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A Family of Derivative Free Optimal Fourth Order Methods for Computing Multiple Roots

Sunil Kumar [1], Deepak Kumar [2,*], Janak Raj Sharma [1] and Lorentz Jäntschi [3,4,*]

1 Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal, Sangrur 148106, India; sfageria1988@gmail.com (S.K.); jrshira@yahoo.co.in (J.R.S.)
2 Department of Mathematics, Chandigarh University, 140413, Mohali, NH-95 Chandigarh-Ludhiana Highway, Punjab 140413, India
3 Department of Physics and Chemistry, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania
4 Institute of Doctoral Studies, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
* Correspondence: deepak.babbi@gmail.com (D.K.); lorentz.jantschi@gmail.com (L.J.)

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Abstract: Many optimal order multiple root techniques, which use derivatives in the algorithm, have been proposed in literature. But contrarily, derivative free optimal order techniques for multiple root are almost nonexistent. By this as an inspirational factor, here we present a family of optimal fourth order derivative-free techniques for computing multiple roots of nonlinear equations. At the beginning the convergence analysis is executed for particular values of multiplicity afterwards it concludes in general form. Behl et. al derivative-free method is seen as special case of the family. Moreover, the applicability and comparison is demonstrated on different nonlinear problems that certifies the efficient convergent nature of the new methods. Finally, we conclude that our new methods consume the lowest CPU time as compared to the existing ones. This illuminates the theoretical outcomes to a great extent of this study.

Keywords: multiple root; king-traub conjecture; derivative-free method; nonlinear equation

MSC: 65H05; 41A25; 49M15

1. Introduction

Construction of optimal higher-order methods, in the sense of Kung-Traub conjecture [1], free from the derivatives, is always required for the multiple roots of nonlinear function of the form $\chi(x) = 0$ with multiplicity θ, i.e. $\chi^{(j)}(\alpha) = 0, j = 0, 1, 2, \ldots, \theta - 1$ and $\chi^{(\theta)}(\alpha) \neq 0$. The well-known Newton's method [2] is one of the simplest method for obtaining multiple roots of the nonlinear function, which is given by

$$x_{t+1} = x_t - \theta \frac{\chi(x_t)}{\chi'(x_t)}, \quad t = 0, 1, 2, \ldots. \tag{1}$$

Numerous higher order methods, have been developed in literature by Dong [3], Geum et al. [4], Hansen [5], Li et al. [6,7], Neta [8], Osada [9], Sharifi et al. [10], Sharma and Sharma [11], Zhou et al. [12], Victory and Neta [13], Agarwal et al. [14] and Soleymani et al. [15]. Such methods require the evaluations of derivatives. The without derivative methods are important in case where derivative χ' of χ is very small or is costly to evaluate. One such without derivative method is the Traub-Steffensen method [16] which used

$$\chi'(x_t) \simeq \frac{\chi(x_t + b\chi(x_t)) - \chi(x_t)}{b\chi(x_t)}, \quad b \in \mathbb{R} - \{0\},$$

or

$$\chi'(x_t) \simeq \chi[w_t, x_t],$$

for the derivative χ' in Newton method (1). Here $w_t = x_t + b\chi(x_t)$ and $\chi[w, x] = \frac{\chi(w)-\chi(x)}{w-x}$ is divided difference. Then method (1) takes the form of

$$x_{t+1} = x_t - \theta \frac{\chi(x_t)}{\chi[w_t, x_t]}. \tag{2}$$

Very recently, researchers have proposed some higher order derivative free methods. For example; Kumar et al. [17] have developed quadratically convergent method, Sharma et al. [18,19], Kumar et al. [20] and Behl et al. [21] developed fourth methods, and Sharma et al. [22] developed eighth order methods for computing the multiple solutions. The methods of [17–22] require two, three and four function evaluations per step and, therefore, according to Kung-Traub conjecture these possess optimal convergence [1]. Our main objective of this work is to develop derivative-free multiple root methods of high computational efficiency, which may attain a high convergence order using as small number of function evaluations as possible. Consequently, we develop a class of two-step derivative-free methods with fourth order of convergence. The presented scheme requires three function evaluations per step and, hence, it satisfy optimal criteria [1]. The methodology is based on the classical Traub-Steffensen method (2) and further modified by employing Traub-Steffensen-like iteration in the second step.

2. Construction of Method

Consider the following two-step iterative scheme $\theta \geq 2$:

$$\begin{aligned} z_t &= x_t - \theta \frac{\chi(x_t)}{\chi[w_t, x_t]}, \\ x_{t+1} &= z_t - \theta \frac{H(s_t, k_t)}{1 - 2s_t} \frac{\chi(x_t)}{\chi[w_t, x_t]}, \end{aligned} \tag{3}$$

where $s_t = \sqrt[\theta]{\frac{\chi(z_t)}{\chi(x_t)}}$, $k_t = \sqrt[\theta]{\frac{\chi(z_t)}{\chi(w_t)}}$ and $H : \mathbb{C}^2 \to \mathbb{C}$ is analytic in a neighborhood of $(0, 0)$. The second step is weighted by the factor $H(s, k)$, so we can call it weight factor or more appropriately weight function.

In Theorems 1–3, we demonstrate that the presented iterative scheme (3) attains highest fourth-order of convergence, without adding any extra evaluation of function or its derivative.

Theorem 1. *Assume that $\chi : \mathbb{C} \to \mathbb{C}$ is an analytic function in a domain containing a multiple zero (say, α) with multiplicity $\theta = 2$. Suppose that the initial point x_0 is close enough to α, then the convergence order of the formula (3) is at least 4, provided that $H_{00} = 0$, $H_{10} = \frac{1}{2}$, $H_{01} = \frac{1}{2}$, $H_{20} = -H_{02} - 2H_{11}$, $H_{02} \in \mathbb{R}$ and $H_{11} \in \mathbb{R}$, where $H_{ij} = \frac{\partial^{i+j}}{\partial s^i \partial k^j} H(s_t, k_t)|_{(s_t=0, k_t=0)}$, for $0 \leq i, j \leq 2$.*

Proof. Assume that the error at t-th stage is $e_t = x_t - \alpha$. Using the Taylor's expansion of $\chi(x_t)$ about α and keeping into mind that $\chi(\alpha) = 0$, $\chi'(\alpha) = 0$ and $\chi''(\alpha) \neq 0$, we have

$$\chi(x_t) = \frac{\chi''(\alpha)}{2!} e_t^2 \left(1 + B_1 e_t + B_2 e_t^2 + B_3 e_t^3 + B_4 e_t^4 + \cdots\right), \tag{4}$$

where $B_n = \frac{2!}{(2+n)!} \frac{\chi^{(2+n)}(\alpha)}{\chi''(\alpha)}$ for $n \in \mathbb{N}$.

Similarly $\chi(w_t)$ about α, we have

$$\chi(w_t) = \frac{\chi''(\alpha)}{2!} e_{w_t}^2 \left(1 + B_1 e_{w_t} + B_2 e_{w_t}^2 + B_3 e_{w_t}^3 + B_4 e_{w_t}^4 + \cdots\right), \tag{5}$$

where $e_{w_t} = w_t - \alpha = e_t + \frac{b\chi''(\alpha)}{2!} e_t^2 \left(1 + B_1 e_t + B_2 e_t^2 + B_3 e_t^3 + B_4 e_t^4 + \cdots\right)$.

Then the first step of (3) yields

$$\begin{aligned} e_{z_t} &= z_t - \alpha \\ &= \frac{1}{2}\Big(\frac{b\chi''(\alpha)}{2} + B_1\Big)e_t^2 - \frac{1}{16}\big((b\chi''(\alpha))^2 - 8b\chi''(\alpha)B_1 + 12B_1^2 - 16B_2\big)e_t^3 + \frac{1}{64}\big((b\chi''(\alpha))^3 \\ &\quad - 20b\chi''(\alpha)B_1^2 + 72B_1^3 + 64b\chi''(\alpha)B_2 - 10B_1\big((b\chi''(\alpha))^2 + 16B_2\big) + 96B_3\big)e_t^4 + O(e_t^5). \end{aligned} \tag{6}$$

Expanding $\chi(z_t)$ about α, it follows that

$$\chi(z_t) = \frac{\chi''(\alpha)}{2!}e_{z_t}^2\big(1 + B_1 e_{z_t} + B_2 e_{z_t}^2 + B_3 e_{z_t}^3 + B_4 e_{z_t}^4 + \cdots\big). \tag{7}$$

Using (4), (5) and (7) in s_t and k_t, after some simple calculations we have

$$\begin{aligned} s_t &= \frac{1}{2}\Big(\frac{b\chi''(\alpha)}{2} + B_1\Big)e_t - \frac{1}{16}\big((b\chi''(\alpha))^2 - 6b\chi''(\alpha)B_1 + 16(B_1^2 - B_2)\big)e_t^2 + \frac{1}{64}\big((b\chi''(\alpha))^3 \\ &\quad - 22b\chi''(\alpha)B_1^2 + 4\big(29B_1^3 + 14b\chi''(\alpha)B_2\big) - 2B_1\big(3(b\chi''(\alpha))^2 + 104B_2\big) + 96B_3\big)e_t^3 + O(e_t^4) \end{aligned} \tag{8}$$

and

$$\begin{aligned} k_t &= \frac{1}{2}\Big(\frac{b\chi''(\alpha)}{2} + B_1\Big)e_t - \frac{1}{16}\big(3(b\chi''(\alpha))^2 - 2b\chi''(\alpha)B_1 + 16(B_1^2 - B_2)\big)e_t^2 + \frac{1}{64}\big(7(b\chi''(\alpha))^3 \\ &\quad + 24b\chi''(\alpha)B_2 - 14b\chi''(\alpha)B_1^2 + 116B_1^3 - 2B_1\big(11(b\chi''(\alpha))^2 + 104B_2\big) + 96B_3\big)e_t^3 + O(e_t^4). \end{aligned} \tag{9}$$

Taylor expansion of $H(s_t, k_t)$ in the neighborhood of $(0,0)$ is

$$H(s_t, k_t) \approx H_{00} + s_t H_{10} + k_t H_{01} + \frac{1}{2}s_t^2 H_{20} + s_t k_t H_{11} + \frac{1}{2}k_t^2 H_{02}. \tag{10}$$

Using (4)-(10) in the second step of (3), then we have

$$e_{t+1} = -H_{00}e_t - \frac{1}{2}\big(H_{00} + H_{01} + H_{10} - 1\big)\Big(\frac{b\chi''(\alpha)}{2} + B_1\Big)e_t^2 + \sum_{n=1}^{2}\psi_n e_t^{n+2} + O(e_t^5), \tag{11}$$

where $\psi_n = \psi_n(b, B_1, B_2, B_3, H_{00}, H_{10}, H_{01}, H_{20}, H_{11}, H_{02})$.

We will get at least fourth order if we set coefficients of e_t, e_t^2 and e_t^3 simultaneously equal to zero. Then, we have

$$H_{00} = 0, \;\; H_{10} = \frac{1}{2}, \;\; H_{01} = \frac{1}{2}, \;\; H_{20} = -H_{02} - 2H_{11}. \tag{12}$$

Now using equation (12) in (11), we have

$$e_{t+1} = \frac{1}{16}\Big(\frac{b\chi''(\alpha)}{2} + B_1\Big)\Big(b\chi''(\alpha)(2H_{02} + 2H_{11} - 3)B_1 + 3B_1^2 + \big((b\chi''(\alpha))^2(H_{11} + H_{02} - 1) - 4B_2\big)\Big)e_t^4 + O(e_t^5). \tag{13}$$

Thus, the theorem is proved. □

Theorem 2. *Using the hypotheses of Theorem 1, the order of convergence of scheme (3) for the case $\theta = 3$ is at least 4, if $H_{00} = 0$, $H_{10} = 1 - H_{01}$, $H_{20} = -H_{02} - 2H_{11}$, $H_{01} \in \mathbb{R}$, $H_{02} \in \mathbb{R}$ and $H_{11} \in \mathbb{R}$.*

Proof. Keeping into mind that $\chi(\alpha) = 0$, $\chi'(\alpha) = 0$, $\chi''(\alpha) = 0$, and $\chi'''(\alpha) \neq 0$, then we have

$$\chi(x_t) = \frac{\chi'''(\alpha)}{3!}e_t^3\big(1 + \bar{B}_1 e_t + \bar{B}_2 e_t^2 + \bar{B}_3 e_t^3 + \bar{B}_4 e_t^4 + \cdots\big), \tag{14}$$

where $\bar{B}_n = \frac{3!}{(3+n)!}\frac{\chi^{(3+n)}(\alpha)}{\chi'''(\alpha)}$ for $n \in \mathbb{N}$.

Similarly, $\chi(w_t)$ about α

$$\chi(w_t) = \frac{\chi'''(\alpha)}{3!}e_{w_t}^3\big(1 + \bar{B}_1 e_{w_t} + \bar{B}_2 e_{w_t}^2 + \bar{B}_3 e_{w_t}^3 + \bar{B}_4 e_{w_t}^4 + \cdots\big), \tag{15}$$

where $e_{w_t} = w_t - \alpha = e_t + \frac{b\chi'''(\alpha)}{2!} e_t^3 \left(1 + \bar{B}_1 e_t + \bar{B}_2 e_t^2 + \bar{B}_3 e_t^3 + \bar{B}_4 e_t^4 + \cdots\right)$.

Then the first step of (3) yields

$$\begin{aligned} e_{z_t} &= z_t - \alpha \\ &= \frac{\bar{B}_1}{3} e_t^2 + \left(\frac{b\chi'''(\alpha)}{6} - \frac{4}{9}\bar{B}_1^2 + \frac{2}{3}\bar{B}_2\right) e_t^3 + \left(\frac{16}{27}\bar{B}_1^3 + \frac{1}{9}\bar{B}_1(2b\chi'''(\alpha) - 13\bar{B}_2) + \bar{B}_3\right) e_t^4 + O(e_t^5). \end{aligned} \tag{16}$$

Expanding $\chi(z_t)$ about α, it follows that

$$\chi(z_t) = \frac{\chi'''(\alpha)}{3!} e_{z_t}^3 \left(1 + \bar{B}_1 e_{z_t} + \bar{B}_2 e_{z_t}^2 + \bar{B}_3 e_{z_t}^3 + \bar{B}_4 e_{z_t}^4 + \cdots\right). \tag{17}$$

Using (14), (15) and (17) in s_t and k_t, after some simple calculations we have

$$s_t = \frac{\bar{B}_1}{3} e_t + \left(\frac{b\chi'''(\alpha)}{6} - \frac{5}{9}\bar{B}_1^2 + \frac{2}{3}\bar{B}_2\right) e_t^2 + \left(\frac{23}{27}\bar{B}_1^3 + \bar{B}_1\left(\frac{b\chi'''(\alpha)}{6} - \frac{16}{9}\bar{B}_2\right) + \bar{B}_3\right) e_t^3 + O(e_t^4) \tag{18}$$

and

$$k_t = \frac{\bar{B}_1}{3} e_t + \left(\frac{b\chi'''(\alpha)}{6} - \frac{5}{9}\bar{B}_1^2 + \frac{2}{3}\bar{B}_2\right) e_t^2 + \left(\frac{23}{27}\bar{B}_1^3 + \frac{2}{9}\bar{B}_1\left(\frac{b\chi'''(\alpha)}{2} - 8\bar{B}_2\right) + \bar{B}_3\right) e_t^3 + O(e_t^4). \tag{19}$$

Using (10) and (14)–(19) in the second step of (3), then we have

$$e_{k+1} = -H_{00} e_t - \frac{1}{3}(H_{00} + H_{01} + H_{10} - 1) e_t^2 + \sum_{n=1}^{2} \phi_n e_t^{n+2} + O(e_t^5), \tag{20}$$

where $\phi_n = \phi_n(b, \bar{B}_1, \bar{B}_2, \bar{B}_3, H_{00}, H_{10}, H_{01}, H_{20}, H_{11}, H_{02})$.

If we set coefficients of e_t, e_t^2 and e_t^3 simultaneously equal to zero. Then we have

$$H_{00} = 0, \;\; H_{10} = 1 - H_{01}, \;\; H_{20} = -H_{02} - 2H_{11}. \tag{21}$$

Now using equation (21) in (20), we have

$$e_{t+1} = \frac{\bar{B}_1}{27}\left(\frac{3}{2} b\chi'''(\alpha)(H_{01} - 1) + 2\bar{B}_1^2 - 3\bar{B}_2\right) e_t^4 + O(e_t^5). \tag{22}$$

Thus, the theorem is proved. □

Remark 1. From above results we observe that the number of conditions on H_{ij} is 4, 3 corresponding to $\theta = 2, 3$ to obtain the fourth convergence order of the method (3). Their error equations also contain the term involving the parameter b. However, for the cases $\theta \geq 4$, it has been seen that the error equation in each such case does not contain b term. We shall prove this fact in the next section.

3. Main Result

We shall prove the order of convergence of scheme (3) for $\theta \geq 4$ by the following theorem:

Theorem 3. *Using the hypotheses of Theorem 1, the order of convergence of scheme (3) for the cases $\theta \geq 4$ is at least 4, if $H_{00} = 0$, $H_{10} = 1 - H_{01}$, $H_{20} = -H_{02} - 2H_{11}$, $H_{01} \in \mathbb{R}$, $H_{02} \in \mathbb{R}$ and $H_{11} \in \mathbb{R}$. Moreover, error in the scheme is given by*

$$e_{t+1} = \frac{1}{2\theta^3}\left((1+\theta)\bar{\bar{B}}_1^3 - 2\theta\bar{\bar{B}}_1\bar{\bar{B}}_2\right) e_t^4 + O(e_t^5).$$

Proof. Keeping into mind that $\chi^{(j)}(\alpha) = 0, j = 0, \ldots, \theta - 1$ and $\chi^{(\theta)}(\alpha) \neq 0$, then developing $\chi(x_t)$ about α in the Taylor's series

$$\chi(x_t) = \frac{\chi^{\theta}(\alpha)}{\theta!} e_t^{\theta}\left(1 + \bar{\bar{B}}_1 e_t + \bar{\bar{B}}_2 e_t^2 + \bar{\bar{B}}_3 e_t^3 + \bar{\bar{B}}_4 e_t^4 + \cdots\right), \tag{23}$$

where $\bar{\bar{B}}_n = \frac{\theta!}{(\theta+n)!}\frac{\chi^{(\theta+n)}(\alpha)}{\chi^{(\theta)}(\alpha)}$ for $n \in \mathbb{N}$.

Also from the expansion of $\chi(w_t)$ about α, it follows that

$$\chi(w_t) = \frac{\chi^{\theta}(\alpha)}{\theta!} e_{w_t}^{\theta}\left(1 + \bar{\bar{B}}_1 e_{w_t} + \bar{\bar{B}}_2 e_{w_t}^2 + \bar{\bar{B}}_3 e_{w_t}^3 + \bar{\bar{B}}_4 e_{w_t}^4 + \cdots\right), \tag{24}$$

where $e_{w_t} = w_t - \alpha = e_t + \frac{\beta f^{\theta}(\alpha)}{\theta!} e_t^{\theta}\left(1 + \bar{\bar{B}}_1 e_t + \bar{\bar{B}}_2 e_t^2 + \bar{\bar{B}}_3 e_t^3 + \bar{\bar{B}}_4 e_t^4 + \cdots\right)$.

From the first step of (3)

$$e_{z_t} = \begin{cases} \frac{\bar{\bar{B}}_1}{4} e_t^2 + \frac{1}{16}\left(4\bar{\bar{B}}_2 - 3\bar{\bar{B}}_1^2\right)e_t^3 + \left(\frac{25}{64}\bar{\bar{B}}_1^3 - \bar{\bar{B}}_1\bar{\bar{B}}_2 + \frac{1}{16}\left(b\chi^{(4)}(\alpha) + 12\bar{\bar{B}}_3\right)\right)e_t^4 + O(e_t^5), & \text{if } \theta = 4. \\ \frac{\bar{\bar{B}}_1}{\theta} e_t^2 + \frac{1}{\theta^2}\left(2\theta\bar{\bar{B}}_2 - (1+\theta)\bar{\bar{B}}_1^2\right)e_t^3 + \frac{1}{\theta^3}\left((1+\theta)^2\bar{\bar{B}}_1^3 - \theta(4+3\theta)\bar{\bar{B}}_1\bar{\bar{B}}_2 + 3\theta^2\bar{\bar{B}}_3\right)e_t^4 + O(e_t^5), \\ \text{if } \theta \geq 5. \end{cases} \tag{25}$$

Expansion of $\chi(z_t)$ around α yields

$$\chi(z_t) = \frac{\chi^{(\theta)}(\alpha)}{\theta!} e_{z_t}^{\theta}\left(1 + \bar{\bar{B}}_1 e_{z_t} + \bar{\bar{B}}_2 e_{z_t}^2 + \bar{\bar{B}}_3 e_{z_t}^3 + \bar{\bar{B}}_4 e_{z_t}^4 + \cdots\right). \tag{26}$$

Using (23), (24) and (26) in the expressions of s_t and k_t, we have that

$$s_k = \begin{cases} \frac{\bar{\bar{B}}_1}{4} e_t + \frac{1}{8}\left(4\bar{\bar{B}}_2 - 3\bar{\bar{B}}_1^2\right)e_t^2 + \frac{1}{128}\left(67\bar{\bar{B}}_1^3 - 152\bar{\bar{B}}_1\bar{\bar{B}}_2 + 8\left(b\chi^{(4)}(\alpha) + 12\bar{\bar{B}}_3\right)\right)e_t^3 + O(e_t^4), & \text{if } \theta = 4. \\ \frac{\bar{\bar{B}}_1}{\theta} e_t + \frac{1}{\theta^2}\left(2\theta\bar{\bar{B}}_2 - (2+\theta)\bar{\bar{B}}_1^2\right)e_t^2 + \frac{1}{2\theta^3}\left((2\theta^2 + 7\theta + 7)\bar{\bar{B}}_1^3 - 2\theta(7+3\theta)\bar{\bar{B}}_1\bar{\bar{B}}_2 + 6\theta^2\bar{\bar{B}}_3\right)e_t^3 \\ +O(e_t^4), \text{ if } \theta \geq 5 \end{cases} \tag{27}$$

and

$$k_t = \begin{cases} \frac{\bar{\bar{B}}_1}{4} e_t + \frac{1}{8}\left(4\bar{\bar{B}}_2 - 3\bar{\bar{B}}_1^2\right)e_t^2 + \frac{1}{128}\left(67\bar{\bar{B}}_1^3 - 152\bar{\bar{B}}_1\bar{\bar{B}}_2 + 8\left(b\chi^{(4)}(\alpha) + 12\bar{\bar{B}}_3\right)\right)e_t^3 + O(e_t^4), & \text{if } \theta = 4. \\ \frac{\bar{\bar{B}}_1}{\theta} e_t + \frac{1}{\theta^2}\left(2\theta\bar{\bar{B}}_2 - (2+\theta)\bar{\bar{B}}_1^2\right)e_t^2 + \frac{1}{2\theta^3}\left((2\theta^2 + 7\theta + 7)\bar{\bar{B}}_1^3 - 2\theta(7+3\theta)\bar{\bar{B}}_1\bar{\bar{B}}_2 + 6\theta^2\bar{\bar{B}}_3\right)e_t^3 \\ +O(e_t^4), \text{ if } \theta \geq 5. \end{cases} \tag{28}$$

Inserting (10) and (23)–(28) in the second step of (3), it follows that

$$e_{t+1} = -H_{00}e_t + \frac{1}{\theta}\left((H_{00} + H_{01} + H_{10} - 1)\bar{\bar{B}}_1\right)e_t^2 + \sum_{n=1}^{2}\varphi_n e_t^{n+2} + O(e_t^5). \tag{29}$$

where $\varphi_n = \varphi_n(b, \bar{\bar{B}}_1, \bar{\bar{B}}_2, \bar{\bar{B}}_3, H_{00}, H_{10}, H_{01}, H_{20}, H_{11}, H_{02})$, for $\theta = 4$ and $\varphi_n = \varphi_n(\bar{\bar{B}}_1, \bar{\bar{B}}_2, \bar{\bar{B}}_3, H_{00}, H_{10}, H_{01}, H_{20}, H_{11}, H_{02})$, for $\theta \geq 5$.

If the coefficients of e_t, e_t^2 and e_t^3 vanish then we have

$$H_{00} = 0, \quad H_{10} = 1 - H_{01}, \quad H_{20} = -H_{02} - 2H_{11}. \tag{30}$$

Then, error equation (29) is given by

$$e_{t+1} = \frac{1}{2\theta^3}\left((1+\theta)\bar{\bar{B}}_1^3 - 2\theta\bar{\bar{B}}_1\bar{\bar{B}}_2\right)e_t^4 + O(e_t^5). \tag{31}$$

Thus, the theorem is proved. □

Remark 2. This fourth order convergence rate is achieved by using only $\chi(x_t)$, $\chi(w_t)$ and $\chi(z_t)$ per iteration. Therefore, the scheme (3) is optimal by the Kung-Traub conjecture [1].

Remark 3. Note that parameter b, which is utilized in w_t, shows up just in the error equations of the cases $\theta = 2, 3$ yet not for $\theta \geq 4$. We have seen that this parameter appears in the coefficients of e_t^5 and higher order. However, we do not need such terms in order to show the required fourth order convergence.

Some Special Cases

Based on the forms of function $H(s,k)$ that satisfy the conditions of Theorems 1–3. Then, we get a new optimal family of order fourth as follows:

$$\begin{aligned} z_t &= x_t - \theta \frac{\chi(x_t)}{\chi[w_t, x_t]}, \\ x_{t+1} &= z_t - \theta \Big(\frac{s_t(1 - H_{01}) + k_t H_{01} - \frac{1}{2} s_t^2 (H_{02} + 2H_{11}) + s_t k_t H_{11} + \frac{1}{2} k_t^2 H_{02}}{1 - 2s_t} \Big) \frac{\chi(x_t)}{\chi[w_t, x_t]}. \end{aligned} \tag{32}$$

(1) For $H_{01} = \frac{1}{2}$, $H_{02} = 0$ and $H_{11} = 0$ in expression (32), we have

$$x_{t+1} = z_t - \theta \frac{s_t + k_t}{2(1 - 2s_t)} \frac{\chi(x_t)}{\chi[w_t, x_t]}. \tag{33}$$

It is important to note that the above method (33) is Behl et al. method [21]. This shows that Behl et al. method [21] is the special case of our family (32).

(2) If $H_{01} = \frac{1}{2}$, $H_{02} = 0$ and $H_{11} = 1$ in expression (32), we have

$$x_{t+1} = z_t - \theta \frac{s_t - 2s_t^2 + k_t + 2s_t k_t}{2(1 - 2s_t)} \frac{\chi(x_t)}{\chi[w_t, x_t]}. \tag{34}$$

(3) If $H_{01} = \frac{1}{2}$, $H_{02} = -1$ and $H_{11} = 0$ in expression (32), we get

$$x_{t+1} = z_t - \theta \frac{s_t + s_t^2 + k_t - k_t^2}{2(1 - 2s_t)} \frac{\chi(x_t)}{\chi[w_t, x_t]}. \tag{35}$$

(4) Let $H_{01} = \frac{1}{2}$, $H_{02} = 1$ and $H_{11} = -1$ in expression (32), we obtain

$$x_{t+1} = z_t - \theta \frac{s_t + s_t^2 + k_t - 2s_t k_t + k_t^2}{2(1 - 2s_t)} \frac{\chi(x_t)}{\chi[w_t, x_t]}. \tag{36}$$

(5) Let $H_{01} = \frac{\theta - 1}{2}$, $H_{02} = 0$ and $H_{11} = 0$ in expression (32), we have

$$x_{t+1} = z_t - \theta \frac{(3 - \theta)s_t + (\theta - 1)k_t}{2(1 - 2s_t)} \frac{\chi(x_t)}{\chi[w_t, x_t]}. \tag{37}$$

In above each case $z_t = x_t - \theta \frac{\chi(x_t)}{\chi[w_t, x_t]}$. For future reference the proposed methods (33), (34),(35), (36) and (37) are denoted by BM, NM1, NM2, NM3 and NM4, respectively.

4. Numerical Results

In order to validate of theoretical results that have been proven in previous sections, the new methods BM, NM1, NM2, NM3 and NM4 are checked numerically by imposing them on some nonlinear equations. Moreover, they are also compared with some existing derivative free optimal fourth order methods. We consider, for example, the methods by Sharma et al. [18,19] and Kumar et al. [20]. The methods are expressed as follows:

Method by Sharma et al. [18] (SK1):

$$\begin{aligned} z_t &= x_t - \theta \frac{\chi(x_t)}{\chi[w_t, x_t]}, \\ x_{t+1} &= z_t - \left(s_t + (\theta - 1)k_t + \theta s_t^2 + \theta s_t k_t \right) \frac{\chi(x_t)}{\chi[w_t, x_t]}. \end{aligned}$$

Method by Sharma et al. [19] (SK2):

$$z_t = x_t - \theta \frac{\chi(x_t)}{\chi[w_t, x_t]},$$
$$x_{t+1} = z_t - \left(\frac{\theta}{2}h + 3\theta\frac{h^2}{2}\right)\left(1 + \frac{1}{d_t}\right)\frac{\chi(x_t)}{\chi[w_t, x_t]},$$

where

$$h = \frac{s_t}{1+s_t}, \quad \text{and} \quad d_t = \sqrt[\theta]{\frac{\chi(w_t)}{\chi(x_t)}}.$$

Method by Kumar et al. [20] (KM):

$$z_t = x_t - \theta \frac{\chi(x_t)}{\chi[w_t, x_t]},$$
$$x_{t+1} = z_t - \frac{(\theta+2)s_t}{1-2s_t}\frac{\chi(x_t)}{\chi[w_t, x_t] + \chi[w_t, z_t]}.$$

Computational work is compiled in the programming software, e.g. Mathematica [23]. Performance of the new methods is tested by selecting value of the parameter $b = 0.01$. The tabulated results obtained by the methods for each problem include (i) number of iterations (t) required to obtain the solution using the stopping criterion $|x_{t+1} - x_t| + |\chi(x_t)| < 10^{-100}$, (ii) estimated error $|x_{t+1} - x_t|$ in the first three iterations, (iii) calculated convergence order (CCO) and (iv) elapsed time (CPU time in seconds), which is measured by the command "TimeUsed[ ]" (Table 1. The calculated convergence order (CCO) to confirm the theoretical convergence order is calculated by the formula (see [24])

$$\text{CCO} = \frac{\log|(x_{t+2}-\alpha)/(x_{t+1}-\alpha)|}{\log|(x_{t+1}-\alpha)/(x_t-\alpha)|}, \quad \text{for each } t = 1, 2, \ldots \tag{38}$$

Table 1. Following problems are considered in this paper.

Problems	**Root**	**Multiplicity**	**Initial Guess**
Isothermal continuous stirred tank reactor problem [25]			
$\chi_1(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875$	-2.85	2	-2.7
Van der Waals problem [26]			
$\chi_2(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675$	1.75	2	2
Planck law radiation problem [27]			
$\chi_3(x) = \left(e^{-x} - 1 + \frac{x}{5}\right)^3$	4.9651142317...	3	5.5
Manning problem for isentropic supersonic flow [28]			
$\chi_4(x) = \left[\tan^{-1}\left(\frac{\sqrt{5}}{2}\right) - \tan^{-1}(\sqrt{x^2-1}) + \sqrt{6}\left(\tan^{-1}\left(\sqrt{\frac{x^2-1}{6}}\right) - \tan^{-1}\left(\frac{1}{2}\sqrt{\frac{5}{6}}\right)\right) - \frac{11}{63}\right]^4$	1.8411294068...	4	1.2
Standard test problem [20]			
$\chi_5(x) = x(x^2+1)(2e^{x^2+1} + x^2 - 1)\cosh^3\left(\frac{\pi x}{2}\right)$	i	5	1.2i
Clustering problem [29]			
$\chi_6(x) = (x-2)^{15}(x-4)^5(x-3)^{10}(x-1)^{20}$	1	20	0.7

From the computed results in Table 2, we can observe the good convergence behavior of the proposed methods like that of existing methods. This also explains stable nature of the methods. It is also clear that the approximations to the solutions by the proposed methods have greater or equal accuracy than those computed by existing methods. We display the value 0 of $|x_{t+1} - x_t|$ at the stage when stopping criterion $|x_{t+1} - x_t| + |\chi(x_t)| < 10^{-100}$ has been satisfied. From the calculation of computational order of convergence shown in each table, we verify the fourth order of convergence. The efficient nature of presented methods can be verified by the fact that the amount of CPU time consumed by the methods is less than that of the time taken by existing methods. This conclusion is also confirmed by similar numerical experiments on many other different problems.

Table 2. Numerical results for problems.

Methods	t	$\lvert x_2 - x_1 \rvert$	$\lvert x_3 - x_2 \rvert$	$\lvert x_4 - x_3 \rvert$	**CCO**	**CPU-Time**
Problem $-\chi_1(x)$						
SK1	4	5.02×10^{-3}	4.91×10^{-12}	4.84×10^{-48}	4.000	0.0812
SK2	4	5.02×10^{-3}	5.00×10^{-12}	5.34×10^{-48}	4.000	0.0853
KM	4	5.02×10^{-3}	4.83×10^{-12}	4.83×10^{-48}	4.000	0.0798
BM	4	5.02×10^{-3}	4.84×10^{-12}	4.48×10^{-48}	4.000	0.0788
NM1	4	5.02×10^{-3}	4.85×10^{-12}	4.54×10^{-48}	4.000	0.0778
NM2	4	5.02×10^{-3}	4.82×10^{-12}	4.41×10^{-48}	4.000	0.0779
NM3	4	5.02×10^{-3}	4.84×10^{-12}	4.48×10^{-48}	4.000	0.0784
NM4	4	5.02×10^{-3}	4.84×10^{-12}	4.48×10^{-48}	4.000	0.0783
Problem $-\chi_2(x)$						
SK1	6	3.03×10^{-2}	1.26×10^{-3}	5.30×10^{-8}	4.000	0.0724
SK2	6	3.40×10^{-2}	2.14×10^{-3}	6.88×10^{-7}	4.000	0.0942
KM	5	2.25×10^{-2}	2.69×10^{-4}	2.37×10^{-11}	4.000	0.0704
BM	5	2.34×10^{-2}	3.43×10^{-4}	9.30×10^{-11}	4.000	0.0692
NM1	5	2.34×10^{-2}	3.44×10^{-4}	9.34×10^{-11}	4.000	0.0654
NM2	5	2.34×10^{-2}	3.43×10^{-4}	9.26×10^{-11}	4.000	0.0472
NM3	5	2.34×10^{-2}	3.43×10^{-4}	9.30×10^{-11}	4.000	0.0494
NM4	5	2.34×10^{-2}	3.43×10^{-4}	9.30×10^{-11}	4.000	0.0502
Problem $-\chi_3(x)$						
SK1	3	5.56×10^{-6}	1.32×10^{-25}	0	4.000	0.4962
SK2	3	6.34×10^{-6}	2.70×10^{-25}	0	4.000	0.4726
KM	3	4.93×10^{-6}	6.76×10^{-26}	0	4.000	0.4137
BM	3	4.91×10^{-6}	6.62×10^{-26}	0	4.000	0.4232
NM1	3	4.91×10^{-6}	6.62×10^{-26}	0	4.000	0.4062
NM2	3	4.91×10^{-6}	6.61×10^{-26}	0	4.000	0.4204
NM3	3	4.91×10^{-6}	6.62×10^{-26}	0	4.000	0.4247
NM4	3	4.94×10^{-6}	6.86×10^{-26}	0	4.000	0.4251
Problem $-\chi_4(x)$						
SK1	5	2.88×10^{-1}	2.21×10^{-2}	3.24×10^{-9}	4.000	3.3120
SK2	5	2.73×10^{-1}	1.97×10^{-2}	2.84×10^{-9}	4.000	3.2642
KM	5	3.11×10^{-1}	2.60×10^{-2}	4.32×10^{-9}	4.000	3.3230
BM	5	3.11×10^{-1}	2.60×10^{-2}	4.31×10^{-9}	4.000	3.2114
NM1	5	3.11×10^{-1}	2.60×10^{-2}	4.31×10^{-9}	4.000	3.1423
NM2	5	3.11×10^{-1}	2.60×10^{-2}	4.31×10^{-9}	4.000	3.1876
NM3	5	3.11×10^{-1}	2.60×10^{-2}	4.31×10^{-9}	4.000	3.2591
NM4	5	3.11×10^{-1}	2.60×10^{-2}	4.32×10^{-9}	4.000	2.9642
Problem $-\chi_5(x)$						
SK1	4	7.14×10^{-5}	5.13×10^{-18}	1.36×10^{-70}	4.000	0.5691
SK2	4	7.93×10^{-5}	1.16×10^{-17}	5.21×10^{-69}	4.000	0.5724
KM	4	6.43×10^{-5}	2.07×10^{-18}	2.22×10^{-72}	4.000	0.5772
BM	4	6.66×10^{-5}	2.38×10^{-18}	3.91×10^{-72}	4.000	0.5547
NM1	4	6.65×10^{-5}	2.37×10^{-18}	3.84×10^{-72}	4.000	0.5462
NM2	4	6.67×10^{-5}	2.39×10^{-18}	3.98×10^{-72}	4.000	0.5531
NM3	4	6.66×10^{-5}	2.38×10^{-18}	3.91×10^{-72}	4.000	0.5684
NM4	4	6.12×10^{-5}	1.70×10^{-18}	1.00×10^{-72}	4.000	0.5642

Problem $-\chi_6(x)$						
SK1	4	9.74×10^{-3}	5.21×10^{-8}	4.57×10^{-29}	4.000	0.1377
SK2	5	1.39×10^{-2}	4.13×10^{-7}	3.65×10^{-25}	4.000	0.1421
KM	4	3.41×10^{-3}	1.50×10^{-10}	5.63×10^{-40}	4.000	0.1324
BM	4	3.42×10^{-3}	1.51×10^{-10}	5.86×10^{-40}	4.000	0.1257
NM1	4	3.41×10^{-3}	1.51×10^{-10}	5.83×10^{-40}	4.000	0.1246
NM2	4	3.42×10^{-3}	1.51×10^{-10}	5.89×10^{-40}	4.000	0.1098
NM3	4	3.42×10^{-3}	1.51×10^{-10}	5.86×10^{-40}	4.000	0.1249
NM4	4	3.35×10^{-3}	1.40×10^{-10}	4.34×10^{-40}	4.000	0.0914

5. Conclusions

In the foregoing study, we have proposed a family of fourth order derivative-free numerical methods for solving nonlinear equations with multiple roots of known multiplicity. Analysis of the convergence has been carried out, which proves the order four under standard assumptions of the function whose zeros we are looking for. In addition, our proposed scheme also satisfies the Kung-Traub hypothesis of optimal order of convergence. Some special cases have been discussed. These are employed to solve nonlinear equations including those arising in practical problems. The new methods are compared with existing techniques of same order. We conclude the work with a remark that derivative-free methods are good alternatives to Newton-type schemes in the cases when derivatives are expensive to compute or difficult to obtain.

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