in achieving satisfactory performances, to reveal the connections with Metzler structure of system matrices, a rather common notation of them is Metzlerian systems. Serviceable publications to systems control design are focused on methods for stabilization, investigating memoryless controllers even at the cost that necessary sophisticated techniques are applied for positivity and constraints representation [2,3]. To achieve the closed-loop system state positivity with respect to parameter boundaries, semi-definite programming [4], implementation of non-symmetrical bounds [5], and combined linear programming method [6] are used, but not solving in general the specific problems concerned with solver interactions and parameter constraints. The main area of applicability are switched systems [7] and multi agent systems [8].

Many studies exist concerning PID control, where of interest are stabilizing tasks, formulated with inclusion of additional performance requirements and constraints. The resulting closed-loop system possesses stable system responses, and, if design is covered by adequate matrix formulation, it establishes computational efficiency and variable constraints [9], as well as effective computational design schemes by LMI-based formulations, directly connected with stability and robustness [10]. Unfortunately, most results related to those above mentioned for general linear systems are not directly applicable to linear Metzlerian systems [11].

Because control algorithms for Metzlerian linear positive systems are static, and consequently a static error remains in the closed circuit when used, one of the motivating factors for this paper is a formulation of control law based on a dynamic PID controller. In this context, the paper proposes a new methodological way in design of PID control for single input, single output (SISO) linear Mezlerian continuous-time systems. To find out what restrictions must be placed on parameters of PID control law as well as on the convergence analysis to turn them into a Metzler and Hurwitz closed-loop system matrix structure, a more general design task is analyzed and formulated.

Concerning only with asymptotically stable solutions, as is often the case, the article is a follow-up of the authors' previous paper [12], which introduces a representation of a Metzlerian system parameter constraints using linear matrix inequalities (LMI) and diagonal principle in stabilization is exploited. Such formulation is mathematically represented through state–space models with a strictly Metzler system matrix and supported by a minimal number of LMIs, defining structural constraints. As a result, it is documented that a solution for Metzler or purely Metzler system matrices may not exist (compare [5]) because of other structural constraints defined by the number and structure of zero elements in these matrices, and in dependence on elements of the input system matrix.

Then, reflecting these specific conditions, an adaptation presented in this paper for PID control design of linear Mezlerian SISO systems is original and primary. Since the D-part of PID is always bandwidth limited, such part of control law can refine system matrix parameter constraints in design task. This is accomplished by assuming that a suitable equivalent system exists and the resulting PI design bilinear task can be tackled using LMIs and linear matrix equality (LME) approach.

The remaining part of this paper is organized as follows. To present the reasoning path, brief comments on SISO linear Mezlerian systems are given in Section 2. The proposed LMI technique, enforcing conditions on the PID control law design, with the main theorem characterizing the system's behavior, is stated in Section 3, focusing on parametric features in PID control design for SISO Metzlerian systems, basic constitutive relations concerning the D-part of control law and a suitable way to translate synthesis into a feasibility problem that involves system parametric constraints and turning this approach into LMI based design formulation, conditioned by one LME. Conforming these results, Section 4 follows with an illustrative numerical example. Finally, Section 5 discuss the results and their interpretation, to set a straightforward manner point of view for the conclusions in Section 6.

Throughout the paper, the following notations are used: $\boldsymbol{x}^{\mathrm{T}}$, $\boldsymbol{X}^{\mathrm{T}}$ denotes the transpose of the vector $\boldsymbol{x}$, and the matrix $\boldsymbol{X}$, respectively, the indication $\boldsymbol{X}^{h\mathrm{T}}$ means transpose of the h-th power of a square matrix $\boldsymbol{X}$, the notation $\boldsymbol{X} \otimes \boldsymbol{Y}$ represents Kronecker product (tensor product) of two real matrices $\boldsymbol{X}, \boldsymbol{Y}$, $diag[\,\cdot\,]$ outlines a diagonal form, $\rho(\boldsymbol{X})$ identifies set of related matrix eigenvalues, labeling of matrix $\boldsymbol{X} \succ 0$ means its positive definiteness, $\boldsymbol{I}_n \in \mathbb{R}^{n \times n}$ is unit matrix, $a \in \mathbb{R}_+$ is nonnegative real scalar, $(\mathbb{R}_+^{n \times r})$, $\mathbb{R}^{n \times r}$ refers to the set of $n \times r$ (nonnegative) real matrices and $\mathbb{M}_{-+}^{n \times n}$ means the set of (strictly) Metzler square matrices, respectively.

2. Problem Formulation and Preliminary

Continuous-time, time-invariant linear Metzlerian SISO systems admit the description

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{b}u(t)\,, \tag{1}$$

$$y(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{q}(t)\,, \tag{2}$$

where $\boldsymbol{q}(t) \in \mathbb{R}_+^n$, $u(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}_+$ are the system state vector, control input, and measurable output, respectively.

To efficiently introduce the proposed methodology, some definitions and lemmas, borrowed from the properties of Metzlerian systems, are presented first.

Definition 1 ([13])**.** *A square matrix $\boldsymbol{A} \in \mathbb{R}_{-+}^{n \times n}$ is pure Metzler if its diagonal elements are negative and its off-diagonal elements are nonnegative. A square matrix A is called strictly Metzler if its diagonal elements are negative and its off-diagonal elements are positive. A Metzler matrix is stable if it is Hurwitz.*

Remark 1. *A strictly Metzler matrix* $\boldsymbol{A} \in \mathbb{R}_{-+}^{n\times n}$ *implies* n^2 *structural constraints*

$$a_{ii} < 0 \,\forall\, i = 1,\ldots n, \quad a_{ij,i\neq j} > 0 \,\forall\, i,j = 1,\ldots n\,. \tag{3}$$

If the vectors $\boldsymbol{b} \in \mathbb{R}_{+}^{n}$, $\boldsymbol{c} \in \mathbb{R}_{+}^{n}$ are nonnegative in control synthesis or in observer design, negative feedback reduces (nonnegative or positive) off-diagonal elements of the Metler matrix $\boldsymbol{A} \in \mathbb{R}_{-+}^{n\times n}$, and the analogous structural constraints resulting from Metzler matrix structure must be included in synthesis conditions to keep the resulting structure Metzler. Moreover, since a square matrix $\boldsymbol{X}$ and its inverse have nonnegative structure, if $\boldsymbol{X}$ is positively definite diagonal, to implement structural constraints through linear matrix inequalities [12], the LMI based design conditions for Metzlerian systems are formulated using positive definite diagonal matrix variables and the term "diagonal stability" is used [14]. If $\boldsymbol{A} \in \mathbb{R}_{-+}^{n\times n}$ is only pure Metzler, the synthesis conditions has to reflect further structural constraints, includable in design by related structured diagonal matrix variables [15].

Proposition 1 ([1])**.** *A solution* $\boldsymbol{q}(t)$ *of (1) for* $t \geq 0$ *is asymptotically stable and positive if* $\boldsymbol{A} \in \mathbb{R}_{-+}^{n\times n}$ *is a stable Metzler matrix;* $\boldsymbol{b} \in \mathbb{R}_{+}^{n}$ *is nonnegative matrix and the state vector* $\boldsymbol{q}(t) \in \mathbb{R}_{+}^{n}$ *for given* $\boldsymbol{u}(t) \in \mathbb{R}_{+}$ *and* $\boldsymbol{q}(0) \in \mathbb{R}_{+}$*. The linear system in (1) and (2) is asymptotically stable and positive if* $\boldsymbol{A} \in \mathbb{R}_{-+}^{n\times n}$ *is a stable Metzler matrix,* $\boldsymbol{b} \in \mathbb{R}_{+}^{n}$ *and* $\boldsymbol{c} \in \mathbb{R}_{+}^{n}$ *are nonnegative matrices, and the output vector* $\boldsymbol{y}(t) \in \mathbb{R}_{+}$ *for all* $\boldsymbol{u}(t) \in \mathbb{R}_{+}$ *and* $\boldsymbol{q}(0) \in \mathbb{R}_{+}$*.*

Definition 2 ([16])**.** *A matrix* $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ *is a permutation matrix if exactly one item in each column and row is equal to 1 and all other elements are equal to 0.*

Remark 2. *Considering Definition 2 and pondering a diagonal* $\boldsymbol{Y} \in \mathbb{R}^{n\times n}$ *such that*

$$\boldsymbol{Y} = diag \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}, \tag{4}$$

yields

$$\boldsymbol{L}^{\mathrm{T}}\boldsymbol{Y}\boldsymbol{L} = diag \begin{bmatrix} y_2 & \cdots & y_n & y_1 \end{bmatrix}, \tag{5}$$

if $\boldsymbol{L}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ *takes the circulant form*

$$\boldsymbol{L}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_{n-1} \\ 1 & \boldsymbol{0} \end{bmatrix}. \tag{6}$$

Lemma 1 (adapted from [12])**.** *Letting a matrix* $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$ *be strictly Metzler, then it is Hurwitz if and only if there exists a positive definite diagonal matrix* $\boldsymbol{P} \in \mathbb{R}_{+}^{n\times n}$ *such that the following set of linear matrix inequalities is feasible for* $i = 1,2,\ldots n$ *and* $h = 1,2,\ldots n-1$*,*

$$\boldsymbol{P} \succ 0\,, \tag{7}$$

$$\boldsymbol{A}(i,i)_{(1\leftrightarrow n)/n}\boldsymbol{P} \prec 0\,, \tag{8}$$

$$\boldsymbol{L}^{h}\boldsymbol{A}(i,i+h)_{(1\leftrightarrow n)/n}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} \succ 0\,, \tag{9}$$

$$\boldsymbol{P}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A}\boldsymbol{P} \prec 0\,, \tag{10}$$

where, computing with the circulant $\boldsymbol{L} \in \mathbb{R}_{+}^{n\times n}$ *and sequentially fixing of* $h \in \langle 1, n-1 \rangle$*,*

$$\boldsymbol{A}(i,i+h)_{(1\leftrightarrow n)/n} = diag \begin{bmatrix} a_{1,1+h} & \cdots & a_{n-h,n} & a_{n-h+1,1} & \cdots & a_{n,h} \end{bmatrix}. \tag{11}$$

Note that LMI (8) reflects structural constraints for elements on the main diagonal of strictly Metzler $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$, the set of LMIs (9) reflects structural constraints for off-diagonal elements of strictly Metzler $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$, and (10) guaranties that $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$ is also Hurwitz.

The notation $\Delta = (1 \leftrightarrow n)/n$ symbolically expresses that the indexing of the elements in the set of diagonal matrices (11) is bound to the sum of modulo n for given h. Interested readers are referred for further details to [12].

Lemma 2 (adapted from [17]). *Let a square real $n \times n$ matrix $\boldsymbol{\Lambda}$ be partitioned as*

$$\boldsymbol{\Lambda} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}\boldsymbol{C}\,, \tag{12}$$

where $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}_{+}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}_{+}^{m\times n}$, and $\boldsymbol{D} \in \mathbb{R}_{+}^{m\times m}$, while $\boldsymbol{A}$ is strictly Metzler. Then, $\boldsymbol{\Lambda}$ is strictly Metzler if, equivalently,

(i)

$$a_{ii} - \boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{D}\boldsymbol{c}_i < 0 \text{ for all } i = 1,\dots,n\,, \quad a_{ij} - \boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{K}\boldsymbol{c}_j > 0 \text{ for all } i,j = 1,\dots n,\ i \neq j\,. \tag{13}$$

(ii)

$$\boldsymbol{A}(i,i)_{(1\leftrightarrow n)/n} - \boldsymbol{B}_d\boldsymbol{D}_d\boldsymbol{C}_d \prec 0\,, \quad \boldsymbol{A}(i,i+h)_{(1\leftrightarrow n)/n} - \boldsymbol{B}_d\boldsymbol{D}_d\boldsymbol{C}_{dh} \succ 0\,, \tag{14}$$

where

$$\boldsymbol{B}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n\end{bmatrix}, \quad \boldsymbol{B}_d = diag\begin{bmatrix}\boldsymbol{b}_1^{\mathrm{T}} & \cdots & \boldsymbol{b}_n^{\mathrm{T}}\end{bmatrix}, \quad \boldsymbol{D}_d = \boldsymbol{I}_n \otimes \boldsymbol{K}\,, \tag{15}$$

$$\boldsymbol{C} = \begin{bmatrix}\boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n\end{bmatrix}, \quad \boldsymbol{C}_d = diag\begin{bmatrix}\boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n\end{bmatrix}, \quad \boldsymbol{C}_{dh} = \boldsymbol{S}^{h\mathrm{T}}\boldsymbol{C}_d\boldsymbol{L}^h, \quad \boldsymbol{S} = \boldsymbol{L} \otimes \boldsymbol{I}_m\,. \tag{16}$$

Moreover,

$$\boldsymbol{\Lambda} = \sum_{h=0}^{n-1}(\boldsymbol{A}(i,i+h)_{(1\leftrightarrow n)/n} - \boldsymbol{B}_d\boldsymbol{D}_d\boldsymbol{C}_{dh})\boldsymbol{L}^{h\mathrm{T}}\,. \tag{17}$$

Lemma 3 ([18] Lyapunov inequality). *Considering the autonomous subsystem*

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t)\,, \tag{18}$$

the following statements are equivalent from the point of asymptotic stability:

(i) (18) is asymptotically stable.
(ii) $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is Hurwitz.
(iii) There exists a symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ satisfying

$$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \succ 0\,, \quad \boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^{\mathrm{T}} \prec 0\,. \tag{19}$$

Lemma 4 ([19]). *A square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is called a Hurwitz matrix (a stable matrix) if every eigenvalue of $\boldsymbol{A}$ has a strictly negative real part. Then, the autonomous system (18) is asymptotically stable. Denoting that*

$$P(s) = \det(s\boldsymbol{I}_n - \boldsymbol{A}) \tag{20}$$

is characteristic polynomial of the transfer function related to (18), the eigenvalue spectrum $\rho(\boldsymbol{A})$ is the set of roots of the characteristic equation $P(s) = 0$.

Definition 3 ([13]). *Letting $\boldsymbol{U} \in \mathbb{R}^{m\times m}$, $\boldsymbol{V} \in \mathbb{R}^{n\times n}$, then the (mn)-dimensional matrix, called the Kronecker product of $\boldsymbol{U}$ and $\boldsymbol{V}$, is constructed as*

$$\boldsymbol{U} \otimes \boldsymbol{V} = \begin{bmatrix}\{u_{ij}\boldsymbol{V}\}_{i,j=1}^{m}\end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix}\{u_{ij}\}_{i,j=1}^{m}\end{bmatrix}. \tag{21}$$

It is convenient at this point to underline that it prioritizes the following Kronecker product properties [20]

$$(\boldsymbol{I}_n \otimes \boldsymbol{U})(\boldsymbol{V} \otimes \boldsymbol{I}_m) = (\boldsymbol{V} \otimes \boldsymbol{I}_m)(\boldsymbol{I}_n \otimes \boldsymbol{U})\,, \tag{22}$$

$$(\boldsymbol{U} \otimes \boldsymbol{V})^{-1} = \boldsymbol{U}^{-1} \otimes \boldsymbol{V}^{-1}, \tag{23}$$

$$(\boldsymbol{U} \otimes \boldsymbol{V})^{\mathrm{T}} = \boldsymbol{U}^{\mathrm{T}} \otimes \boldsymbol{V}^{\mathrm{T}}. \tag{24}$$

Since the positivity of the systems is defined by a nonnegative system state, nonnegative system input and output parameters, and a Metzler system matrix structure, it is necessary to proceed from this fact and take into account the above-stated structural constraints also in the synthesis of the PID controller to preserve the system positive properties. Because this introduces added limitations in the synthesis conditions with addition to the parametric redundancy, the aim is to propose a methodology that would be sufficiently general with respect to the structure of the measured system state variables but also effective in terms of closed-loop positivity.

3. Main Results

To develop a constructive method for synthesis of PID controllers for given system class, it is necessary to establish a direct consequence of control parameters and Metzler matrix parametric constraints on asymptotic stability and system trajectories. This is the role of this section.

3.1. Parametric features in PID Control Design for SISO Metzlerian Systems

If the Metzlerian system in (1) and (2) is operating under PID control, then, respecting the positiveness of Metzlerian system variables, the continuous-time control algorithm can be considered as

$$u(t) = k_P e_r(t) + k_I \int_0^t \boldsymbol{c}_p^{\mathrm{T}} \boldsymbol{q}(\tau) d\tau - k_D \dot{e}_r(t)\,, \tag{25}$$

where $w_r \in \mathbb{R}_+^r$ is a constant positive reference signal, $e_r(t) \in \mathbb{R}$ is the tracking error defined as

$$e_r(t) = w_r - y(t)\,, \tag{26}$$

and $k_P, k_I, k_D \in \mathbb{R}_+$ are parameters of the PID controller. If there is further assumed

$$w_r(t) = \boldsymbol{c}^{\mathrm{T}} w, \quad \boldsymbol{e}(t) = \boldsymbol{w} - \boldsymbol{q}(t), \quad e_r(t) = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{e}(t)\,, \tag{27}$$

it is not difficult to verify that in dependence to (27)

$$\begin{aligned} u(t) = &\quad k_P \boldsymbol{c}^{\mathrm{T}} \boldsymbol{e}(t) + k_I p(t) - k_D \boldsymbol{c}^{\mathrm{T}} \dot{\boldsymbol{e}}(t) \\ &= -k_P \boldsymbol{c}^{\mathrm{T}} \boldsymbol{q}(t) + k_I p(t) + k_D \boldsymbol{c}^{\mathrm{T}} \dot{\boldsymbol{q}}(t) + k_P \boldsymbol{c}^{\mathrm{T}} \boldsymbol{w}\,, \end{aligned} \tag{28}$$

while the input to the integrator $\dot{p}(t) \in \mathbb{R}$ is

$$\dot{p}(t) = \boldsymbol{c}_p^{\mathrm{T}} \boldsymbol{q}(t)\,, \tag{29}$$

and for implementation of (25) discrete-time realizations of PID controller law can be used.

Note that, in general, $\boldsymbol{c}_p^{\mathrm{T}} = \boldsymbol{c}^{\mathrm{T}}$ can be set, which means to append to the integrator input all state variables involved in the system output projection. If this is not the case, $\boldsymbol{c}_p^{\mathrm{T}}$ reflects projection (a measurable subset) of the system state variables.

Since PID synthesis theory for standard linear systems cannot be used in PID parameter synthesis for Metzlerian systems, the role of constraints in design is analyzed and given at first. Thus, a consequence of certainty equivalence is the assembled system structure

$$\begin{bmatrix} \boldsymbol{I}_n - \boldsymbol{b} k_D \boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} k_P & \boldsymbol{0} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} w_r \\ p(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{A} - \boldsymbol{b} k_P \boldsymbol{c}^{\mathrm{T}} & \boldsymbol{b} k_I \\ \boldsymbol{c}_p^{\mathrm{T}} & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ p(t) \end{bmatrix}, \tag{30}$$

$$\begin{bmatrix} y(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ p(t) \end{bmatrix}, \tag{31}$$

and introducing the composite variables and matrix parameters

$$\boldsymbol{q}^{\circ}(t) = \begin{bmatrix} \boldsymbol{q}(t) \\ p(t) \end{bmatrix}, \quad \boldsymbol{w}^{\circ}(t) = \begin{bmatrix} w_r \\ p(t) \end{bmatrix}, \quad \boldsymbol{y}^{\circ}(t) = \begin{bmatrix} y(t) \\ p(t) \end{bmatrix}, \tag{32}$$

$$\boldsymbol{A}^{\diamond} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{b}k_P\boldsymbol{c}^{\mathrm{T}} & \boldsymbol{b}k_I \\ \boldsymbol{c}_p^{\mathrm{T}} & -1 \end{bmatrix}, \boldsymbol{B}^{\diamond} = \begin{bmatrix} \boldsymbol{b}k_P & \boldsymbol{0} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix}, \boldsymbol{E}^{\diamond} = \begin{bmatrix} \boldsymbol{I}_n - \boldsymbol{b}k_D\boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix}, \boldsymbol{C}^{\circ} = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix}, \tag{33}$$

the corresponding closed loop system description is

$$\boldsymbol{E}^{\diamond}\dot{\boldsymbol{q}}^{\circ}(t) = \boldsymbol{A}^{\diamond}\boldsymbol{q}^{\circ}(t) + \boldsymbol{B}^{\diamond}\boldsymbol{w}^{\circ}(t), \tag{34}$$

$$\boldsymbol{y}^{\circ}(t) = \boldsymbol{C}^{\circ}\boldsymbol{q}^{\circ}(t). \tag{35}$$

Since $\boldsymbol{B}^{\diamond}$, $\boldsymbol{C}^{\circ}$ are nonnegative, to obtain Metzlerian structures in control if $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ are bound with a Metzler system and k_P, k_I, k_D are positive, then $\boldsymbol{A}^{\diamond}$ is Metzler if $(\boldsymbol{A} - \boldsymbol{b}k_P\boldsymbol{c}^{\mathrm{T}})$ is (strictly) Metzler and $(\boldsymbol{I}_n - \boldsymbol{b}k_D\boldsymbol{c}^{\mathrm{T}})$ is regular.

The control problem can be formulated considering structure of the static output control by the parameterizations

$$\boldsymbol{A}^{\diamond} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{c}_p^{\mathrm{T}} & -1 \end{bmatrix} - \begin{bmatrix} \boldsymbol{b} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_P & -k_I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix}, \tag{36}$$

$$\boldsymbol{E}^{\diamond} = \begin{bmatrix} \boldsymbol{I}_n & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{b} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -k_D & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix}. \tag{37}$$

It is then possible to write in a straightforward manner that

$$(\boldsymbol{I}^{\circ} + \boldsymbol{B}^{\circ}\boldsymbol{K}_D^{\circ}\boldsymbol{C}^{\circ})\dot{\boldsymbol{q}}^{\circ}(t) = (\boldsymbol{A}^{\circ} - \boldsymbol{B}^{\circ}\boldsymbol{K}_{PI}^{\circ}\boldsymbol{C}^{\circ})\boldsymbol{q}^{\circ}(t) + \boldsymbol{B}^{\diamond}\boldsymbol{w}^{\circ}, \tag{38}$$

where

$$\boldsymbol{A}^{\circ} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{c}_p^{\mathrm{T}} & -1 \end{bmatrix}, \quad \boldsymbol{B}^{\circ} = \begin{bmatrix} \boldsymbol{b} & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{C}^{\circ} = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix}, \tag{39}$$

$$\boldsymbol{I}^{\circ} = \begin{bmatrix} \boldsymbol{I}_n & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 0 \end{bmatrix}, \quad \boldsymbol{K}_{PI}^{\circ} = \begin{bmatrix} k_P & -k_I \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{K}_D^{\circ} = \begin{bmatrix} -k_D & 0 \\ 0 & 1 \end{bmatrix}. \tag{40}$$

This is the basic feature formulation in order to convexify the considered synthesis problem.

3.2. Basic Constitutive Relations

Consider a sub-class of linear SISO Metzlerian systems, where

$$\boldsymbol{c}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad \boldsymbol{b}^{\mathrm{T}} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}, \tag{41}$$

and $\boldsymbol{b} \in \mathbb{R}_+^n$ is nonnegative vector. Then, using the Sherman–Morrison–Woodbury formula [21], (37) implies

$$(\boldsymbol{I}^{\circ} + \boldsymbol{B}^{\circ}\boldsymbol{K}_D^{\circ}\boldsymbol{C}^{\circ})^{-1} = \boldsymbol{I}^{\circ} - \boldsymbol{B}^{\circ}(\boldsymbol{K}_D^{\circ-1} + \boldsymbol{C}^{\circ}\boldsymbol{B}^{\circ})^{-1}\boldsymbol{C}^{\circ}, \tag{42}$$

with SISO system parameter representation

$$\boldsymbol{C}^{\circ}\boldsymbol{B}^{\circ} = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{0} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{43}$$

$$\boldsymbol{K}_D^{\circ -1} = \begin{bmatrix} -k_D^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{K}_D^{\circ -1} + \boldsymbol{C}^\circ \boldsymbol{B}^\circ = \begin{bmatrix} b_1 - k_D^{-1} & 0 \\ 0 & 1 \end{bmatrix}. \tag{44}$$

Considering that $k_D^{-1} > b_1$, then with $-\lambda^{-1} = -k_D^{-1} + b_1 < 0$ it yields

$$\boldsymbol{E}^{\circ -1} = \begin{bmatrix} \boldsymbol{I}_n + \lambda \boldsymbol{b}\boldsymbol{c}^{\mathrm{T}} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1+\lambda b_1) & 0 & \cdots & 0 & 0 \\ \lambda b_2 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \\ \lambda b_n & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 1 \end{bmatrix}, \tag{45}$$

and, because

$$\det(s\boldsymbol{I}^\circ - \boldsymbol{E}^{\circ -1}) = (s - (1+\lambda b_1))(s-1)^n, \tag{46}$$

$\boldsymbol{E}^{\circ -1}$ is positive definite if $1 + \lambda b_1 > 0$.

Prompted by the observation inversion of positive definite matrices [16], it implies that in this case $\boldsymbol{E}^\circ$ is also positive definite. Thus, for positive $k_D > 0$, $b_1 > 0$, the relation

$$1 + \lambda b_1 = 1 + \frac{b_1}{k_D^{-1} - b_1} = \frac{k_D^{-1}}{k_D^{-1} - b_1} > 0 \tag{47}$$

implies $k_D^{-1} > b_1$.

If for example it is considered that

$$\boldsymbol{c}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{48}$$

then, analogously,

$$\boldsymbol{K}_D^{\circ -1} + \boldsymbol{C}^\circ \boldsymbol{B}^\circ = \begin{bmatrix} b_1 + b_2 - k_D^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \tag{49}$$

and with

$$-\lambda^{-1} = -k_D^{-1} + b_1 + b_2 < 0\,, \tag{50}$$

then

$$\boldsymbol{E}^{\circ -1} = \begin{bmatrix} (1+\lambda b_1) & \lambda b_1 & 0 & \cdots & 0 & 0 \\ \lambda b_2 & (1+\lambda b_2) & 0 & & 0 & 0 \\ \lambda b_3 & \lambda b_3 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \\ \lambda b_n & \lambda b_n & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{51}$$

Then, using the property of the block matrix determinant with

$$\boldsymbol{Y}(\lambda) = \begin{bmatrix} (1+\lambda b_1) & \lambda b_1 \\ \lambda b_2 & (1+\lambda b_2) \end{bmatrix} \tag{52}$$

yields

$$\det(s\boldsymbol{I}^\circ - \boldsymbol{E}^{\circ -1}) = \det(s\boldsymbol{I}_2 - \boldsymbol{Y}(\lambda))(s-1)^{n-1}, \tag{53}$$

where

$$\det(s\boldsymbol{I}_2 - \boldsymbol{Y}(\lambda)) = (s-1)^2 - (s-1)\lambda(b_1+b_2)\,, \tag{54}$$

and, consequently,

$$\det(s\boldsymbol{I}^\circ - \boldsymbol{E}^{\circ -1}) = (s - (1+\lambda(b_1+b_2)))(s-1)^n. \tag{55}$$

Thus, $\boldsymbol{E}^{\circ-1}$ is positive definite if with respect to (50)

$$1+\lambda(b_1+b_2)=1+\frac{b_1+b_2}{k_D^{-1}-(b_1+b_2)}=\frac{k_D^{-1}}{k_D^{-1}-(b_1+b_2)}>0\,. \tag{56}$$

It can be generalized that, if

$$1+\lambda\sum_{h=1}^{n}b_hc_h=\frac{k_D^{-1}}{k_D^{-1}-\sum\limits_{h=1}^{n}b_hc_h}=\frac{k_D^{-1}}{k_D^{-1}-\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}}>0\,, \tag{57}$$

$$-\lambda^{-1}=-k_D^{-1}+\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}<0\,, \tag{58}$$

then

$$P_D(s)=\det(s\boldsymbol{I}^{\circ}-\boldsymbol{E}^{\circ-1})=\Big(s-\frac{k_D^{-1}}{k_D^{-1}-\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}}\Big)(s-1)^n, \tag{59}$$

$$\boldsymbol{E}^{\circ-1}=\begin{bmatrix}\boldsymbol{I}_n+\lambda\boldsymbol{b}\boldsymbol{c}^{\mathrm{T}} & 0\\ 0 & 1\end{bmatrix}. \tag{60}$$

Analyzing the matrix product $\boldsymbol{E}^{\diamond-1}\boldsymbol{B}^{\circ}$, yields, in general,

$$\boldsymbol{E}^{\diamond-1}\boldsymbol{B}^{\circ}=\begin{bmatrix}\boldsymbol{I}_n+\lambda\boldsymbol{b}\boldsymbol{c}^{\mathrm{T}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{b} & \boldsymbol{0}\\ \boldsymbol{0} & 1\end{bmatrix}=\begin{bmatrix}(\boldsymbol{I}_n+\lambda\boldsymbol{b}\boldsymbol{c}^{\mathrm{T}})\boldsymbol{b} & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\boldsymbol{b}_1^{\bullet} & 0\\ 0 & 1\end{bmatrix}=\boldsymbol{B}^{\bullet}, \tag{61}$$

and, since $\boldsymbol{b}$ is nonnegative, $\boldsymbol{b}^{\bullet}$ as well as $\boldsymbol{B}_v^{\bullet}$ are nonnegative.

Analogously, it yields

$$\boldsymbol{E}^{\diamond-1}\boldsymbol{B}^{\diamond}=\begin{bmatrix}\boldsymbol{I}_n+\lambda\boldsymbol{b}\boldsymbol{c}^{\mathrm{T}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{b}k_P & \boldsymbol{0}\\ \boldsymbol{0}^{\mathrm{T}} & 1\end{bmatrix}=\begin{bmatrix}(\boldsymbol{I}_n+\lambda\boldsymbol{b}\boldsymbol{c}^{\mathrm{T}})\boldsymbol{b}k_P & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\boldsymbol{b}^{\bullet}k_P & 0\\ 0 & 1\end{bmatrix}=\boldsymbol{B}_w^{\bullet}, \tag{62}$$

and, since $\boldsymbol{b}^{\bullet}$ is nonnegative, $\boldsymbol{B}_w^{\bullet}$ is nonnegative.

The structure of the matrix product $\boldsymbol{E}^{\diamond-1}\boldsymbol{A}^{\circ}$ is

$$\begin{aligned}\boldsymbol{E}^{\diamond-1}\boldsymbol{A}^{\circ}&=\begin{bmatrix}\boldsymbol{I}_n+\lambda\boldsymbol{b}\boldsymbol{c}^{\mathrm{T}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{c}_p^{\mathrm{T}} & -1\end{bmatrix}\\ &=\begin{bmatrix}(\boldsymbol{I}_n+\lambda\boldsymbol{b}\boldsymbol{c}^{\mathrm{T}})\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{c}_p^{\mathrm{T}} & -1\end{bmatrix}\\ &=\begin{bmatrix}\boldsymbol{A}+\lambda\boldsymbol{b}\begin{bmatrix}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{a}_1 & \cdots & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{a}_n\end{bmatrix} & \boldsymbol{0}\\ \boldsymbol{c}_p^{\mathrm{T}} & -1\end{bmatrix},\end{aligned} \tag{63}$$

that is

$$\boldsymbol{E}^{\diamond-1}\boldsymbol{A}^{\circ}=\begin{bmatrix}\boldsymbol{A}_1^{\bullet} & \boldsymbol{0}\\ \boldsymbol{c}_p^{\mathrm{T}} & -1\end{bmatrix}=\boldsymbol{A}^{\bullet}\,, \tag{64}$$

where

$$\boldsymbol{A}_1^{\bullet}=\boldsymbol{A}+\lambda\boldsymbol{b}\begin{bmatrix}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{a}_1 & \cdots & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{a}_n\end{bmatrix}. \tag{65}$$

Thus, the descriptor form (38) for SISO systems can be transformed to the following regular form

$$\dot{\boldsymbol{q}}^{\circ}(t)=(\boldsymbol{A}^{\bullet}-\boldsymbol{B}^{\bullet}\boldsymbol{K}_{PI}^{\circ}\boldsymbol{C}^{\circ})\boldsymbol{q}^{\circ}(t)+\boldsymbol{B}_w^{\bullet}\boldsymbol{w}^{\circ}(t)=\boldsymbol{A}_c^{\bullet}\boldsymbol{q}^{\circ}(t)+\boldsymbol{B}_w^{\bullet}\boldsymbol{w}^{\circ}(t)\,, \tag{66}$$

underlining that such system description is of a bilinear structure.

Remark 3. *It can be remarked that (57) implies the sufficient constraint on the gain k_D in synthesis of PID control for SISO Metzlerian systems, guaranteeing that resulting $\boldsymbol{B}^\bullet$ and $\boldsymbol{B}^\bullet_w$ are nonnegative. This in turn means that, for a positive k_D thus limited, it is sufficient to include in the synthesis only the parametric constraints resulting from the Metzler structure of $\boldsymbol{A}^\bullet_c = \boldsymbol{A}^\bullet - \boldsymbol{B}^\bullet \boldsymbol{K}^\circ_{PI} \boldsymbol{C}^\circ$ with relation to given Lemma 1.*

From the relation (65), it can be concluded to the limit case when all elements of the vector $\boldsymbol{c}^{\mathrm{T}}$ are equal to one, all elements of $\boldsymbol{b}$ are positive, and pure (strictly) Metzler matrix $\boldsymbol{A}$ is diagonally dominant. In such a case, a negative value will be added to each element of the matrix $\boldsymbol{A}$, which may destroy the Metzler structure of $\boldsymbol{A}^\bullet_1$, leading to juncture where a nonnegative synthesis solution does not exist. This results in it being extremely important in choosing the measured system state, i.e., the structure of nonnegative vector $\boldsymbol{c}^{\mathrm{T}}$, as well as to application of a structure of $\boldsymbol{c}^{\mathrm{T}}_p$, different from $\boldsymbol{c}^{\mathrm{T}}$.

To provide constraint limitations, constraint structures need to reflect linear matrix inequality forms, but the structure of $\boldsymbol{A}^\bullet_c$ is essentially bilinear. In the following, this bilinear limitation is eliminated applying one way of linearization, where structural LMIs are conditioned by one linear matrix equality (see, e.g., [17]). This principle is explicated at the point of application in the following subsection.

3.3. PID Control Design

Exploiting diagonal stabilization in accession to control of linear strictly Metzlerian structures [12], the following matrix parameter expressions need to be applied, considering the suitable fixing $\lambda > 0$ in such a way that $\boldsymbol{A}^\bullet \in \mathbb{M}^{(n+1)\times(n+1)}_{-+}$, $\boldsymbol{B}^\bullet \in \mathbb{R}^{(n+1)\times 2}_{+}$, and $\boldsymbol{C}^\circ \in \mathbb{R}^{2\times(n+1)}_{+}$. These are represented in coincidence with Lemmas 1 and 2 as follows:

$$\boldsymbol{A}^\bullet = \begin{bmatrix} -a^\bullet_{11} & a^\bullet_{12} & \cdots & a^\bullet_{1n} & 0 \\ a^\bullet_{21} & -a^\bullet_{22} & \cdots & a^\bullet_{2n} & 0 \\ \vdots & & \ddots & & \\ a^\bullet_{n1} & a^\bullet_{n2} & \cdots & -a^\bullet_{nn} & 0 \\ c_{p1} & c_{p2} & \cdots & c_{pn} & -1 \end{bmatrix} = \begin{bmatrix} -a^\bullet_{11} & a^\bullet_{12} & \cdots & a^\bullet_{1n} & a^\bullet_{1,n+1} \\ a^\bullet_{21} & -a^\bullet_{22} & \cdots & a^\bullet_{2n} & a^\bullet_{2,n+1} \\ \vdots & & \ddots & & \\ a^\bullet_{n1} & a^\bullet_{n2} & \cdots & -a^\bullet_{nn} & a^\bullet_{n,n+1} \\ a^\bullet_{n+1,1} & a^\bullet_{n+1,2} & \cdots & a^\bullet_{n+1,1} & -a^\bullet_{n+1,n+1} \end{bmatrix}, \tag{67}$$

$$\boldsymbol{A}^\bullet(i,i)_\Delta = \mathrm{diag}\begin{bmatrix} -a^\bullet_{1,1} & \cdots & -a^\bullet_{nn} & -a^\bullet_{n+1,n+1} \end{bmatrix}, \tag{68}$$

$$\boldsymbol{A}^\bullet(i,i+h)_\Delta = \mathrm{diag}\begin{bmatrix} a^\bullet_{1,1+h} & \cdots & a^\bullet_{n-h,n+1} & a^\bullet_{n-h+1,1} & \cdots & a^\bullet_{n+1,h} \end{bmatrix}, \tag{69}$$

where $\Delta = (1 \leftrightarrow (n+1))/(n+1)$, $h = 1, 2, \ldots, n$,

$$\boldsymbol{B}^\bullet = \begin{bmatrix} \begin{bmatrix} b^\bullet_{11} & 0 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} b^\bullet_{1n} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}^{\bullet\mathrm{T}}_1 \\ \vdots \\ \boldsymbol{b}^{\bullet\mathrm{T}}_n \\ \boldsymbol{b}^{\bullet\mathrm{T}}_{n+1} \end{bmatrix}, \qquad \boldsymbol{B}^\bullet_d = \mathrm{diag}\begin{bmatrix} \boldsymbol{b}^{\bullet\mathrm{T}}_1 & \cdots & \boldsymbol{b}^{\bullet\mathrm{T}}_n & \boldsymbol{b}^{\bullet\mathrm{T}}_{n+1} \end{bmatrix}, \tag{70}$$

$$\boldsymbol{C}^\circ = \begin{bmatrix} \boldsymbol{c}^{\mathrm{T}} & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}^\circ_1 & \cdots & \boldsymbol{c}^\circ_n & \boldsymbol{c}^\circ_{n+1} \end{bmatrix}, \qquad \boldsymbol{C}^\bullet_d = \mathrm{diag}\begin{bmatrix} \boldsymbol{c}^\circ_1 & \cdots & \boldsymbol{c}^\circ_n & \boldsymbol{c}^\circ_{n+1} \end{bmatrix}. \tag{71}$$

The supported matrix structure are constructed as

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_n & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{S} = \boldsymbol{L} \otimes \boldsymbol{I}_2, \quad \boldsymbol{J}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{I}_2 & \cdots & \boldsymbol{I}_2 \end{bmatrix}, \quad \boldsymbol{C}^\bullet_{dh} = \boldsymbol{S}^{h\mathrm{T}} \boldsymbol{C}^\bullet_d \boldsymbol{L}^h, \tag{72}$$

and the expected performance of the gain matrix is

$$\boldsymbol{K}^\circ = \begin{bmatrix} k_P & -k_I \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{K}^\bullet_d = \mathrm{diag}\begin{bmatrix} \boldsymbol{K}^\circ & \cdots & \boldsymbol{K}^\circ & \boldsymbol{K}^\circ \end{bmatrix} = \boldsymbol{I}_{n+1} \otimes \boldsymbol{K}^\circ. \tag{73}$$

Theorem 1. *The closed-loop built on (1), (2) under PID control (25) is stable if, for a given positive scalar* $\lambda \in \mathbb{R}_+$*, the matrix* $\boldsymbol{A}^{\bullet} \in \mathbb{M}_{-+}^{(n+1)\times(n+1)}$ *is Metzler,* $\boldsymbol{B}^{\bullet}, \boldsymbol{B}_w^{\bullet} \in \mathbb{R}_+^{(n+1)\times 2}$ *are nonnegative, and there exist positive definite diagonal matrices* $\boldsymbol{P} \in \mathbb{R}_+^{(n+1)\times(n+1)}$ *and* $\boldsymbol{H} \in \mathbb{R}_+^{2\times 2}$ *and a nonnegative matrix* $\boldsymbol{R} \in \mathbb{R}_+^{2\times 2}$ *such that for* $h = 1, 2, \ldots n$*,*

$$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \succ 0\,, \qquad \boldsymbol{H} = \boldsymbol{H}^{\mathrm{T}} \succ 0\,, \tag{74}$$

$$\boldsymbol{A}(i,i)_{\Delta}\boldsymbol{P} - \boldsymbol{B}_d^{\bullet}\boldsymbol{R}_d^{\bullet}\boldsymbol{C}_d^{\bullet} \prec 0\,, \tag{75}$$

$$\boldsymbol{L}^h\boldsymbol{A}(i,i{+}h)_{\Delta}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{P} - \boldsymbol{L}^h\boldsymbol{B}_d^{\bullet}\boldsymbol{S}^{h\mathrm{T}}\boldsymbol{R}_d^{\bullet}\boldsymbol{C}_d^{\bullet} \succ 0\,, \tag{76}$$

$$\boldsymbol{A}^{\bullet}\boldsymbol{P}{+}\boldsymbol{P}\boldsymbol{A}^{\bullet\mathrm{T}} - \boldsymbol{B}^{\bullet}\boldsymbol{R}_d^{\bullet}\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{C}_d^{\bullet} - \boldsymbol{C}_d^{\bullet\mathrm{T}}\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{R}_d^{\bullet\mathrm{T}}\boldsymbol{B}^{\bullet\mathrm{T}} \prec 0\,, \tag{77}$$

$$\boldsymbol{C}_d^{\bullet}\boldsymbol{P} = \boldsymbol{H}_d\boldsymbol{C}_d^{\bullet}\,, \tag{78}$$

where

$$\boldsymbol{R}_d = \boldsymbol{I}_{n+1} \otimes \boldsymbol{R}\,, \qquad \boldsymbol{H}_d = \boldsymbol{I}_{n+1} \otimes \boldsymbol{H}\,, \tag{79}$$

and the specific causal relations are pre-considered in (67)–(72).

Within a feasible solution with λ *fixing* k_D*, the gain* $\boldsymbol{K}^{\circ} \in \mathbb{R}_+^{r\times n}$ *representing not fixed design parameters is*

$$\boldsymbol{K}^{\circ} = \begin{bmatrix} k_P & -k_I \\ 0 & 0 \end{bmatrix} = \boldsymbol{R}\boldsymbol{H}^{-1}. \tag{80}$$

Proof of Theorem 1. For a stable realization of $\boldsymbol{A}_c^{\bullet}$, it yields according to Lyapunov inequality (19) and relation (17)

$$\begin{aligned} &\boldsymbol{A}_c^{\bullet}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_c^{\bullet\mathrm{T}} = \\ &= \sum_{h=0}^{n}(\boldsymbol{A}^{\bullet}(i,i+h)_{\Delta}\boldsymbol{L}^{h\mathrm{T}} - \boldsymbol{B}_d^{\bullet}\boldsymbol{K}_d^{\bullet}\boldsymbol{C}_{dh}^{\bullet}\boldsymbol{L}^{h\mathrm{T}})\boldsymbol{P} + \sum_{h=0}^{n}\boldsymbol{P}(\boldsymbol{A}^{\bullet}(i,i+h)_{\Delta}\boldsymbol{L}^{h\mathrm{T}} - \boldsymbol{B}_d^{\bullet}\boldsymbol{K}_d^{\bullet}\boldsymbol{C}_{dh}^{\bullet}\boldsymbol{L}^{h\mathrm{T}})^{\mathrm{T}} \prec 0\,. \end{aligned} \tag{81}$$

Then, using (22), it can proceed that

$$\begin{aligned} \boldsymbol{B}_d^{\bullet}\boldsymbol{K}_d^{\bullet}\boldsymbol{C}_{dh}^{\bullet}\boldsymbol{L}^{h\mathrm{T}} &= \boldsymbol{B}_d^{\bullet}\boldsymbol{K}_d^{\bullet}\boldsymbol{S}^{h\mathrm{T}}\boldsymbol{C}_d^{\bullet}\boldsymbol{L}^h\boldsymbol{L}^{h\mathrm{T}} \\ &= \boldsymbol{B}_d^{\bullet}(\boldsymbol{I}_{n+1} \otimes \boldsymbol{K}^{\circ})(\boldsymbol{L}^{h\mathrm{T}} \otimes \boldsymbol{I}_2)\boldsymbol{C}_d^{\bullet} \\ &= \boldsymbol{B}_d^{\bullet}(\boldsymbol{L}^{h\mathrm{T}} \otimes \boldsymbol{I}_2)(\boldsymbol{I}_{n+1} \otimes \boldsymbol{K}^{\circ})\boldsymbol{C}_d^{\bullet} \\ &= \boldsymbol{B}_d^{\bullet}\boldsymbol{S}^{h\mathrm{T}}\boldsymbol{K}_d^{\bullet}\boldsymbol{C}_d^{\bullet}\,, \end{aligned} \tag{82}$$

and the product $\boldsymbol{K}_d^{\bullet}\boldsymbol{C}_d^{\bullet}$ can be written as follows

$$\begin{aligned} \boldsymbol{K}_d^{\bullet}\boldsymbol{C}_d^{\bullet} &= \begin{bmatrix} \boldsymbol{K}^{\circ} & & \\ & \ddots & \\ & & \boldsymbol{K}^{\circ} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_1^{\circ} & & \\ & \ddots & \\ & & \boldsymbol{c}_{n+1}^{\circ} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{K}^{\circ}\boldsymbol{H} & & \\ & \ddots & \\ & & \boldsymbol{K}^{\circ}\boldsymbol{H} \end{bmatrix} \begin{bmatrix} \boldsymbol{H}^{-1} & & \\ & \ddots & \\ & & \boldsymbol{H}^{-1} \end{bmatrix} \boldsymbol{C}_d^{\bullet} \\ &= \boldsymbol{R}_d\boldsymbol{H}_d^{-1}\boldsymbol{C}_d^{\bullet}\,, \end{aligned} \tag{83}$$

where

$$\boldsymbol{R} = \boldsymbol{K}^{\circ}\boldsymbol{H}\,. \tag{84}$$

Thus, (14) can be reflected as

$$\boldsymbol{A}^{\bullet}(i,i)_{\Delta}\boldsymbol{P} - \boldsymbol{B}_d^{\bullet}\boldsymbol{K}_d^{\bullet}\boldsymbol{H}_d\boldsymbol{H}_d^{-1}\boldsymbol{C}_d^{\bullet}\boldsymbol{P} \prec 0\,, \tag{85}$$

$$\boldsymbol{A}^{\bullet}(i,i+h)_{\Delta}\boldsymbol{L}^{hT}\boldsymbol{P} - \boldsymbol{B}_d^{\bullet}\boldsymbol{S}^{hT}\boldsymbol{K}_d^{\bullet}\boldsymbol{H}_d\boldsymbol{H}_d^{-1}\boldsymbol{C}_d^{\bullet}\boldsymbol{P} \succ 0\,, \tag{86}$$

and prescribing that

$$\boldsymbol{H}_d^{-1}\boldsymbol{C}_d^{\bullet} = \boldsymbol{C}_d^{\bullet}\boldsymbol{P}^{-1}, \tag{87}$$

then (85) and (86) imply (75), (76), and (87) give (78), while the left multiplication (86) by $\boldsymbol{L}^h$ retains the set of diagonal LMIs.

To avoid the need to introduce additional structured variables into the design conditions [15], realization of (10) is required to be relaxed as

$$\begin{aligned}
\boldsymbol{A}_c^{\bullet}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_c^{\bullet\mathrm{T}} &= (\boldsymbol{A}^{\bullet} - \boldsymbol{B}^{\bullet}\boldsymbol{K}^{\circ}\boldsymbol{C}^{\circ})\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}^{\bullet} - \boldsymbol{B}^{\bullet}\boldsymbol{K}^{\circ}\boldsymbol{C}^{\circ})^{\mathrm{T}} \\
&= \boldsymbol{A}^{\bullet}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^{\bullet\mathrm{T}} - \boldsymbol{B}^{\bullet}\boldsymbol{R}\boldsymbol{C}^{\circ} - \boldsymbol{C}^{\circ\mathrm{T}}\boldsymbol{R}\boldsymbol{B}^{\bullet\mathrm{T}} \\
&= \boldsymbol{A}^{\bullet}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^{\bullet\mathrm{T}} - \boldsymbol{B}^{\bullet}\boldsymbol{R}_d\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{C}_d^{\bullet} - \boldsymbol{C}_d^{\bullet}\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{R}_d\boldsymbol{B}^{\bullet} \\
&\prec 0\,,
\end{aligned} \tag{88}$$

where it is considered that (78) implies $\boldsymbol{C}^{\circ}\boldsymbol{P} = \boldsymbol{H}\boldsymbol{C}^{\circ}$. Thus, (88) leads to inequality (77), verifying system stability. This ends the proof. □

Going on in this direction, additional constraints can be introduced to be able to solve the given set of LMIs if $\boldsymbol{A}_1^{\bullet}$ is pure Metzler. However, such a situation can be avoided by another pre-setting of the related parameter λ.

4. Illustrative Numerical Example

The unstable Metzlerian linear system in (1) and (2) is considered with the following matrix parameters

$$\boldsymbol{A} = \begin{bmatrix} -3.380 & 2.208 & 4.715 & 2.676 \\ 1.881 & -4.290 & 2.050 & 0.675 \\ 2.067 & 4.273 & -6.654 & 2.893 \\ 1.148 & 2.273 & 1.343 & -2.104 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0.0189 \\ 0.0203 \\ 0.0315 \\ 0.0170 \end{bmatrix},$$

$$\boldsymbol{c}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{c}_p^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b} = 0.0706\,.$$

Fixing by $k_D = 2$ the tuning parameter $-\lambda = -0.4294$, then $\Delta = (1 \leftrightarrow 5)/5$,

$$\boldsymbol{E}^{\circ -1} = \begin{bmatrix} 1.0081 & 0.0081 & 0.0081 & 0 & 0 \\ 0.0087 & 1.0087 & 0.0087 & 0 & 0 \\ 0.0135 & 0.0135 & 1.0135 & 0 & 0 \\ 0.0073 & 0.0073 & 0.0073 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{A}^{\bullet} = \begin{bmatrix} -3.3754 & 2.2258 & 4.7159 & 2.7266 & 0 \\ 1.8860 & -4.2709 & 2.0510 & 0.7294 & 0 \\ 2.0747 & 4.3026 & -6.6525 & 2.9773 & 0 \\ 1.1521 & 2.2890 & 1.3438 & -2.0584 & 0 \\ 1.0000 & 0.0000 & 0.0000 & 0.0000 & -1 \end{bmatrix},$$

$$\boldsymbol{B}^{\bullet} = \begin{bmatrix} 0.0189 & 0 \\ 0.0203 & 0 \\ 0.0315 & 0 \\ 0.0170 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1^{\bullet\mathrm{T}} \\ \boldsymbol{b}_2^{\bullet\mathrm{T}} \\ \boldsymbol{b}_3^{\bullet\mathrm{T}} \\ \boldsymbol{b}_4^{\bullet\mathrm{T}} \\ \boldsymbol{b}_5^{\bullet\mathrm{T}} \end{bmatrix}, \ \boldsymbol{B}_d^{\bullet} = \mathrm{diag}\begin{bmatrix} \boldsymbol{b}_1^{\bullet\mathrm{T}} & \boldsymbol{b}_2^{\bullet\mathrm{T}} & \boldsymbol{b}_3^{\bullet\mathrm{T}} & \boldsymbol{b}_4^{\bullet\mathrm{T}} & \boldsymbol{b}_5^{\bullet\mathrm{T}} \end{bmatrix}, \ \boldsymbol{C}_d^{\bullet} = \mathrm{diag}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{A}(i,i)_{\Delta} = -\mathrm{diag}\begin{bmatrix} 3.3754 & 4.2709 & 6.6525 & 2.0584 & 1 \end{bmatrix},$$

$$
\begin{aligned}
\boldsymbol{A}(i,i+1)_{\Delta} &= \operatorname{diag}\begin{bmatrix}2.2258 & 2.0510 & 2.9773 & 0 & 1\end{bmatrix},\\
\boldsymbol{A}(i,i+2)_{\Delta} &= \operatorname{diag}\begin{bmatrix}4.7159 & 0.7294 & 0 & 1.1521 & 0\end{bmatrix},\\
\boldsymbol{A}(i,i+3)_{\Delta} &= \operatorname{diag}\begin{bmatrix}2.7266 & 0 & 2.0747 & 2.2890 & 0\end{bmatrix},\\
\boldsymbol{A}(i,i+4)_{\Delta} &= \operatorname{diag}\begin{bmatrix}0 & 1.8860 & 4.3026 & 1.3438 & 0\end{bmatrix}.
\end{aligned}
$$

Defining

$$
\boldsymbol{L} = \begin{bmatrix}\boldsymbol{0}^{\mathrm{T}} & 1\\ \boldsymbol{I}_4 & \boldsymbol{0}\end{bmatrix}, \quad \boldsymbol{S} = \boldsymbol{L}\otimes\boldsymbol{I}_2, \quad \boldsymbol{J}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{I}_2 & \boldsymbol{I}_2 & \boldsymbol{I}_2 & \boldsymbol{I}_2 & \boldsymbol{I}_2\end{bmatrix},
$$

the solution of (74)–(78), obtained using SeDuMi package for Matlab [22], is represented by the set of matrix variables

$$
\boldsymbol{P} = \operatorname{diag}\begin{bmatrix}0.1125 & 0.1125 & 0.1125 & 0.0688 & 0.7296\end{bmatrix},
$$

$$
\boldsymbol{R} = \begin{bmatrix}6.3210 & -2.2280\\ 0 & 0\end{bmatrix}, \quad \boldsymbol{H} = \begin{bmatrix}0.1125 & 0\\ 0 & 0.7296\end{bmatrix},
$$

which implies

$$
\boldsymbol{K}^{\circ} = \begin{bmatrix}k_P & -k_I\\ 0 & 0\end{bmatrix} = \begin{bmatrix}56.1789 & -3.0536\\ 0 & 0\end{bmatrix}, \quad k_P = 56.1789, \quad k_I = 3.0536, \quad k_D = 2\,.
$$

Analyzing the final result, the closed-loop system matrix $\boldsymbol{A}_c^{\bullet}$ takes the structure

$$
\boldsymbol{A}_c^{\bullet} = \begin{bmatrix}
-4.4682 & 1.1329 & 3.6231 & 2.7266 & 0.0594\\
0.7109 & -5.4459 & 0.8759 & 0.7294 & 0.0639\\
0.2537 & 2.4816 & -8.4735 & 2.9773 & 0.0990\\
0.1676 & 1.3044 & 0.3592 & -2.0584 & 0.0535\\
1.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0000
\end{bmatrix}.
$$

It is pure Metzler and Hurwitz with the stable eigenvalue spectrum

$$
\rho(\boldsymbol{A}_{vc}^{\bullet}) = \left\{\ -0.6213 \quad -1.2802 \quad -4.4730 \quad -6.1352 \quad -8.9363\ \right\}.
$$

Although the matrix $\boldsymbol{A}_c^{\bullet}$ is not diagonally dominant, its structure and eigenvalues guarantee that, with these PID controller parameters, the nonnegative closed-loop state performance are achieved. The purpose of the example is primarily to illustrate the desired design procedure.

Purely real negative eigenvalues are conditional on the use of positive systems because they guarantee aperiodic positive trajectories of state variables with a nonnegative initial state of the system. However, they do not guarantee an overshoot during their evolutions, which sometimes needs to be suppressed. Unfortunately, standard methods for tuning PID controller parameters [23,24] for these structures cannot be used. Methods based on the principle of D-stability circle region [25,26] come into consideration but, due to the bilinear structure of the synthesis conditions, it is not possible to guarantee an optimal overshoot suppression using this approach as well. This sub-area of synthesis will therefore be preferred in authors' future research.

5. Discussion

Using the control law (28), the D-part so defined PID control law supports Metzlerian structure, as can be seen by comparing $\boldsymbol{A}^{\bullet}$ and $\boldsymbol{A}$. The analysis shows that, with this PID controller structure, a closed-loop with a Metzler structure of the dynamics matrix can be expected.

Both in general and in Metzler systems, such a synthesis task has many degrees of freedom in defining structures of the $\boldsymbol{c}^{\mathrm{T}}$ and $\boldsymbol{c}_p^{\mathrm{T}}$ matrices. For the given system, the $\boldsymbol{c}^{\mathrm{T}}$ structure considered has proved to be advantageous. Comparable results can be obtained constructing others by permuting its non-zero elements. The structure of the matrix $\boldsymbol{c}_p^{\mathrm{T}}$ is chosen to demonstrate solutions with minimal impact of I-part, where the I-part can be enhanced by using a matrix $\boldsymbol{c}_p^{\mathrm{T}}$ with several non-zero components. Note that the strictly Metzler structure of $\boldsymbol{A}_c^{\bullet}$ is potentially obtained if the vector $\boldsymbol{c}^{\mathrm{T}}$ is positive.

To the best of authors' knowledge, no comparable results are available for design of PID control of SISO Metzlerian linear systems. In the authors' opinion, the proposed method is one which gives through constraint limits in conditions for a class of switched positive systems. Exposing the principle details the approach can be adapt to study PI control of SISO strictly Metzlerian linear systems, where similar results can be expected.

6. Conclusions

This paper completes a design method for synthesis of PID control for SISO Metzlerian continuous-time linear systems. The equivalent Metzler structure is proposed for representing an unstable Metzlerian system, exploiting a fixed D-part gain. The newly formulated exposition of the problem treatments the existing freedom, provided by measurement assignment through output vector structure to find a solvable matrix representation. Maintaining system parametric constraints by the set of LMIs, the design conditions are completed by Lyapunov matrix inequality, guaranteeing closed-loop asymptotic stability within a feasible solution.

Since the analysis is linear, one can see evidently the dependence of the resulting PID gains on Metzler parameters of the system. The proposed approach lends itself to algorithm formalization through LMIs and, even given structural constraints, it can be expected that the proposed design conditions are applicable to a wide variety of Metzlerian systems. The theory yields results that have otherwise not been derived for these systems' PID control. The development of an approach for Metzlerian systems with extended set parametric constraints is a topic of future research.

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Abbreviations

The following abbreviations are used in this manuscript:

LME	Linear Matrix Equality
LMI	Linear Matrix Inequality
PID controller	Proportional-Integral-Derivative controller
SISO system	Single-Input Single-Output system

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