Article

Stability of Maximum Functional Equation and Some Properties of Groups

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Abstract: In this research paper, we deal with the problem of determining the function $\chi : G \rightarrow \mathbb{R}$, which is the solution to the maximum functional equation (MFE) $\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y)$, when the domain is a discretely normed abelian group or any arbitrary group G . We also analyse the stability of the maximum functional equation $\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y)$ and its solutions for the function $\chi : G \rightarrow \mathbb{R}$, where G be any group and also investigate the connection of the stability with commutators and free abelian group K that can be embedded into a group G .

Keywords: maximum functional equations; discretely normed abelian group; stability of functional equation

MSC: 20D60; 54E35; 11T71

1. Literature Review

In [1], Volkmann proved that every function χ defined on an abelian group G can be described in the form $\chi(x) = |\alpha(x)|$, which is the solution of the maximum FE

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y), \quad (1)$$

where $\alpha : G \rightarrow \mathbb{R}$ is an additive function. Consequently, in [2], Toborg showed that Equation (1) also characterises the absolute value of additive functions when G be any group. Their's main theorem is stated, as follows:

Theorem 1 (Toborg [2]). *Let G be any group (G is an abelian group (Volkmann [1])), then a function $\chi : G \rightarrow \mathbb{R}$ fulfills (1) if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$, such that $\chi(x) = |\alpha(x)|$ for any $x \in G$.*

We recommend the readers to see [3,4], and the references cited therein in order to obtain comprehensive results related to functional Equation (1), which characterises the additive function's absolute value.

According to Simon and Volkmann [5], solutions of the MFE

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y) \quad (2)$$

with the additional assumption that G is an additive abelian group was exhibited in the theorem, which is stated as:

Theorem 2 ([5], Theorem 2). Assume that G is an additive abelian group, whose each element is divisible by 6 (divisible by 2 and 3), and then a function $\chi : G \rightarrow \mathbb{R}$ fulfills

$$\max\{\chi(x+y), \chi(x-y)\} = \chi(x)\chi(y)$$

if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$ such that $\chi(x) = 0$ or $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G$.

Jarczyk and Volkman [6] demonstrated the stability results of MFE (1) on additive abelian groups. Consequently, Badora et al. [7] also generalized their results in order to prove the stability for a certain class of groupoids. In [3], Gilanyi et al. determined the stability results of maximum functional equation $\max\{\chi((x \circ y) \circ y), \chi(x)\} = \chi(x \circ y) + \chi(y)$ on a square-symmetric groupoid. Consequently, they also examined the stability of maximum equation $\max\{\chi(x+y), \chi(x-y)\} = \chi(x) + \chi(y)$ on additive abelian groups. Various appropriate and useful results regarding stability can be found in papers by Przebieracz [8,9].

This paper is arranged, as follows: in Section 2, we prove the functional Equation (1) for any arbitrary group G without any characterisation of an additive function's absolute value. Besides, we also demonstrate some consistent and useful results concerning the normal subgroup of G .

In Section 3, we analyse the functional Equation (2) in order to obtain its solution. For this purpose, we drop the additional assumption that G is an additive abelian group and is divisible by 6. We present the generalization of Theorem 2 by proposing a discretely normed abelian group G . Moreover, we investigate the functional Equation (2) for any arbitrary group G , which satisfies the Kannappan condition [10].

Section 4 deals with the stability results of the MFE (1) for a function $\chi : G \rightarrow \mathbb{R}$, where G be any group and find some useful connection of stability with commutators and embedding of a group.

2. Solutions of the Functional Equation (1)

Throughout this article, let G be any group, and 1 is considered to be the identity element of a group G .

Definition 1 ([10]). Let G be an arbitrary group, we say a function $\chi : G \rightarrow \mathbb{R}$ satisfies the Kannappan condition if

$$\chi(uvx) = \chi(uxv) \quad \text{for all } u, v, x \in G.$$

Theorem 3. Let G be a group and function $\chi : G \rightarrow \mathbb{R}$ satisfies the Kannappan condition then

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y) \quad \text{for all } x, y \in G \quad (3)$$

if and only if

$$\min\{\chi(xy), \chi(xy^{-1})\} = |\chi(x) - \chi(y)| \quad \text{for all } x, y \in G \quad (4)$$

and also satisfies

$$\chi(x^2) = 2\chi(x) \quad \text{for any } x \in G. \quad (5)$$

Proof. Assume that χ satisfies Equation (3), and then by setting $x = y = 1$ in (3), we can conclude that $\chi(1) = 0$. It is also clear that $\chi(x) \geq 0$ and $\chi(x^2) = 2\chi(x)$ for any $x \in G$. The proof of Equation (4) consists of the following simple computation:

$$\begin{aligned}
\max\{\chi(xy), \chi(xy^{-1})\} + \min\{\chi(xy), \chi(xy^{-1})\} &= \chi(xy) + \chi(xy^{-1}) \\
\chi(x) + \chi(y) + \min\{\chi(xy), \chi(xy^{-1})\} &= \chi(xy) + \chi(xy^{-1}) \\
\chi(x) + \chi(y) + \min\{\chi(xy), \chi(xy^{-1})\} &= \max\{\chi(xyxy^{-1}), \chi(xy y x^{-1})\} \\
\chi(x) + \chi(y) + \min\{\chi(xy), \chi(xy^{-1})\} &= \max\{\chi(x^2), \chi(y^2)\} \\
\min\{\chi(xy), \chi(xy^{-1})\} &= \max\{2\chi(x), 2\chi(y)\} - \chi(x) - \chi(y) \\
\min\{\chi(xy), \chi(xy^{-1})\} &= |\chi(x) - \chi(y)|.
\end{aligned}$$

Conversely, assume that Equations (4) and (5) hold. Subsequently, we can obtain

$$\max\{\chi(xy), \chi(xy^{-1})\} - \min\{\chi(xy), \chi(xy^{-1})\} = |\chi(xy) - \chi(xy^{-1})|.$$

Using Equation (4) in the following computation, we get that

$$\begin{aligned}
\max\{\chi(xy), \chi(xy^{-1})\} - |\chi(x) - \chi(y)| &= |\chi(xy) - \chi(xy^{-1})| \\
&= \min\{\chi(xyxy^{-1}), \chi(xy y x^{-1})\} \\
&= \min\{\chi(x^2), \chi(y^2)\} \\
\max\{\chi(xy), \chi(xy^{-1})\} &= 2 \min\{\chi(x), \chi(y)\} + |\chi(x) - \chi(y)| \\
\max\{\chi(xy), \chi(xy^{-1})\} &= \chi(x) + \chi(y).
\end{aligned}$$

□

Corollary 1. Let G be a group and function $\chi : G \rightarrow \mathbb{R}$ satisfies the Kannappan condition, and then χ satisfies the Equation (3) if and only if

$$\chi(x) + \chi(y) = \chi(xy) + \chi(xy^{-1}) - |\chi(x) - \chi(y)| \quad \text{for any } x, y \in G \quad (6)$$

and also $\chi(1) = 0$.

Proof. Suppose that $\chi(1) = 0$ and Equation (6) holds. Subsequently, (6) can be written as

$$2 \max\{\chi(x), \chi(y)\} = \chi(xy) + \chi(xy^{-1}) \quad \text{for any } x, y \in G. \quad (7)$$

Because $\chi(1) = 0$, then, for $x = y$, we can obtain that $\chi(x^2) = 2\chi(x)$. Afterwards, writing xy instead of x and xy^{-1} instead of y in (7), we can conclude that

$$\begin{aligned}
2 \max\{\chi(xy), \chi(xy^{-1})\} &= \chi(xyxy^{-1}) + \chi(xy^2x^{-1}) \\
&= \frac{1}{2}\{\chi(x^2) + \chi(y^2)\} \\
&= \frac{1}{2}\{2\chi(x) + 2\chi(y)\} \\
\max\{\chi(xy), \chi(xy^{-1})\} &= \chi(x) + \chi(y).
\end{aligned}$$

Conversely, assume that maximum Equation (3) holds, then, by Theorem 3, we can determine that $\chi(1) = 0$ and Equation (6) holds. □

Corollary 2. Let G be a group and function $\chi : G \rightarrow \mathbb{R}$ satisfies the Kannappan condition, and then χ is a solution of equation

$$\min\{\chi(xy), \chi(xy^{-1})\} = |\chi(x) - \chi(y)| \quad \text{for any } x, y \in G$$

and also satisfies

$$\chi(x^2) = 2\chi(x) \quad \text{for any } x \in G$$

if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$ such that $\chi(x) = |\alpha(x)|$ for any $x \in G$.

Theorem 4. Let G be a group and function $\chi : G \rightarrow \mathbb{R}$ satisfies the Kannappan condition. If the Equations (4) and (5) are satisfied, then there exists a normal subgroup $N_\chi = \{v \in G \mid \chi(v) = 0\}$ of G .

Proof. Because the function $\chi : G \rightarrow \mathbb{R}$ satisfies the Equations (4) and (5), then by Theorem 3

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y) \quad \text{for any } x, y \in G.$$

When we investigate Equations (3) and (4), then we can see that either $\chi(xy) = |\chi(x) - \chi(y)|$ then $\chi(xy^{-1}) = \chi(x) + \chi(y)$ or $\chi(xy^{-1}) = |\chi(x) - \chi(y)|$ then $\chi(xy) = \chi(x) + \chi(y)$. First, writing v instead of y and assume that $\chi(x) \neq 0$ for some $x \in G$ and also $\chi(xv) = \chi(xv^{-1})$. Subsequently, we can compute that $\chi(x) + \chi(v) = |\chi(x) - \chi(v)|$. From Equation (3), we can easily deduce that $\chi(v), \chi(x) \geq 0$. Our assumption that $\chi(x) \neq 0$ forces $\chi(v)$ to be 0.

From Equation (5), we can compute that $\chi(1) = 0$, therefore $1 \in N_\chi$. Let $u, v \in N_\chi$. We can also deduce from Equation (3) that χ is even, so we have $\chi(v^{-1}) = \chi(v) = 0$, therefore $v^{-1} \in N_\chi$. Additionally, $0 = \chi(u) + \chi(v) = \max\{\chi(uv), \chi(uv^{-1})\} \geq \chi(uv)$, but $\chi(x) \geq 0$ for all $x \in G$; therefore, we can obtain that $\chi(uv) = 0$, so $uv \in N_\chi$. Hence N_χ is a subgroup of G . Let $v \in N_\chi$ and $x \in G$, then Equation (3) yields that

$$\begin{aligned} \chi(x) + \chi(x^{-1}vx) &= \max\{\chi(xx^{-1}vx), \chi(xx^{-1}v^{-1}x)\} \\ &= \max\{\chi(vx), \chi(v^{-1}x)\} \\ &= \max\{\chi(x^{-1}v^{-1}), \chi(x^{-1}v)\} \\ &= \chi(x^{-1}) + \chi(v) \\ \chi(x) + \chi(x^{-1}vx) &= \chi(x) + \chi(v) \\ \chi(x^{-1}vx) &= \chi(v), \end{aligned}$$

which implies that $\chi(x^{-1}vx) = \chi(v)$ for all $x \in G$ and $v \in N_\chi$. Hence, N_χ is a normal subgroup of G . $\square$

Corollary 3. For any group G , let a function χ on group G satisfying Equation (3), then $\chi : G/N_\chi \rightarrow \mathbb{R}$ also satisfies

$$\chi(xv) = \chi(x) \quad \text{for any } x \in G, v \in N_\chi. \quad (8)$$

Additionally, G/N_χ is an abelian quotient group.

Proof. Let $x \in G$ be an arbitrary and assume that $v \in N_\chi$. We can deduce from (3)

$$\begin{aligned} \chi(xv) &\leq \max\{\chi(xv), \chi(xv^{-1})\} = \chi(x) + \chi(v) = \chi(x) \\ &\leq \max\{\chi(xv^2), \chi(x)\} = \chi(xv) + \chi(v) = \chi(xv) \end{aligned}$$

that $\chi(xv) = \chi(x) + \chi(v)$. Hence, $\chi(xv) = \chi(x)$ for any $x \in G$ and $v \in N_\chi$. Moreover, we already proved that $\chi(x^{-1}vx) = \chi(v)$ for any $x \in G$ and $v \in N_\chi$, then, by replacing v with xv , we can evaluate that $\chi(vx) = \chi(vx)$ for any $x \in G$ and $v \in N_\chi$. Additionally, N_χ is a normal subgroup of G ; therefore, G/N_χ is an abelian quotient group. $\square$

Corollary 4. Let G be any group and function $\chi : G \rightarrow \mathbb{R}$ satisfy the Kannappan condition. If the maximum functional Equation (3) holds, then $\min\{\chi(xy), \chi(xy^{-1})\} \in N_\chi$ for every $x, y \in G$.

Proof. Assume that χ satisfies the Equation (3) and Kannappan condition. Subsequently, from Theorem 3, we can compute that

$$\min\{\chi(xy), \chi(xy^{-1})\} = \chi(x) - \chi(y), \quad (\text{when } \chi(x) \geq \chi(y))$$

or

$$\min\{\chi(xy), \chi(xy^{-1})\} = \chi(y) - \chi(x), \quad (\text{when } \chi(x) < \chi(y))$$

combining both cases, we can see that $2 \min\{\chi(xy), \chi(xy^{-1})\} = 0$, therefore $\min\{\chi(xy), \chi(xy^{-1})\} = 0$. In either case, it follows that $\min\{\chi(xy), \chi(xy^{-1})\} \in N_\chi$ for every $x, y \in G$. $\square$

3. Solutions of the Functional Equation (2)

Theorem 5. Let G be any arbitrary group, then a function $\chi : G \rightarrow \mathbb{R}$ satisfies

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y) \quad \text{for any } x, y \in G \quad (9)$$

if and only if simultaneously $\chi(x) = 0$ or

$$\min\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y)^{-1} \vee [\chi(x)\chi(y)^{-1}]^{-1} \quad \text{for any } x, y \in G \quad (10)$$

holds and satisfies

$$\chi(x^2) = \chi(x)^2 \quad \text{for all } x \in G. \quad (11)$$

Proof. Suppose that χ satisfies Equation (9). Assume 1 as neutral element of group G , then by putting $x = y = 1$ in (9), we can obtain that $\chi(1)\chi(1) = \chi(1)$, then $\chi(1) = 0$ or $\chi(1) = 1$. Let $\chi(1) = 0$, then by Equation (9) we can compute $\chi(x) = 0$ for any $x \in G$. Assume that $\chi(1) = 1$, then from (9), It is easy to see that $\chi(x^{-1}) = \chi(x) \geq 0$, $\chi(x^2) \geq 1$ and $\chi(x^2) = \chi(x)^2$ for any $x \in G$. The proof of Equation (10) consists of the following simple computation:

$$\begin{aligned} \max\{\chi(xy), \chi(xy^{-1})\} \cdot \min\{\chi(xy), \chi(xy^{-1})\} &= \chi(xy)\chi(xy^{-1}) \\ \chi(x)\chi(y) \cdot \min\{\chi(xy), \chi(xy^{-1})\} &= \chi(xy)\chi(xy^{-1}) \\ \chi(x)\chi(y) \cdot \min\{\chi(xy), \chi(xy^{-1})\} &= \max\{\chi(xyxy^{-1}), \chi(xy y x^{-1})\} \\ \chi(x)\chi(y) \cdot \min\{\chi(xy), \chi(xy^{-1})\} &= \max\{\chi(x^2), \chi(y^2)\} \\ \min\{\chi(xy), \chi(xy^{-1})\} &= \max\{\chi(x)^2, \chi(y)^2\} \chi(x)^{-1} \chi(y)^{-1} \\ \min\{\chi(xy), \chi(xy^{-1})\} &= \chi(x)\chi(y)^{-1} \quad (\text{when } \chi(x) \geq \chi(y)) \\ &\vee [\chi(x)\chi(y)^{-1}]^{-1} \quad (\text{when } \chi(x) < \chi(y)). \end{aligned}$$

Conversely, suppose that the Equations (10) and (11) are satisfied and $\chi(x) \neq 0$. Subsequently, it can be determined that

$$\begin{aligned} \max\{\chi(xy), \chi(xy^{-1})\} \cdot \min\{\chi(xy), \chi(xy^{-1})\}^{-1} &= \chi(xy)\chi(xy^{-1})^{-1} \\ &\vee \chi(xy)^{-1}\chi(xy^{-1}). \end{aligned}$$

Here, we have two cases, in the first case, using Equation (10) and (11), we derive the required result, as follows:

$$\begin{aligned}
\max\{\chi(xy), \chi(xy^{-1})\} &= \chi(xy)\chi(xy^{-1})^{-1} \cdot \min\{\chi(xy), \chi(xy^{-1})\} \\
&= \chi(xy)\chi(xy^{-1})^{-1}\chi(x)\chi(y)^{-1} && (\text{when } \chi(x) \geq \chi(y)) \\
&= \min\{\chi(xyx y^{-1}), \chi(xyy x^{-1})\}\chi(x)\chi(y)^{-1} \\
&= \min\{\chi(x^2), \chi(y^2)\}\chi(x)\chi(y)^{-1} \\
&= \min\{\chi(x)^2, \chi(y)^2\}\chi(x)\chi(y)^{-1} \\
&= \chi(y)^2\chi(x)\chi(y)^{-1} && (\text{when } \chi(x) \geq \chi(y)) \\
\max\{\chi(xy), \chi(xy^{-1})\} &= \chi(x)\chi(y).
\end{aligned}$$

From second case, when $\chi(x) < \chi(y)$, then we can also get that $\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y)$. $\square$

Corollary 5. Let G be a group and function $\chi : G \rightarrow \mathbb{R}$ satisfies the maximum Equation (9) if and only if

$$\chi(xy)\chi(xy^{-1}) = \chi(x)\chi(y) \min\{\chi(xy), \chi(xy^{-1})\} \quad \text{for any } x, y \in G \quad (12)$$

and also $\chi(1) = 1$.

Corollary 6. Let G be a group and function $\chi : G \rightarrow \mathbb{R}$ is a solution of Equation (10) satisfies (11) if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$, such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G$.

Definition 2 ([11]). Let G be an abelian group. Subsequently, a function $\chi : G \rightarrow \mathbb{R}$ is said to be a discrete norm on G if there exists some $\alpha > 0$ such that $\chi(x) > \alpha$ whenever x is not identity element of G . Afterwards, (G, χ) is called the discretely normed abelian group [11].

Simon and Volkmann [5] proved Theorem 2 with additional assumption that G is an additive abelian group and is divisible by 6, but we present the generalization of Theorem 2 by introducing the notion of discretely normed abelian group G , as follows:

Theorem 6. Assume that (G, χ) be a discretely normed abelian group, then a function $\chi : G \rightarrow \mathbb{R}$ fulfills (9) if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$, such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G \setminus \{1\}$.

Proof. Because G is a discretely normed abelian group, therefore there exists a discrete norm function $\chi : G \rightarrow \mathbb{R}$ such that $\chi(x) > 0$ whenever $x \in G \setminus \{1\}$. Assume that $\eta(x) = \log \chi(x)$, then applying the main theorem of Toborg [2], a function $\chi : G \rightarrow \mathbb{R}$ satisfies

$$\max\{\eta(xy), \eta(xy^{-1})\} = \eta(x) + \eta(y)$$

if and only if there exist an additive function $\alpha : G \rightarrow \mathbb{R}$ such that $\eta(x) = |\alpha(x)|$. Subsequently, we have

$$\max\{\log \chi(xy), \log \chi(xy^{-1})\} = \log \chi(x) + \log \chi(y)$$

if and only if $\log \chi(x) = |\alpha(x)|$, which implies that

$$\log \max\{\chi(xy), \chi(xy^{-1})\} = \log \chi(x)\chi(y),$$

thus, we conclude that

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y)$$

if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$, such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G \setminus \{1\}$. $\square$

Corollary 7. Let (G, χ) be a discretely normed abelian group, then a function χ is a solution of Equation (10) satisfying (11) if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$, such that $\chi(x) = e^{|\alpha(x)|}$ for any non-identity element $x \in G$.

Proof. From Theorem 6, we concluded that maximum functional Equation (9) holds; therefore, using Theorem 5, we can also obtain required proof. $\square$

We have well-known theorem presented by Steprāns Juris in [11] about a group G , which is a discretely normed abelian group. Therefore, we have following corollaries.

Corollary 8. For free abelian group G , a function χ is a solution of Equation (9) if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$, such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G \setminus \{1\}$.

Corollary 9. Suppose that G is a free abelian group, then χ is a solution of functional Equation (10) satisfying (11) if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$, such that $\chi(x) = e^{|\alpha(x)|}$ for any $x \in G \setminus \{1\}$.

Theorem 7. Let G be any group and let a function $\chi : G \rightarrow \mathbb{R}$ is a solution of Equation (10) and (11), which is not identically zero, then there exists a normal subgroup $H_\chi = \{v \in G \mid \chi(v^2) = 1\}$ of G .

Proof. Because the function $\chi : G \rightarrow \mathbb{R}$ satisfying the Equations (10) and (11), then by Theorem 5, we can obtain that

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y) \quad \text{for all } x, y \in G. \quad (13)$$

For neutral element 1, we can write $\chi(1^2) = \chi(1) = 1$, then $1 \in H_\chi$. Let $v \in H_\chi$, then $\chi(v^2) = 1$. From Equation (13), follows immediately that χ is even; therefore, $\chi(v^{-2}) = \chi(v^2) = 1$, hence $v^{-1} \in H_\chi$. Let $u, v \in H_\chi$, therefore, by property of H_χ , $\chi(u^2) = 1$ and $\chi(v^2) = 1$. We can deduce from maximum Equation (13) that

$$\begin{aligned} \chi(u^2v^2) &= \chi(u^2v^2)\chi(v^2) \\ &= \max\{\chi(u^2v^4), \chi(u^2)\} \\ &\geq \chi(u^2) \\ \chi(u^2v^2) &\geq \chi(u^2)\chi(v^2). \end{aligned} \quad (14)$$

$$\begin{aligned} \chi(u^2)\chi(v^2) &= \max\{\chi(u^2v^2), \chi(u^2v^{-2})\} \\ &\geq \chi(u^2v^2) \\ \chi(u^2)\chi(v^2) &\geq \chi(u^2v^2). \end{aligned} \quad (15)$$

From inequalities (14) and (15), we can calculate $\chi(u^2v^2) = \chi(u^2)\chi(v^2) = 1$, which implies that $uv \in H_\chi$. Therefore, H_χ is a subgroup of G . Additionally, Equation (13) yields that χ is central, therefore $\chi(xv) = \chi(vx)$, for every $x \in G$ and $v \in H_\chi$. Hence, H_χ is a normal subgroup of G . $\square$

Corollary 10. For any group G , let a function χ on group G satisfying (13), which is not identically zero, then $\chi : G/H_\chi \rightarrow \mathbb{R}$ also satisfies

$$\chi(xv) = \chi(x)\chi(v) \quad \text{for any } x \in G, v \in H_\chi.$$

Moreover,

$$\chi(xv^2) = \chi(v^2x) = \chi(x) \quad \text{for any } x \in G, v \in H_\chi. \quad (16)$$

Proof. Let $x \in G$ be an arbitrary and assume that $v \in H_\chi$, then $\chi(v^2) = 1$. We can deduce from (13) that

$$\chi(xv)\chi(v) = \max\{\chi(xv^2), \chi(x)\} \geq \chi(x)\chi(v^2),$$

which provides that $\chi(xv)\chi(v) \geq \chi(x)\chi(v^2)$. Applying condition (11), we compute $\chi(xv) \geq \chi(x)\chi(v)$. Additionally, from (13), we have $\chi(x)\chi(v) \geq \chi(xv)$, so we can evaluate $\chi(xv) = \chi(x)\chi(v)$ for any $x \in G$ and $v \in H_\chi$.

Because H_χ is a subgroup of G , then $v^2 \in H_\chi$, so we can see that $\chi(xv^2) = \chi(x)\chi(v^2)$ for any $x \in G$ and $v^2 \in H_\chi$, therefore $\chi(xv^2) = \chi(x)$. Writing x instead of y and v instead of x in (13), we can conclude $\chi(x^{-1}vx) = \chi(v)$ for any $x \in G$, then writing xv^2 instead of v , we have $\chi(v^2x) = \chi(xv^2) = \chi(x)$. $\square$

Corollary 11. Let G be a group and function $\chi : G \rightarrow \mathbb{R}$ satisfies the Kannappan condition. If the maximum functional Equation (13) holds, then $\min\{\chi(xy), \chi(xy^{-1})\} \in H_\chi$ for any $x, y \in G$.

Proof. Assume that χ satisfies the Equation (9) and Kannappan condition. Subsequently, from Theorem 5, we can conclude that

$$\min\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y)^{-1}, \quad (\text{when } \chi(x) \geq \chi(y))$$

or

$$\min\{\chi(xy), \chi(xy^{-1})\} = [\chi(x)\chi(y)^{-1}]^{-1}, \quad (\text{when } \chi(x) < \chi(y))$$

combining both cases, we can see that $\min\{\chi(xy), \chi(xy^{-1})\} = [\min\{\chi(xy), \chi(xy^{-1})\}]^{-1}$, therefore $\min\{\chi(xy), \chi(xy^{-1})\}^2 = 1$. In either case, it follows that $\min\{\chi(xy), \chi(xy^{-1})\} \in H_\chi$ for every $x, y \in G$. $\square$

4. Stability of Maximum Functional Equation (1)

In order to check the stability of MFE (1) in two-variables x and y , at the first stage, we put $y = x$ in (1) and derive the stability of MFE (1) in a single variable, as follows:

Theorem 8. Let G be any group and a function $\eta : G \rightarrow \mathbb{R}$ satisfies

$$|\max\{\eta(x^2), \eta(1)\} - 2\eta(x)| \leq \lambda \quad \text{for any } x \in G, \quad (17)$$

for some $\lambda \geq 0$. Subsequently, we can obtain a solution $\chi : G \rightarrow \mathbb{R}$ of

$$\max\{\chi(x^2), \chi(1)\} = 2\chi(x) \quad \text{for any } x \in G \quad (18)$$

such that

$$-3\lambda \leq \chi(x) - \eta(x) \leq \lambda \quad \text{for any } x \in G. \quad (19)$$

Additionally, χ is given by

$$\chi(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \eta(x^{2^n}) \quad x \in G. \quad (20)$$

Additionally, by (18), χ is uniquely determined, and by (19), the requirement of $\chi - \eta$ to be bounded is also satisfied.

Proof. First, put $x = 1$ in (17), then we have $|\max\{\eta(1), \eta(1)\} - 2\eta(1)| \leq \lambda$, which implies that $|\eta(1)| \leq \lambda$. Furthermore, (17) also implies that

$$\begin{aligned} -\lambda + 2\eta(x) &\leq \max\{\eta(x^2), \eta(1)\} \leq \lambda + 2\eta(x) \\ -\lambda &\leq \eta(1) \leq \lambda + 2\eta(x) \\ -\lambda &\leq \eta(x) \quad \text{for any } x \in G. \end{aligned} \quad (21)$$

Writing x^2 instead of x , then also $-\lambda \leq \eta(x^2)$, so we can get that

$$\begin{aligned} \eta(1) &\leq \lambda = 2\lambda - \lambda \leq 2\lambda + \eta(x^2) \\ \eta(1) &\leq 2\lambda + \eta(x^2) \quad \text{for any } x \in G. \end{aligned} \quad (22)$$

From (17), we have

$$\begin{aligned} 2\eta(x) &\leq \lambda + \max\{\eta(x^2), \eta(1)\} \\ 2\eta(x) &\leq \lambda + \eta(1) \leq 3\lambda + \eta(x^2) \quad (\text{using (22)}) \\ \text{or } 2\eta(x) &\leq \lambda + \eta(x^2) \leq 3\lambda + \eta(x^2). \end{aligned}$$

Combining both cases, we can conclude

$$-3\lambda \leq \eta(x^2) - 2\eta(x). \quad (23)$$

From (17), we have $\max\{\eta(x^2), \eta(1)\} \leq \lambda + 2\eta(x)$, which is only possible when

$$\eta(x^2) - 2\eta(x) \leq \lambda. \quad (24)$$

From inequalities (23) and (24), we have

$$-3\lambda \leq \eta(x^2) - 2\eta(x) \leq \lambda \quad \text{for all } x \in G. \quad (25)$$

When we observe (25), it can be seen that the function $\chi : G \rightarrow \mathbb{R}$ presented in (20) exists and this χ satisfies

$$\chi(x^2) = 2\chi(x) \quad \text{for all } x \in G, \quad (26)$$

also χ fulfills (19). Furthermore, writing x^{2^n} instead of x in (21), dividing by 2^n and taking a limit $n \rightarrow \infty$ and, using (20), we can obtain that $\chi(x) \geq 0$ for any $x \in G$. Hence we can obtain (18) from (26). Furthermore, when we consider (18), then we can easily see the uniqueness of χ , due to the result that $\chi(x^2) = 2\chi(x) \geq 0$ for any $x \in G$.

By utilizing Theorem (8), we are going to derive the stability of Equation (1) in two-variables x and y . $\square$

Theorem 9. Let G be any group and function $\eta : G \rightarrow \mathbb{R}$ satisfies

$$|\max\{\eta(xy), \eta(xy^{-1})\} - \eta(x) - \eta(y)| \leq \lambda \quad \text{for any } x, y \in G, \quad (27)$$

for some $\lambda \geq 0$. Subsequently, we can evaluate a unique solution $\chi : G \rightarrow \mathbb{R}$ of

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y) \quad \text{for any } x, y \in G \quad (28)$$

such that

$$-3\lambda \leq \chi(x) - \eta(x) \leq \lambda \quad \text{for any } x \in G. \quad (29)$$

Additionally,

$$\chi(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \eta(x^{2^n}) \quad \text{for any } x \in G. \quad (30)$$

Proof. Applying the Theorem (8), we can easily prove the required results. First, writing x instead of y in (27), then, from Theorem (8), we can obtain a function $\chi : G \rightarrow \mathbb{R}$. We also need to prove that this function χ satisfies (28). To obtain the required maximum functional Equation (28), we need to write x^{2^n} instead of x and y^{2^n} instead of y in (27), dividing inequality by 2^n and, applying limit $n \rightarrow \infty$ and also using Equation (30). $\square$

Theorem 10. Let G be a group and function $\eta : G \rightarrow \mathbb{R}$ fulfills (27), then, for any $x, y \in G$, it holds the condition

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} [\max\{\eta(x^{2^n} y^{2^n}), \eta(x^{2^n} y^{-2^n})\} - \max\{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\}] = 0 \quad (31)$$

if and only if the Equation (28) is satisfied.

Proof. Assume that maximum Equation (28) is satisfied. For every $x, y \in G$, we have

$$\begin{aligned} & |\max\{\eta(x^{2^n} y^{2^n}), \eta(x^{2^n} y^{-2^n})\} - \max\{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\}| \\ & \leq |\max\{\eta(x^{2^n} y^{2^n}), \eta(x^{2^n} y^{-2^n})\} - \eta(x^{2^n}) - \eta(y^{2^n})| \\ & + |\max\{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\} - \eta(x^{2^n}) - \eta(y^{2^n})| \\ & \leq \lambda + |\max\{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\} - \eta(x^{2^n}) - \eta(y^{2^n})|. \end{aligned}$$

To obtain the required result, first, we divide both sides by 2^n , apply limit as $n \rightarrow \infty$ and utilize the given condition (28), we obtain that, for any $x, y \in G$, we can get $\lim_{n \rightarrow \infty} \frac{1}{2^n} [\max\{\eta(x^{2^n} y^{2^n}), \eta(x^{2^n} y^{-2^n})\} - \max\{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\}] = 0$.

Conversely, assume that condition (31) holds. Putting x^{2^n} and y^{2^n} in (27) instead of x and y , respectively, then dividing by 2^n and taking limit $n \rightarrow \infty$, we can compute

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2^n} [\max\{\eta(x^{2^n} y^{2^n}), \eta(x^{2^n} y^{-2^n})\}] &= \chi(x) + \chi(y), \\ \lim_{n \rightarrow \infty} \frac{1}{2^n} \max\{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\} &= \max\{\chi(xy), \chi(xy^{-1})\}. \end{aligned}$$

Given condition (31), yielding that $\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y)$.

Furthermore, the given condition (31) for function η is not directly associated to the properties of a group G , but we can see some useful results regarding group G . The resulting condition presented below is equivalent to the above condition (31), as there exists a subsequence of $\mathbb{N}$, such that

$$\lim_{n \rightarrow \infty} \frac{1}{2^{m(n)}} [\max\{\eta(x^{2^{m(n)}} y^{2^{m(n)}}), \eta(x^{2^{m(n)}} y^{-2^{m(n)}})\} - \max\{\eta([xy]^{2^{m(n)}}), \eta([xy^{-1}]^{2^{m(n)}})\}] = 0.$$

This refers to the function η and completely satisfactory, due to the fact that both limits

$$\lim_{n \rightarrow \infty} \frac{1}{2^{m(n)}} \max\{\eta(x^{2^{m(n)}} y^{2^{m(n)}}), \eta(x^{2^{m(n)}} y^{-2^{m(n)}})\}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{2^{m(n)}} \max\{\eta([xy]^{2^{m(n)}}), \eta([xy^{-1}]^{2^{m(n)}})\}$$

are finite and exists. $\square$

Corollary 12. A function $\eta : G \rightarrow \mathbb{R}$ fulfills (18), then it satisfies the condition

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} [\max\{\eta(x^{2^n} y^{2^n}), \eta(x^{2^n} y^{-2^n})\} - \eta([xy]^{2^n})] = 0 \quad \text{for all } x, y \in G \quad (32)$$

if and only if $\chi(xy) = \chi(x) + \chi(y)$ for any $x, y \in G$.

Remark 1. (i) The given condition (31) holds when group G is an n —Abelian group (for any integer n , a group G is called an n —Abelian group if $(ab)^n = a^n b^n$, for any $a, b \in G$, see [12,13]).

(ii) The condition (31) also satisfies when group G belongs to the class C_n for any natural number $n \in \mathbb{N}$ (for all $n \in \mathbb{N}$, C_n is denoted as the class of groups, which satisfies the relation $b^n a^n = a^n b^n$).

(iii) When χ is central, then condition (31) also holds.

Theorem 11. If maximum functional Equation (1) is stable on group G , then every free abelian group K can be embedded into a group G .

Proof. Assume that $\eta : G \rightarrow \mathbb{R}$ and let $\lambda \geq 0$ be such that

$$|\max\{\eta(xy), \eta(xy^{-1})\} - \eta(x) - \eta(y)| \leq \lambda \quad \text{for any } x, y \in G. \quad (33)$$

Because every free abelian group is a torsion-free, K is torsion-free group. Subsequently, from HNN-extensions, any torsion-free group K can be embedded into a group G , if for all $k \in K$, there exists $g \in G$ such that $gkg^{-1} = k^2$ [14,15]. Assume that K is embedded into a group G . Subsequently, inequality (33) can be rewritten as

$$|\max\{\eta(kg), \eta(kg^{-1})\} - \eta(k) - \eta(g)| \leq \lambda \quad \text{for any } k, g \in G. \quad (34)$$

We need to show that η is bounded. The result is obvious for $\lambda = 0$, so let λ be positive. More specifically, we prove that $|\eta(k)| < 2\lambda$ for all $k \in G$. Afterwards, from (34), we have two possibilities, either $|\eta(kg^{-1}) - \eta(k) - \eta(g)| \leq \lambda$ or $|\eta(kg) - \eta(k) - \eta(g)| \leq \lambda$. Let us consider first possibility and put $k = g = 1$, then we can conclude that $|\eta(1)| \leq \lambda$. Put $g = k$, then we have

$$\begin{aligned} |\eta(1) - 2\eta(k)| &\leq \lambda \\ |2\eta(k) - \eta(1)| &\leq \lambda \\ 2|\eta(k)| &\leq \lambda + |\eta(1)| \\ 2|\eta(k)| &\leq \lambda + \lambda \\ |\eta(k)| &\leq \lambda. \end{aligned}$$

For second possibility, we have

$$|\eta(kg) - \eta(k) - \eta(g)| \leq \lambda. \quad (35)$$

On the contrary, let $|\eta(k)| \geq 2\lambda$ for some arbitrary $k \in G$. Put $g = k$ in (35), then we have

$$\begin{aligned} |\eta(k^2) - 2\eta(k)| &\leq \lambda \\ |2\eta(k) - \eta(k^2)| &\leq \lambda \\ 2|\eta(k)| - \lambda &\leq |\eta(k^2)| \\ 3\lambda &\leq |\eta(k^2)|. \end{aligned}$$

Again, put $g = k^2$ in (35), then

$$\begin{aligned} |\eta(k^3) - \eta(k) - \eta(k^2)| &\leq \lambda \\ |\eta(k) + \eta(k^2) - \eta(k^3)| &\leq \lambda \\ |\eta(k)| + |\eta(k^2)| &\leq \lambda + |\eta(k^3)| \\ 5\lambda - \lambda &\leq |\eta(k^3)| \\ 4\lambda &\leq |\eta(k^3)|. \end{aligned}$$

Repeating again, for $g = k^3$, we can determine $|\eta(k^4)| \geq 4\lambda$. Continuing this process, we can evaluate that

$$(m+1)\lambda \leq |\eta(k^m)|, \quad \text{for } m=1, 2, \dots,$$

which determines that when m varies, then $\eta(k^m)$ is unbounded.

Furthermore, pick an arbitrary element $g \in G$, such that $k^2 = gkg^{-1}$. Subsequently, for every integer $m > 0$, we have $k^{2m} = gk^m g^{-1}$. Therefore, for every m , put $g = k^m$ and $k = k^m$ in (35), then

$$\begin{aligned} |\eta(k^{2m}) - 2\eta(k^m)| &\leq \lambda \\ |\eta(gk^m g^{-1}) - 2\eta(k^m)| &\leq \lambda. \end{aligned} \quad (36)$$

Additionally, inequality (35) follows that

$$|\eta(gk^m g^{-1}) - \eta(g) - \eta(k^m g^{-1})| \leq \lambda \quad \text{and} \quad |\eta(k^m g^{-1}) - \eta(k^m) - \eta(g^{-1})| \leq \lambda,$$

thus, we can conclude that

$$\begin{aligned} |\eta(gk^m g^{-1}) - \eta(g) - \eta(k^m) - \eta(g^{-1})| &\leq |\eta(gk^m g^{-1}) - \eta(g) - \eta(k^m g^{-1})| \\ &\quad + |\eta(k^m g^{-1}) - \eta(k^m) - \eta(g^{-1})| \\ |\eta(gk^m g^{-1}) - \eta(g) - \eta(k^m) - \eta(g^{-1})| &\leq 2\lambda. \end{aligned} \quad (37)$$

From (36) and (37), we have

$$\begin{aligned} |\eta(gk^m g^{-1}) - 2\eta(k^m) + \eta(k^m)| &\leq 2\lambda + |\eta(g)| + |\eta(g^{-1})| \\ |\eta(k^m)| &\leq 2\lambda + |\eta(g)| + |\eta(g^{-1})| + |\eta(gk^m g^{-1}) - 2\eta(k^m)| \\ |\eta(k^m)| &\leq 5\lambda \quad \text{for } m=1, 2, \dots, \end{aligned}$$

which gives a contradiction. Hence, η is bounded, which completes the required proof. $\square$

Corollary 13. *If maximum functional Equation (1) is stable on group G , then any discretely normed abelian group K can be embedded into a group G .*

Proof. Because K is a discretely normed abelian group, then, by applying theorem from [11], K is a free group; hence it can be embedded into a group G . $\square$

Theorem 12. A function $\eta : G \rightarrow \mathbb{R}$ satisfying (27) is bounded on the commutator group G_1 of the subgroup G' of G if the condition (31) is satisfied.

Proof. Assume that condition (31) is satisfied. Subsequently, by Theorem 10, we can get that $\chi(x) + \chi(y) = \max\{\chi(xy), \chi(xy^{-1})\}$ holds for any $x, y \in G$. Because this maximum equation holds; therefore, by Theorem 1, there exists an additive function $\alpha : G \rightarrow \mathbb{R}$, such that $\chi(x) = |\alpha(x)|$ for any $x \in G$. For $a, b \in G$, take $a^{-1}b^{-1}ab \in G$, then

$$\begin{aligned}\chi(a^{-1}b^{-1}ab) &= |\alpha(a^{-1}b^{-1}ab)| = |\alpha(a^{-1}) + \alpha(b^{-1}) + \alpha(a) + \alpha(b)| \\ &= |\alpha(a^{-1}a) + \alpha(b^{-1}b)| = |\alpha(1) + \alpha(1)| = |0 + 0| \\ \chi(a^{-1}b^{-1}ab) &= 0.\end{aligned}$$

Because χ is zero on the commutator group G_1 of the subgroup G' of G , η is bounded on commutator group G_1 . $\square$

Corollary 14. If a function $\eta : G \rightarrow \mathbb{R}$ satisfies the condition

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} [\max\{\eta(x^{2^n}y^{2^n}), \eta(x^{2^n}y^{-2^n})\} - \eta([xy]^{2^n})] = 0 \quad \text{for any } x, y \in G,$$

then η is bounded on the commutator group G_1 .

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