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On the Differential Equation Governing Torqued Vector Fields on a Riemannian Manifold

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Abstract: In this article, we show that the presence of a torqued vector field on a Riemannian manifold can be used to obtain rigidity results for Riemannian manifolds of constant curvature. More precisely, we show that there is no torqued vector field on n-sphere $\mathbf{S}^n(c)$. A nontrivial example of torqued vector field is constructed on an open subset of the Euclidean space $\mathbf{E}^n$ whose torqued function and torqued form are nowhere zero. It is shown that owing to topology of the Euclidean space $\mathbf{E}^n$, this type of torqued vector fields could not be extended globally to $\mathbf{E}^n$. Finally, we find a necessary and sufficient condition for a torqued vector field on a compact Riemannian manifold to be a concircular vector field.

Keywords: torqued vector fields; torqued form; torqued function ; euclidean space; sphere

MSC: 53C20; 53C21; 53C24

1. Introduction

Differential equations on complete and connected Riemannian manifolds were used by Obata (cf. [1,2]), who observed that a necessary and sufficient condition for an m-dimensional complete and connected Riemannian manifold (M, g) to be isometric to m-sphere $\mathbf{S}^m(c)$ of constant curvature c is that it admits a nontrivial solution of the differential equation

$$H_f = -cfg,$$

where H_f is the Hessian of the smooth function f. It is well known that mathematical problems involving the Hessian of unknown functions are tied to real models appearing in real applications. We refer to [3,4] for analyses tied with real mechanical models involving Hessian matrices in the related formulations. Similarly in [5], it has been observed that a necessary and sufficient condition for an m-dimensional complete and connected Riemannian manifold (M, g) to be isometric to the Euclidean space $\mathbf{E}^m$ is that it admits a nontrivial solution of the differential equation

$$H_f = cg,$$

for a nonzero constant c. Indeed differential equations have been used in various ways to predict geometry as well as topology of a Riemannian manifold. In particular, differential equations satisfied by certain vector fields such as *conformal vector fields*, *Killing vector fields*, *Jacobi-type vector fields*, *torse-forming vector fields*, *geodesic vector fields* are useful in studying geometry of a Riemannian manifolds (cf. [6–17]). There are special types of conformal vector fields known as concircular vector fields, which are used both in geometry as well as in general theory of relativity (cf. [13,18–20]). Professor Chen obtained an elegant characterization of generalized Robertson-Walker space-times, using concircular vector fields (cf. [18]). Recall that a smooth vector field ξ on a Riemannian manifold (M, g) is called a *concircular vector field* if it satisfies the differential equation

$$\nabla_X \xi = \sigma X, \ X \in \mathfrak{X}(M), \tag{1}$$

where ∇ denotes the Riemannian connection of (M, g), $\sigma : M \to \mathbf{R}$ is a smooth function, and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. The function σ in Equation (1) is called the potential function of ξ (cf. [6,21]).

In [17], Yano introduced the notion of *torse-forming vector fields*, which is a generalization of concircular vector fields. A smooth vector field ξ on a Riemannian manifold (M, g) is called *torse-forming vector field* if it satisfies the differential equation

$$\nabla_X \xi = \sigma X + \alpha(X)\xi, \ X \in \mathfrak{X}(M), \tag{2}$$

where α is a smooth 1-form on M. Torse-forming vector fields are important specially in physics (cf. [13,22]). Chen in [23], initiated a special type of torse-forming vector fields called *torqued vector fields*. A nowhere zero vector field $\mathbf{u}$ on a Riemannian manifold (M, g) is said to a torqued vector field if it satisfies

$$\nabla_X \mathbf{u} = \sigma X + \alpha(X)\mathbf{u}, \ X \in \mathfrak{X}(M) \text{ and } \alpha(\mathbf{u}) = 0. \tag{3}$$

The smooth 1-form α and the smooth function σ in the definition of torqued vector field are called torqued form and torqued function of the torqued vector field $\mathbf{u}$ (cf. [23,24]). Note that if the torqued form $\alpha = 0$, then a torqued vector field is a concircular vector field. In [23], it has been observed that if the torqued vector field $\mathbf{u}$ is a gradient of a smooth function, then it is a concircular vector field. It has been observed that the twisted product $I \times_f M$ of an interval I and an $(n-1)$-dimensional Riemannian manifold M admits a torqued vector field which is not a concircular vector field (cf. [24]).

Most basic among special vector fields are geodesic vector fields. In [9,25,26] it has been shown that geodesic vector fields are useful in characterizing spheres and Euclidean spaces.

One of the interesting questions in geometry of torqued vector fields is to find conditions under which a torqued vector field on a Riemannian manifold is a concircular vector field. First, in this paper, we show that there does not exist a torqued vector field on the m-sphere $\mathbf{S}^m(c)$. Then we construct a torqued vector field on a proper open subset of a Euclidean space $\mathbf{E}^m$ which is not a concircular vector field with nowhere zero torqued function σ and nowhere zero torqued 1-form α. However, on the Euclidean space $\mathbf{E}^m$, we show that such torqued vector field does not exist. Finally in the last section, we find a necessary and sufficient condition for a torqued vector field on a compact Riemannian manifold to be a concircular vector field.

2. Preliminaries

In this section, we introduce the notions, concepts and basic results needed for proving results in the subsequent sections. Let $\mathbf{u}$ be a torqued vector field on an n-dimensional Riemannian manifold (M, g) with torqued function σ and torqued form α. We denote by $\mathbf{v}$ the smooth vector field dual to the torqued 1 -form α, that is, $\alpha(X) = g(\mathbf{v}, X)$, $X \in \mathfrak{X}(M)$.

The curvature tensor field R and the Ricci tensor Ric of (M, g) are given respectively by

$$R(X, Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z \tag{4}$$

and

$$Ric(X, Y) = \sum_{i=1}^{n} g\left(R(e_i, X)Y, e_i\right), \tag{5}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal.

Using symmetry of the Ricci tensor Ric, we get a symmetric operator S called the Ricci operator of M defined by

$$g(SX,Y) = Ric(X,Y)$$

for $X, Y \in \mathfrak{X}(M)$. The scalar curvature τ of M is given by, $\tau = \operatorname{Tr} S$. Note that $\nabla\tau$ the gradient of the scalar curvature τ, satisfies

$$\frac{1}{2}\nabla\tau = \sum_{i=1}^{n} (\nabla S)(e_i, e_i), \tag{6}$$

where ∇S is the covariant derivative of S, defined by

$$(\nabla S)(X,Y) = \nabla_X SY - S\nabla_X Y.$$

Using Equations (3) and (4), we compute

$$R(X,Y)\mathbf{u} = (X\sigma)Y - (Y\sigma)X + \sigma(\alpha(Y)X - \alpha(X)Y) + d\alpha(X,Y)\mathbf{u} \tag{7}$$

for $X, Y \in \mathfrak{X}(M)$, which gives

$$Ric(Y,\mathbf{u}) = -(n-1)(Y\sigma) + (n-1)\sigma\alpha(Y) + d\alpha(\mathbf{u},Y), \tag{8}$$

where $d\alpha$ is differential of the torqued form α.

Now, using the vector field $\mathbf{v}$ dual to torqued form α, we define a symmetric operator A and a skew-symmetric operator Ψ by

$$2g(AX,Y) = (\pounds_{\mathbf{v}} g)(X,Y), \quad 2g(\Psi X,Y) = d\alpha(X,Y), \tag{9}$$

where $\pounds_{\mathbf{v}} g$ is the Lie derivative of g with respect to the vector field $\mathbf{v}$. Then, we have

$$2g(AX,Y) = g(\nabla_X \mathbf{v}, Y) + g(\nabla_Y \mathbf{v}, X) = 2g(\nabla_X \mathbf{v}, Y) - 2g(\Psi X, Y),$$

which gives

$$\nabla_X \mathbf{v} = AX + \Psi X, \quad X \in \mathfrak{X}(M). \tag{10}$$

Using Equations (8) and (9), we conclude

$$S\mathbf{u} = -(n-1)(\nabla\sigma - \sigma\mathbf{v}) + 2\Psi\mathbf{u}, \tag{11}$$

where $\nabla\sigma$ is gradient of torqued function σ.

Note that using definition of torqued vector field, we have $\alpha(\mathbf{u}) = 0$, that is, $g(\mathbf{v},\mathbf{u}) = 0$. Using Equations (3) and (10) in $Xg(\mathbf{v},\mathbf{u}) = 0$, $X \in \mathfrak{X}(M)$, we get

$$g(AX + \Psi X, \mathbf{u}) + g(\mathbf{v}, \sigma X + \alpha(X)\mathbf{u}) = 0,$$

that is,

$$\Psi\mathbf{u} = A\mathbf{u} + \sigma\mathbf{v}. \tag{12}$$

Taking the inner product in Equation (7) with $\mathbf{u}$, we get

$$\|\mathbf{u}\|^2 d\alpha(X,Y) = Y(\sigma)g(X,\mathbf{u}) - X(\sigma)g(Y,\mathbf{u}) + \sigma(\alpha(X)g(Y,\mathbf{u}) - \alpha(Y)g(X,\mathbf{u})),$$

which in view of Equation (9), implies

$$2\|\mathbf{u}\|^2 \Psi X = g(X,\mathbf{u})\nabla\sigma - X(\sigma)\mathbf{u} + \sigma\alpha(X)\mathbf{u} - \sigma g(X,\mathbf{u})\mathbf{v}. \tag{13}$$

Taking $X = \mathbf{u}$, in Equation (13), we have

$$2\|\mathbf{u}\|^2 \Psi\mathbf{u} = \|\mathbf{u}\|^2 (\nabla\sigma - \sigma\mathbf{v}) - \mathbf{u}(\sigma)\mathbf{u}. \tag{14}$$

3. Torqued Vector Fields on Spheres and Euclidean Spaces

In this section, we study torqued vector fields on sphere $\mathbf{S}^n(c)$ of constant curvature c and the Euclidean space $\mathbf{E}^n$. Note that the curvature tensor and the Ricci tensor of the sphere $\mathbf{S}^n(c)$ are given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \quad Ric(X,Y) = (n-1)cg(X,Y). \tag{15}$$

In the following result we show that there does not exist a torqued vector field on the sphere $\mathbf{S}^n(c)$.

Theorem 1. *There does not exist a torqued vector field on the sphere* $\mathbf{S}^n(c)$.

Proof. Suppose $\mathbf{u}$ is a torqued vector field on the sphere $\mathbf{S}^n(c)$. Then using Equation (15), we have $R(\mathbf{v},\mathbf{u})\mathbf{u} = c\|\mathbf{u}\|^2\mathbf{v}$ and, Equation (7) gives

$$c\|\mathbf{u}\|^2\mathbf{v} = \mathbf{v}(\sigma)\mathbf{u} - \mathbf{u}(\sigma)\mathbf{v} - \sigma\|\mathbf{v}\|^2\mathbf{u} + d\alpha(\mathbf{v},\mathbf{u})\mathbf{u}. \tag{16}$$

Taking the inner product in Equation (2) with $\mathbf{v}$, we conclude

$$\|\mathbf{v}\|^2\left(c\|\mathbf{u}\|^2 + \mathbf{u}(\sigma)\right) = 0. \tag{17}$$

As $\mathbf{S}^n(c)$ is connected Equation (17) implies either $\mathbf{v} = 0$ or

$$\mathbf{u}(\sigma) = -c\|\mathbf{u}\|^2. \tag{18}$$

If $\mathbf{v} = 0$, then $\alpha = 0$. Thus, we shall concentrate on the case of Equation (18) and show that it also gives the same conclusion. Using Equation (3), we have div $\mathbf{u} = n\sigma$ and using Equation (18), we have div $(\sigma\mathbf{u}) = -c\|\mathbf{u}\|^2 + n\sigma^2$. Integrating this equation, we conclude

$$c\int_{\mathbf{S}^n(c)}\|\mathbf{u}\|^2 = n\int_{\mathbf{S}^n(c)}\sigma^2. \tag{19}$$

Note that on using Equation (3), we have

$$(\pounds_{\mathbf{u}}g)(X,Y) = 2\sigma g(X,Y) + \alpha(X)g(\mathbf{u},Y) + \alpha(Y)g(\mathbf{u},X). \tag{20}$$

Now, we use a local orthonormal frame $\{e_1,\ldots,e_n\}$ on $\mathbf{S}^n(c)$ and Equations (3) and (20), to compute

$$|\pounds_{\mathbf{u}}g|^2 = \sum_{i,j}\left((\pounds_{\mathbf{u}}g)(e_i,e_j)\right)^2 = 4n\sigma^2 + 4\|\mathbf{u}\|^2\|\mathbf{v}\|^2, \tag{21}$$

where we have used $\alpha(\mathbf{u}) = 0$. Also, we have

$$\|\nabla\mathbf{u}\|^2 = \sum_i g(\nabla_{e_i}\mathbf{u},\nabla_{e_i}\mathbf{u}) = \sum_i\|\sigma e_i + \alpha(e_i)\mathbf{u}\|^2 = n\sigma^2 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2. \tag{22}$$

Next, we use the following integral formula in [20]

$$\int_{\mathbf{S}^n(c)}\left(Ric(\mathbf{u},\mathbf{u}) + \frac{1}{2}|\pounds_{\mathbf{u}}g|^2 - \|\nabla\mathbf{u}\|^2 - (\text{div }\mathbf{u})^2\right) = 0$$

and on inserting Equations (21) and (22) and div $\mathbf{u} = n\sigma$, $Ric(\mathbf{u}, \mathbf{u}) = (n-1)c \|\mathbf{u}\|^2$, we obtain

$$\int_{\mathbf{S}^n(c)} \left((n-1)c \|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - n(n-1)\sigma^2 \right) = 0. \tag{23}$$

Using Equation (19) in Equation (23), we conclude

$$\int_{\mathbf{S}^n(c)} \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 = 0,$$

that is, $\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 = 0$ and as $\mathbf{u}$ is nowhere zero, we get $\mathbf{v} = 0$. This proves that the torqued form $\alpha = 0$ and thus, the skew-symmetric operator $\Psi = 0$. Now, using Equations (14) and (18), we have

$$\|\mathbf{u}\|^2 (\nabla\sigma + c\mathbf{u}) = 0.$$

Since, the torqued vector field $\mathbf{u}$ is nowhere zero, we get $\nabla\sigma + c\mathbf{u} = 0$. However, the sphere $\mathbf{S}^n(c)$ being compact, $\nabla\sigma_p = 0$ for a point $p \in \mathbf{S}^n(c)$ (critical point of σ). Consequently, we get $\mathbf{u}(p) = 0$, a contradiction to the fact that $\mathbf{u}$ is nowhere zero. Hence, there does not exist a torqued vector field on $\mathbf{S}^n(c)$. $\square$

Remark 1. *Note that the unit sphere $\mathbf{S}^n$ is isometric to the warped product $I \times_{\sin t} \mathbf{S}^{n-1}$, where the interval $I = (0, \pi)$ and the isometry is given by the map $f : I \times \mathbf{S}^{n-1} \to \mathbf{S}^n$ defined by (cf. [27])*

$$f(t, u) = (\cos t, \sin t, u).$$

It is interesting to observe that the above warped product is short of being a twisted product and thus our Theorem 1 strengthens that the conclusion of Theorem 3.1 in [25] is sharp.

In the rest of this section, we study torqued vector fields on the Euclidean space $\mathbf{E}^n$. We denote by g the Euclidean metric and by ∇ the Euclidean connection on $\mathbf{E}^n$. The position vector field Γ on $\mathbf{E}^n$ is given by

$$\Gamma = v^1 \frac{\partial}{\partial v^1} + \ldots + v^n \frac{\partial}{\partial v^n},$$

where $v^1, \ldots, v^n$ are Euclidean coordinates. First, we give the following example of a torqued vector field on an open set of $\mathbf{E}^n$.

Example 1. *Consider the open subset*

$$M = \left\{ \left(v^1, \ldots, v^n \right) \in \mathbf{E}^n : v^1 v^n > 0 \right\}$$

and the smooth function f on M defined by

$$f = \cosh\left(\ln \frac{v^1}{v^n} \right).$$

Then it follows that on M, $\Gamma(f) = 0$ and that $f > 0$. Now define a smooth vector field $\mathbf{u}$ on the Riemannian manifold (M, g) by

$$\mathbf{u} = f\Gamma.$$

Then we see that $\mathbf{u}$ is nowhere zero vector field. Using the Euclidean connection ∇ restricted to the open subset M, we get

$$\nabla_X \mathbf{u} = fX + \alpha(X)\mathbf{u}, \quad X \in \mathfrak{X}(M),$$

where $\alpha(X) = X(\ln f)$. Moreover, we have $\alpha(\mathbf{u}) = 0$ and consequently, the vector field $\mathbf{u}$ is a torqued vector field on the Riemannian manifold (M, g). Note that for this torqued vector field the torqued function f is nowhere zero as well as the torqued form α is nowhere zero.

Observe that, as in Example 1, we could construct several torqued vector fields on such open subsets of the Euclidean space $\mathbf{E}^n$ with the property that corresponding torqued function and torqued form are nowhere zero on this open subset. However, in the following result, we see that the topology of the Euclidean space does not allow it to hold globally on $\mathbf{E}^n$ for $n > 2$.

Theorem 2. *There does not exist a torqued vector field on the Euclidean space $\mathbf{E}^n$, $n > 2$, having nowhere zero torqued function and nowhere zero torqued form.*

Proof. Suppose there is a torqued vector field $\mathbf{u}$ on $\mathbf{E}^n$, $n > 2$, with torqued function σ nowhere zero and torqued form α nowhere zero. As the Euclidean space is flat, Equation (8) implies

$$2\Psi\mathbf{u} = (n-1)\left(\nabla\sigma - \sigma\mathbf{v}\right). \tag{24}$$

Taking the inner product with $\mathbf{u}$ in Equation (24), we conclude

$$\mathbf{u}(\sigma) = 0. \tag{25}$$

In view of Equation (25), Equation (14) takes the form

$$2\left\|\mathbf{u}\right\|^2\Psi\mathbf{u} = \left\|\mathbf{u}\right\|^2\left(\nabla\sigma - \sigma\mathbf{v}\right)$$

and as $\mathbf{u}$ is nowhere zero by definition of a torqued vector field, we have

$$2\Psi\mathbf{u} = \nabla\sigma - \sigma\mathbf{v}. \tag{26}$$

Combining Equation (26) with Equation (24), we conclude $(n-2)\left(\nabla\sigma - \sigma\mathbf{v}\right) = 0$ and as $n > 0$, we have

$$\nabla\sigma = \sigma\mathbf{v}. \tag{27}$$

Note that owing to the conditions on the torqued function σ and the torqued form α, the vector field $\sigma\mathbf{v}$ is nowhere zero on the Euclidean space $\mathbf{E}^n$. Thus, as $\nabla\sigma$ nowhere zero, the smooth function $\sigma : \mathbf{E}^n \to \mathbf{E}$ is a submersion and each level set $M_x = \sigma^{-1}\{\sigma(x)\}$, $x \in \mathbf{E}^n$, is an $(n-1)$-dimensional smooth manifold and that it is compact. Now, define the smooth vector field

$$\varsigma = \frac{\nabla\sigma}{\left\|\nabla\sigma\right\|^2}$$

on the Euclidean space $\mathbf{E}^n$. Then it follows that $\varsigma(\sigma) = 1$ and therefore the local one-parameter group of local transformations $\{f_s\}$ of ς has the property

$$\sigma\left(f_s(x)\right) = \sigma(x) + s. \tag{28}$$

Using escape Lemma (cf. [28] p. 446) for the Euclidean space $\mathbf{E}^n$ and Equation (28), we conclude that the vector field ς is complete and that $\{f_s\}$ is one-parameter group of transformations on $\mathbf{E}^n$. Now, define $\varphi : \mathbf{E} \times M_x \to \mathbf{E}^n$ by

$$\varphi(s, u) = f_s(u).$$

It follows that φ is a smooth map and for each $u \in \mathbf{E}^n$, we find $s \in \mathbf{E}$ such that $f_s(u) = y \in M_x$ satisfying $u = f_{-s}(y)$. Thus, $\varphi(-s, y) = u$, that is, φ is an on-to map. Also, for (s_1, u_1), (s_2, u_2) in $\mathbf{E} \times M_x$ such that $\varphi(s_1, u_1) = \varphi(s_2, u_2)$, we have $f_{s_1}(u_1) = f_{s_2}(u_2)$, which on using Equation (28), gives $\sigma(u_1) + s_1 = \sigma(u_2) + s_2$. Since, $u_1, u_2 \in M_x$, we have $\sigma(u_1) = \sigma(u_2)$, and we get $s_1 = s_2$. Consequently, from $f_{s_1}(u_1) = f_{s_2}(u_2)$, we conclude $u_1 = u_2$. This proves that the map φ is one-to-one. Also, the map

$$\varphi^{-1}(u) = (-s, y) = (-s, f_s(u))$$

is smooth. Hence, $\varphi : \mathbf{E} \times M_x \to \mathbf{E}^n$ is a diffeomorphism, where M_x is a compact subset of $\mathbf{E}^n$. Note that $\mathbf{E}^n$ is diffeomorphic to $\mathbf{E} \times \mathbf{E}^{n-1}$ and this will imply that M_x is diffeomorphic to $\mathbf{E}^{n-1}$, which is a contradiction. Hence, there does not exist a torqued vector field on $\mathbf{E}^n$ with nowhere zero torqued function and nowhere zero torqued form. □

4. Torqued Vector Fields on Compact Spaces

In this section, we study torqued vector fields on compact Riemannian manifolds. Let $\mathbf{u}$ be torqued vector field with torqued function σ and torqued form α on n-dimensional compact Riemannian manifold (M, g). Recall that a smooth vector field ξ on a Riemannian manifold (M, g) is said to be incompressible vector field (also known as a solenoidal vector field) if div $\xi = 0$. Indeed any vector field on a Riemannian manifold is sum of a gradient vector field and an incompressible vector field. As we are interested in finding conditions under which a torqued vector field on a Riemannian manifold is a concircular vector field, we prove the following result for torqued vector fields on a compact Riemannian manifold.

Theorem 3. *Let $\mathbf{u}$ be a torqued vector field with torqued form α on an n-dimensional compact Riemannian manifold (M, g). Then the dual vector field $\mathbf{v}$ to torqued form α is incompressible if and only if $\mathbf{u}$ is a concircular vector field.*

Proof. Let $\mathbf{u}$ be a torqued vector field on an n-dimensional compact Riemannian manifold (M, g) such that div $\mathbf{v} = 0$. Then using Equation (10), we conclude

$$TrA = 0, \tag{29}$$

where we have used the fact that Ψ is skew-symmetric. Define a skew-symmetric operator $\mathbf{F}$ on M by

$$\mathbf{F}X = \alpha(X)\mathbf{u} - g\left(X, \mathbf{u}\right)\mathbf{v}, \quad X \in \mathfrak{X}(M). \tag{30}$$

Using Equations (3) and (10), we have

$$\begin{aligned}
\nabla_X \mathbf{F} Y &= X\left(\alpha\left(Y\right)\right)\mathbf{u} + \alpha\left(Y\right)\left(\sigma X + \alpha\left(X\right)\mathbf{u}\right) - g\left(\nabla_X Y, \mathbf{u}\right)\mathbf{v} \\
&\quad - g\left(Y, \sigma X + \alpha(X)\mathbf{u}\right)\mathbf{v} - g\left(Y, \mathbf{u}\right)\left(AX + \Psi X\right) \\
&= g\left(AX + \Psi X, Y\right)\mathbf{u} + \alpha\left(\nabla_X Y\right)\mathbf{u} + \sigma\alpha\left(Y\right)X + \alpha\left(X\right)\alpha\left(Y\right)\mathbf{u} \\
&\quad - g\left(\nabla_X Y, \mathbf{u}\right)\mathbf{v} - g\left(Y, \sigma X + \alpha(X)\mathbf{u}\right)\mathbf{v} - g\left(Y, \mathbf{u}\right)AX - g\left(Y, \mathbf{u}\right)\Psi X
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\nabla \mathbf{F}\right)\left(X, Y\right) &= g\left(AX, Y\right)\mathbf{u} + g\left(\Psi X, Y\right)\mathbf{u} + \sigma\alpha\left(Y\right)X + \alpha\left(X\right)\alpha\left(Y\right)\mathbf{u} \\
&\quad - \sigma g\left(X, Y\right)\mathbf{v} - \alpha(X) g\left(Y, \mathbf{u}\right)\mathbf{v} - g\left(Y, \mathbf{u}\right)AX - g\left(Y, \mathbf{u}\right)\Psi X.
\end{aligned}$$

Now, taking a local orthonormal frame $\{e_1, \ldots, e_n\}$, and using Equation (29), we obtain

$$\sum_i \left(\nabla \mathbf{F}\right)\left(e_i, e_i\right) = \sigma\mathbf{v} + \|\mathbf{v}\|^2\mathbf{u} - n\sigma\mathbf{v} - A\mathbf{u} - \Psi\mathbf{u}. \tag{31}$$

We compute the divergence of the vector field $\mathbf{Fu}$ to find

$$\text{div }(\mathbf{Fu}) = \sum_i g\left(\nabla_{e_i}\mathbf{Fu}, e_i\right) = \sum_i g\left((\nabla\mathbf{F})(e_i, \mathbf{u}) + \mathbf{F}\left(\nabla_{e_i}\mathbf{u}\right), e_i\right). \tag{32}$$

Using skew-symmetry of the operator $\mathbf{F}$ and Equation (3) in Equation (32) , we obtain

$$\text{div }(\mathbf{Fu}) = -\sum_i g\left(\mathbf{u}, (\nabla\mathbf{F})(e_i, e_i)\right) - \sum_i g\left(\sigma e_i + \alpha(e_i)\mathbf{u}, \mathbf{F}e_i\right),$$

that is,

$$\text{div }(\mathbf{Fu}) = -\sum_i g\left(\mathbf{u}, (\nabla\mathbf{F})(e_i, e_i)\right) + g\left(\mathbf{Fu}, \mathbf{v}\right). \tag{33}$$

Inserting Equation (31) in Equation (33) , we have

$$\text{div }(\mathbf{Fu}) = -\|\mathbf{v}\|^2\|\mathbf{u}\|^2 + g\left(A\mathbf{u}, \mathbf{u}\right), \tag{34}$$

where we have used $\alpha(\mathbf{u}) = 0$ and that Ψ is skew-symmetric. Now, using Equation (10), we have $\nabla_{\mathbf{u}}\mathbf{v} = A\mathbf{u} + \Psi\mathbf{u}$ and taking the inner product with $\mathbf{u}$ in this equation, we conclude

$$g\left(A\mathbf{u}, \mathbf{u}\right) = g\left(\nabla_{\mathbf{u}}\mathbf{v}, \mathbf{u}\right). \tag{35}$$

As $\alpha(\mathbf{u}) = 0$, using Equation (3) in Equation (35) , we get

$$g\left(A\mathbf{u}, \mathbf{u}\right) = -g\left(\mathbf{v}, \nabla_{\mathbf{u}}\mathbf{u}\right) = -g\left(\mathbf{v}, \sigma\mathbf{u} + \alpha(\mathbf{u})\mathbf{u}\right) = 0.$$

Thus, the Equation (34) takes the form

$$\text{div }(\mathbf{Fu}) = -\|\mathbf{v}\|^2\|\mathbf{u}\|^2. \tag{36}$$

Integrating Equation (36), we conclude

$$\int_M \|\mathbf{v}\|^2\|\mathbf{u}\|^2 = 0,$$

that is, $\|\mathbf{v}\|^2\|\mathbf{u}\|^2 = 0$. Since, the torqued vector field $\mathbf{u}$ is nowhere zero, we get $\mathbf{v} = 0$, that is, $\alpha = 0$. This proves that $\mathbf{u}$ is concircular vector field.

The converse is trivial, as for the concircular vector field the torqued form is zero and therefore the dual vector field being zero is incompressible. □

Note that an interesting question on torqued vector fields is to find conditions under which they are concircular vector field. In [23], it is shown that a gradient torqued vector field is a concircular vector field. Our Theorem 3 characterizes concircular vector fields using torqued vector fields.

As a consequence of above result we get the following.

Theorem 4. *Let* $\mathbf{u}$ *be a torqued vector field on an* n*-dimensional compact Riemannian manifold* (M, g) *with torqued form* α*. If dual vector field* $\mathbf{v}$ *to torqued form* α *is incompressible, then there exists a point* $p \in M$*, such that* $Ric(u, u)(p) = 0$.

Proof. Suppose $\mathbf{u}$ is a torqued vector field on an n-dimensional compact Riemannian manifold (M, g) with torqued form α such that div $\mathbf{v} = \mathbf{0}$. Then by the previous theorem, we have $\alpha = 0$ and Equation (9) implies $\Psi = 0$. Using Equation (11), we conclude

$$S(\mathbf{u}) = -(n-1)\nabla\sigma. \tag{37}$$

Since, M is compact, at the critical point p of σ, we get $(\nabla\sigma)(p) = 0$. Thus, Equation (37) implies $Ric(u,u)(p) = 0$. $\square$

5. Conclusions

As a particular case of Theorem 4, we observe that there is no torqued vector field with dual vector field to torqued form incompressible on compact Riemannian manifold of positive Ricci curvature. The Riemannian product $\mathbf{S}^n(c_1) \times \mathbf{S}^m(c_2)$ is a compact Riemannian manifold which does not have positive sectional curvature. A natural question is to know whether there exists a torqued vector field on this product with incompressible dual vector field to the torqued form. Further, it will be interesting to know whether there exists a torqued vector field on the hyperbolic space $H^n(-c)$, $c > 0$.

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