

Article

Two Generalizations of Dual-Hyperbolic Balancing Numbers

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Abstract: In this paper, we study two generalizations of dual-hyperbolic balancing numbers: dual-hyperbolic Horadam numbers and dual-hyperbolic k -balancing numbers. We give Catalan's identity, Cassini's identity, and d'Ocagne's identity for them.

Keywords: balancing numbers; Diophantine equation; dual-hyperbolic numbers; Binet formula; Catalan's identity

MSC: 11B37, 11B39

1. Introduction

Let p, q, n be integers. For $n \geq 0$, Horadam (see [1]) defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation:

$$W_n = p \cdot W_{n-1} - q \cdot W_{n-2}, \quad (1)$$

for $n \geq 2$ with arbitrary initial values W_0, W_1 .

The Binet-type formula for the Horadam numbers has the form:

$$W_n = At_1^n + Bt_2^n,$$

where:

$$A = \frac{W_1 - W_0 t_2}{t_1 - t_2}, \quad B = \frac{W_0 t_1 - W_1}{t_1 - t_2}, \quad (2)$$

$$t_1 = \frac{p - \sqrt{p^2 - 4q}}{2}, \quad t_2 = \frac{p + \sqrt{p^2 - 4q}}{2}$$

and $p^2 - 4q \neq 0$. Usually, we assume that t_1, t_2 are two (different) real numbers, though this need not be so; see [1].

For special values of W_0, W_1, p, q , Equation (1) defines special sequences of the Fibonacci type: the Fibonacci numbers $F_n = W_n(0, 1; 1, -1)$, the Lucas numbers $L_n = W_n(2, 1; 1, -1)$, the Pell numbers $P_n = W_n(0, 1; 2, -1)$, the Pell–Lucas numbers $Q_n = W_n(2, 2; 2, -1)$, the Jacobsthal numbers $J_n = W_n(0, 1; 1, -2)$, the Jacobsthal–Lucas numbers $j_n = W_n(2, 1; 1, -2)$, the balancing numbers $B_n = W_n(0, 1; 6, 1)$, the Lucas-balancing numbers $C_n = W_n(1, 3; 6, 1)$, etc.

In this paper, we focus on the balancing numbers, the Lucas-balancing numbers, and some modifications and generalizations of these numbers.

The sequence of balancing numbers, denoted by $\{B_n\}$, was introduced by Behera and Panda in [2]. In [3], Panda introduced the sequence of Lucas-balancing numbers, denoted by $\{C_n\}$ and defined as follows: if B_n is a balancing number, the number C_n for which $(C_n)^2 = 8B_n^2 + 1$ is called a Lucas-balancing number. Recall that a balancing number n with balancer r is the solution of the Diophantine equation:

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r). \quad (3)$$

Cobalancing numbers were defined and introduced in [4] by a modification of Formula (3). The authors called positive integer number n a cobalancing number with cobalancer r if:

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r).$$

Let b_n denote the n th cobalancing number. The n th Lucas-cobalancing number c_n is defined with $(c_n)^2 = 8b_n^2 + 8b_n + 1$; see [5,6].

The balancing, Lucas-balancing, cobalancing, and Lucas-cobalancing numbers fulfill the following recurrence relations:

$$\begin{aligned} B_n &= 6B_{n-1} - B_{n-2} \text{ for } n \geq 2, \text{ with } B_0 = 0, B_1 = 1, \\ C_n &= 6C_{n-1} - C_{n-2} \text{ for } n \geq 2, \text{ with } C_0 = 1, C_1 = 3, \\ b_n &= 6b_{n-1} - b_{n-2} + 2 \text{ for } n \geq 2, \text{ with } b_0 = 0, b_1 = 0, \\ c_n &= 6c_{n-1} - c_{n-2} \text{ for } n \geq 2, \text{ with } c_0 = -1, c_1 = 1. \end{aligned} \quad (4)$$

Note that cobalancing and Lucas-cobalancing numbers were originally defined for $n \geq 1$. Defining $b_0 = 0$ and $c_0 = -1$, which we get by back calculation in recurrences (4), we obtain the same, correctly defined sequences.

The Table 1 includes initial terms of the balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers for $0 \leq n \leq 7$.

Table 1. The values of balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers.

n	0	1	2	3	4	5	6	7
B_n	0	1	6	35	204	1189	6930	40391
C_n	1	3	17	99	577	3363	19601	114243
b_n	0	0	2	14	84	492	2870	16730
c_n	-1	1	7	41	239	1393	8119	47321

The Binet-type formulas for the above-mentioned sequences have the following forms:

$$\begin{aligned} B_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\ C_n &= \frac{\alpha^n + \beta^n}{2}, \\ b_n &= \frac{\alpha^{n-\frac{1}{2}} - \beta^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2}, \\ c_n &= \frac{\alpha^{n-\frac{1}{2}} + \beta^{n-\frac{1}{2}}}{\alpha - \beta}, \end{aligned} \quad (5)$$

for $n \geq 0$, where $\alpha = 3 + 2\sqrt{2}$, $\beta = 3 - 2\sqrt{2}$, $\alpha^{\frac{1}{2}} = 1 + \sqrt{2}$, $\beta^{\frac{1}{2}} = 1 - \sqrt{2}$.

Based on the concept from [7], Özkoç generalized the balancing numbers to k -balancing numbers; see [5].

Let B_n^k denote the n th k -balancing number, C_n^k denote the n th k -Lucas balancing number, b_n^k denote the n th k -cobalancing number, and c_n^k denote the n th k -Lucas cobalancing number, which are the numbers defined by:

$$\begin{aligned} B_n^k &= 6kB_{n-1}^k - B_{n-2}^k \text{ for } n \geq 2, \text{ with } B_0^k = 0, B_1^k = 1, \\ C_n^k &= 6kC_{n-1}^k - C_{n-2}^k \text{ for } n \geq 2, \text{ with } C_0^k = 1, C_1^k = 3, \\ b_n^k &= 6kb_{n-1}^k - b_{n-2}^k + 2 \text{ for } n \geq 2, \text{ with } b_0^k = 0, b_1^k = 0, \\ c_n^k &= 6kc_{n-1}^k - c_{n-2}^k \text{ for } n \geq 2, \text{ with } c_0^k = 6k - 7, c_1^k = 1 \end{aligned}$$

for some positive integer $k \geq 1$. Similarly to the previous considerations, we define additionally $b_0^k = 0$ and $c_0^k = 6k - 7$. For $k = 1$, we obtain classical balancing numbers, Lucas-balancing numbers, etc.

Theorem 1. ([5]) (Binet-type formulas) Let $n \geq 0, k \geq 1$ be integers. Then:

$$B_n^k = \frac{\alpha_k^n - \beta_k^n}{2\sqrt{9k^2 - 1}} = \frac{\alpha_k^n - \beta_k^n}{\alpha_k - \beta_k} \quad (6)$$

and:

$$\begin{aligned} C_n^k &= \frac{(3 - \beta_k)\alpha_k^n - (3 - \alpha_k)\beta_k^n}{2\sqrt{9k^2 - 1}}, \\ b_n^k &= \frac{(\alpha_k + 1)\alpha_k^{n-1} + (\beta_k + 1)\beta_k^{n-1} - 6k - 2}{2(9k^2 - 1)}, \\ c_n^k &= \frac{(7\alpha_k - 1)\alpha_k^{n-2} - (7\beta_k - 1)\beta_k^{n-2}}{2\sqrt{9k^2 - 1}}, \end{aligned}$$

where $\alpha_k = 3k + \sqrt{9k^2 - 1}$, $\beta_k = 3k - \sqrt{9k^2 - 1}$.

Another generalization of the Lucas-balancing numbers was presented in [8]. For integer $k \geq 1$, the sequence of k -Lucas-balancing numbers (written with two hyphens) is defined recursively by:

$$C_{k,n} = 6kC_{k,n-1} - C_{k,n-2} \text{ for } n \geq 2, \text{ with } C_{k,0} = 1, C_{k,1} = 3k.$$

Theorem 2. ([9]) The Binet-type formula for k -Lucas-balancing numbers is:

$$C_{k,n} = \frac{\alpha_k^n + \beta_k^n}{2}$$

for $n \geq 0, k \geq 1$, where $\alpha_k = 3k + \sqrt{9k^2 - 1}$, $\beta_k = 3k - \sqrt{9k^2 - 1}$.

For $k = 1$, we have $C_{1,n} = C_n$. Moreover, $(C_{k,n})^2 = (9k^2 - 1) (B_n^k)^2 + 1$; see [9].

Hyperbolic imaginary unit j was introduced by Cockle (see [10–13]). The set of hyperbolic numbers is defined as:

$$\mathbb{H} = \{a + bj : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

Dual numbers were introduced by Clifford (see [14]). The set of dual numbers is defined as:

$$\mathbb{D} = \{a + c\varepsilon : a, c \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Let $\mathbb{D}\mathbb{H}$ be the set of dual-hyperbolic numbers w of the form:

$$w = a + bj + c\varepsilon + dj\varepsilon,$$

where $a, b, c, d \in \mathbb{R}$ and:

$$\begin{aligned} j^2 &= 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0, j\varepsilon = \varepsilon j, \\ (j\varepsilon)^2 &= 0, \varepsilon(j\varepsilon) = (j\varepsilon)\varepsilon = 0, j(j\varepsilon) = (j\varepsilon)j = \varepsilon. \end{aligned} \quad (7)$$

If $w_1 = a_1 + b_1j + c_1\varepsilon + d_1j\varepsilon$ and $w_2 = a_2 + b_2j + c_2\varepsilon + d_2j\varepsilon$ are any two dual-hyperbolic numbers, then the equality, the addition, the subtraction, the multiplication by the scalar, and the multiplication are defined in the natural way:

equality: $w_1 = w_2$ only if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$,
 addition: $w_1 + w_2 = (a_1 + a_2) + (b_1 + b_2)j + (c_1 + c_2)\varepsilon + (d_1 + d_2)j\varepsilon$,
 subtraction: $w_1 - w_2 = (a_1 - a_2) + (b_1 - b_2)j + (c_1 - c_2)\varepsilon + (d_1 - d_2)j\varepsilon$,
 multiplication by scalar $s \in \mathbb{R}$: $sw_1 = sa_1 + sb_1j + sc_1\varepsilon + sd_1j\varepsilon$,
 multiplication: $w_1 \cdot w_2 = a_1a_2 + b_1b_2 + (a_1b_2 + a_2b_1)j + (a_1c_2 + a_2c_1 + b_1d_2 + b_2d_1)\varepsilon + (a_1d_2 + a_2d_1 + b_1c_2 + b_2c_1)j\varepsilon$.

The dual-hyperbolic numbers form a commutative ring, a real vector space, and an algebra, but every dual-hyperbolic number does not have an inverse, so $(\mathbb{D}\mathbb{H}, +, \cdot)$ is not the field. For more information on the dual-hyperbolic numbers, see [15].

Since the inception of hypercomplex number theory, the authors studied mainly quaternions, octonions, and sedenions with coefficients being words of special integer sequences belonging to the family of Fibonacci sequences, named also as the Fibonacci-type sequences; see for their list [16]. During the development of the theory of hypercomplex numbers, other types of hypercomplex numbers were introduced and studied with the additional restriction that their coefficients are taken from special integer sequences. It suffices to mention papers that have appeared recently. Cihan et al. [17] introduced dual-hyperbolic Fibonacci and Lucas numbers. The dual-hyperbolic Pell numbers (quaternions) were introduced quite recently by Aydın in [18]. In [19], the authors investigated dual-hyperbolic Jacobsthal and Jacobsthal–Lucas numbers. Moreover, to describe their properties we need to have a wide knowledge of the sequences of the Fibonacci type, and we need to base this on some fundamental papers, for example [20,21], as well as that which is absolutely new [22]. Based on these ideas, we define and study dual-hyperbolic balancing numbers and some of their generalizations.

2. Main Results

Let $n \geq 0$ be an integer. The n th dual-hyperbolic balancing number $\mathbb{D}\mathbb{H}B_n$ is defined as:

$$\mathbb{D}\mathbb{H}B_n = B_n + B_{n+1}j + B_{n+2}\varepsilon + B_{n+3}j\varepsilon,$$

where B_n is the n th balancing number and $\varepsilon, j, j\varepsilon$ are dual-hyperbolic units, which satisfy (7).

Based on the relations (4), we define the n th dual-hyperbolic Lucas-balancing number $\mathbb{D}\mathbb{H}C_n$, the n th dual-hyperbolic cobalancing number $\mathbb{D}\mathbb{H}b_n$, and the n th dual-hyperbolic Lucas-cobalancing number $\mathbb{D}\mathbb{H}c_n$ as:

$$\mathbb{D}\mathbb{H}C_n = C_n + C_{n+1}j + C_{n+2}\varepsilon + C_{n+3}j\varepsilon,$$

$$\mathbb{D}\mathbb{H}b_n = b_n + b_{n+1}j + b_{n+2}\varepsilon + b_{n+3}j\varepsilon,$$

$$\mathbb{D}\mathbb{H}c_n = c_n + c_{n+1}j + c_{n+2}\varepsilon + c_{n+3}j\varepsilon,$$

respectively.

In a similar way, we define the n th dual-hyperbolic k -balancing number $\mathbb{DHB}_n^k$, the n th dual-hyperbolic k -Lucas balancing number $\mathbb{DHC}_n^k$, the n th dual-hyperbolic k -cobalancing number $\mathbb{DHB}_n^k$, the n th dual-hyperbolic k -Lucas balancing number $\mathbb{DHC}_n^k$, and the n th dual-hyperbolic k -Lucas-balancing number $\mathbb{DHC}_{k,n}$ as:

$$\begin{aligned}\mathbb{DHB}_n^k &= B_n^k + B_{n+1}^k j + B_{n+2}^k \varepsilon + B_{n+3}^k j \varepsilon, \\ \mathbb{DHC}_n^k &= C_n^k + C_{n+1}^k j + C_{n+2}^k \varepsilon + C_{n+3}^k j \varepsilon, \\ \mathbb{DHB}_n^k &= b_n^k + b_{n+1}^k j + b_{n+2}^k \varepsilon + b_{n+3}^k j \varepsilon, \\ \mathbb{DHC}_n^k &= c_n^k + c_{n+1}^k j + c_{n+2}^k \varepsilon + c_{n+3}^k j \varepsilon, \\ \mathbb{DHC}_{k,n} &= C_{k,n} + C_{k,n+1} j + C_{k,n+2} \varepsilon + C_{k,n+3} j \varepsilon,\end{aligned}$$

respectively.

For $k = 1$, we have $\mathbb{DHB}_n^1 = \mathbb{DHB}_n$, $\mathbb{DHC}_n^1 = \mathbb{DHC}_{1,n} = \mathbb{DHC}_n$, $\mathbb{DHB}_n^1 = \mathbb{DHB}_n$, and $\mathbb{DHC}_n^1 = \mathbb{DHC}_n$.

Theorem 3. (Binet-type formulas) Let $n \geq 0, k \geq 1$ be integers. Then:

$$\begin{aligned}\text{(i)} \quad \mathbb{DHB}_n^k &= \frac{\alpha_k^n \hat{\alpha}_k - \beta_k^n \hat{\beta}_k}{2\sqrt{9k^2 - 1}}, \\ \text{(ii)} \quad \mathbb{DHC}_n^k &= \frac{(3 - \beta_k) \alpha_k^n \hat{\alpha}_k - (3 - \alpha_k) \beta_k^n \hat{\beta}_k}{2\sqrt{9k^2 - 1}}, \\ \text{(iii)} \quad \mathbb{DHB}_n^k &= \frac{(\alpha_k + 1) \alpha_k^{n-1} \hat{\alpha}_k + (\beta_k + 1) \beta_k^{n-1} \hat{\beta}_k - (6k + 2)(1 + j + \varepsilon + j\varepsilon)}{2(9k^2 - 1)}, \\ \text{(iv)} \quad \mathbb{DHC}_n^k &= \frac{(7\alpha_k - 1) \alpha_k^{n-2} \hat{\alpha}_k - (7\beta_k - 1) \beta_k^{n-2} \hat{\beta}_k}{2\sqrt{9k^2 - 1}}, \\ \text{(v)} \quad \mathbb{DHC}_{k,n} &= \frac{\alpha_k^n \hat{\alpha}_k + \beta_k^n \hat{\beta}_k}{2},\end{aligned}$$

where:

$$\alpha_k = 3k + \sqrt{9k^2 - 1}, \quad \beta_k = 3k - \sqrt{9k^2 - 1} \quad (8)$$

and:

$$\hat{\alpha}_k = 1 + \alpha_k j + \alpha_k^2 \varepsilon + \alpha_k^3 j \varepsilon, \quad \hat{\beta}_k = 1 + \beta_k j + \beta_k^2 \varepsilon + \beta_k^3 j \varepsilon. \quad (9)$$

Proof. By Formula (6), we get:

$$\begin{aligned}\mathbb{DHB}_n^k &= B_n^k + B_{n+1}^k j + B_{n+2}^k \varepsilon + B_{n+3}^k j \varepsilon \\ &= \frac{\alpha_k^n - \beta_k^n}{2\sqrt{9k^2 - 1}} + \frac{\alpha_k^{n+1} - \beta_k^{n+1}}{2\sqrt{9k^2 - 1}} j + \frac{\alpha_k^{n+2} - \beta_k^{n+2}}{2\sqrt{9k^2 - 1}} \varepsilon + \frac{\alpha_k^{n+3} - \beta_k^{n+3}}{2\sqrt{9k^2 - 1}} j \varepsilon \\ &= \frac{\alpha_k^n}{2\sqrt{9k^2 - 1}} \left(1 + \alpha_k j + \alpha_k^2 \varepsilon + \alpha_k^3 j \varepsilon \right) \\ &\quad - \frac{\beta_k^n}{2\sqrt{9k^2 - 1}} \left(1 + \beta_k j + \beta_k^2 \varepsilon + \beta_k^3 j \varepsilon \right) \\ &= \frac{\alpha_k^n}{2\sqrt{9k^2 - 1}} \hat{\alpha}_k - \frac{\beta_k^n}{2\sqrt{9k^2 - 1}} \hat{\beta}_k,\end{aligned}$$

and we obtain (i). By the same method, we can prove Formulas (ii)–(v). $\square$

For $k = 1$, we obtain the Binet-type formulas for the dual-hyperbolic balancing numbers, dual-hyperbolic Lucas-balancing numbers, etc.

Corollary 1. Let $n \geq 0$ be an integer. Then:

$$\begin{aligned}\mathbb{DHB}_n &= \frac{\alpha^n \hat{\alpha} - \beta^n \hat{\beta}}{2\sqrt{8}}, \\ \mathbb{DHC}_n &= \frac{\alpha^n \hat{\alpha} + \beta^n \hat{\beta}}{2}, \\ \mathbb{DHB}_n &= \frac{(\alpha + 1)\alpha^{n-1}\hat{\alpha} + (\beta + 1)\beta^{n-1}\hat{\beta} - 8(1 + j + \varepsilon + j\varepsilon)}{16}, \\ \mathbb{DHC}_n &= \frac{(7\alpha - 1)\alpha^{n-2}\hat{\alpha} - (7\beta - 1)\beta^{n-2}\hat{\beta}}{2\sqrt{8}},\end{aligned}$$

where:

$$\begin{aligned}\alpha &= 3 + 2\sqrt{2}, \quad \beta = 3 - 2\sqrt{2}, \\ \hat{\alpha} &= 1 + (3 + \sqrt{8})j + (17 + 6\sqrt{8})\varepsilon + (99 + 35\sqrt{8})j\varepsilon, \\ \hat{\beta} &= 1 + (3 - \sqrt{8})j + (17 - 6\sqrt{8})\varepsilon + (99 - 35\sqrt{8})j\varepsilon.\end{aligned}\tag{10}$$

Now, we will give some identities such as the Catalan-type identity, the Cassini-type identity, and the d'Ocagne-type identity for the dual-hyperbolic k -balancing numbers. These identities can be proven using the Binet-type formula for these numbers.

Theorem 4. (Catalan-type identity for dual-hyperbolic k -balancing numbers) Let $k \geq 1$, $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then:

$$\mathbb{DHB}_{n-r}^k \cdot \mathbb{DHB}_{n+r}^k - \left(\mathbb{DHB}_n^k\right)^2 = \frac{2 - \left(\frac{\beta_k}{\alpha_k}\right)^r - \left(\frac{\alpha_k}{\beta_k}\right)^r}{4(9k^2 - 1)} \hat{\alpha}_k \hat{\beta}_k,$$

where α_k , β_k and $\hat{\alpha}_k$, $\hat{\beta}_k$ are given by (8) and (9), respectively.

Proof. By Formula (i) of Theorem 3, we have:

$$\begin{aligned}&\mathbb{DHB}_{n-r}^k \cdot \mathbb{DHB}_{n+r}^k - \left(\mathbb{DHB}_n^k\right)^2 \\&= \left(\frac{\alpha_k^{n-r} \hat{\alpha}_k - \beta_k^{n-r} \hat{\beta}_k}{2\sqrt{9k^2 - 1}}\right) \cdot \left(\frac{\alpha_k^{n+r} \hat{\alpha}_k - \beta_k^{n+r} \hat{\beta}_k}{2\sqrt{9k^2 - 1}}\right) - \left(\frac{\alpha_k^n \hat{\alpha}_k - \beta_k^n \hat{\beta}_k}{2\sqrt{9k^2 - 1}}\right)^2 \\&= \frac{-\alpha_k^{n-r} \beta_k^{n+r} \hat{\alpha}_k \hat{\beta}_k - \beta_k^{n-r} \alpha_k^{n+r} \hat{\alpha}_k \hat{\beta}_k + 2\alpha_k^n \beta_k^n \hat{\alpha}_k \hat{\beta}_k}{4(9k^2 - 1)} \\&= \frac{\alpha_k^n \beta_k^n \left(2 - \left(\frac{\beta_k}{\alpha_k}\right)^r - \left(\frac{\alpha_k}{\beta_k}\right)^r\right)}{4(9k^2 - 1)} \hat{\alpha}_k \hat{\beta}_k.\end{aligned}$$

Using the fact that $\alpha_k \beta_k = 1$, we obtain the desired formula. $\square$

Note that for $r = 1$, we obtain the Cassini-type identity for the dual-hyperbolic k -balancing numbers.

Corollary 2. (Cassini-type identity for dual-hyperbolic k -balancing numbers) Let $k \geq 1$, $n \geq 1$ be integers. Then:

$$\mathbb{DHB}_{n-1}^k \cdot \mathbb{DHB}_{n+1}^k - \left(\mathbb{DHB}_n^k\right)^2 = -\hat{\alpha}_k \hat{\beta}_k.$$

where $\hat{\alpha}_k, \hat{\beta}_k$ are given by (9).

Proof. By Theorem 4, we have:

$$\mathbb{DHB}_{n-1}^k \cdot \mathbb{DHB}_{n+1}^k - \left(\mathbb{DHB}_n^k\right)^2 = \frac{2 - \frac{\beta_k}{\alpha_k} - \frac{\alpha_k}{\beta_k}}{4(9k^2 - 1)} \hat{\alpha}_k \hat{\beta}_k.$$

Using the fact that:

$$2 - \frac{\beta_k}{\alpha_k} - \frac{\alpha_k}{\beta_k} = \frac{-(\beta_k - \alpha_k)^2}{\alpha_k \beta_k}$$

and $\beta_k - \alpha_k = -2\sqrt{9k^2 - 1}$, $\alpha_k \beta_k = 1$, we get the result. $\square$

Theorem 5. (d'Ocagne-type identity for dual-hyperbolic k -balancing numbers) Let $k \geq 1$, $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then:

$$\mathbb{DHB}_m^k \cdot \mathbb{DHB}_{n+1}^k - \mathbb{DHB}_{m+1}^k \cdot \mathbb{DHB}_n^k = \frac{\alpha_k^{m-n} - \beta_k^{m-n}}{2\sqrt{9k^2 - 1}} \hat{\alpha}_k \hat{\beta}_k,$$

where α_k, β_k and $\hat{\alpha}_k, \hat{\beta}_k$ are given by (8) and (9), respectively.

Proof. By Formula (i) of Theorem 3, we have:

$$\begin{aligned} & \mathbb{DHB}_m^k \cdot \mathbb{DHB}_{n+1}^k - \mathbb{DHB}_{m+1}^k \cdot \mathbb{DHB}_n^k \\ &= \left(\frac{\alpha_k^m \hat{\alpha}_k - \beta_k^m \hat{\beta}_k}{2\sqrt{9k^2 - 1}} \right) \cdot \left(\frac{\alpha_k^{n+1} \hat{\alpha}_k - \beta_k^{n+1} \hat{\beta}_k}{2\sqrt{9k^2 - 1}} \right) \\ & \quad - \left(\frac{\alpha_k^{m+1} \hat{\alpha}_k - \beta_k^{m+1} \hat{\beta}_k}{2\sqrt{9k^2 - 1}} \right) \cdot \left(\frac{\alpha_k^n \hat{\alpha}_k - \beta_k^n \hat{\beta}_k}{2\sqrt{9k^2 - 1}} \right) \\ &= \frac{-\alpha_k^m \beta_k^{n+1} - \beta_k^m \alpha_k^{n+1} + \alpha_k^{m+1} \beta_k^n + \beta_k^{m+1} \alpha_k^n}{4(9k^2 - 1)} \hat{\alpha}_k \hat{\beta}_k \\ &= \frac{\alpha_k^n \beta_k^n (\alpha_k^{m-n+1} + \beta_k^{m-n+1} - \alpha_k^{m-n} \beta_k - \beta_k^{m-n} \alpha_k)}{4(9k^2 - 1)} \hat{\alpha}_k \hat{\beta}_k \\ &= \frac{(\alpha_k \beta_k)^n (\alpha_k^{m-n} (\alpha_k - \beta_k) - \beta_k^{m-n} (\alpha_k - \beta_k))}{4(9k^2 - 1)} \hat{\alpha}_k \hat{\beta}_k \\ &= \frac{(\alpha_k^{m-n} - \beta_k^{m-n}) (\alpha_k - \beta_k)}{4(9k^2 - 1)} \hat{\alpha}_k \hat{\beta}_k = \frac{\alpha_k^{m-n} - \beta_k^{m-n}}{2\sqrt{9k^2 - 1}} \hat{\alpha}_k \hat{\beta}_k, \end{aligned}$$

which ends the proof. $\square$

For $k = 1$, we obtain Catalan-, Cassini-, and d'Ocagne-type identities for the dual-hyperbolic balancing numbers.

Corollary 3. (Catalan-type identity for dual-hyperbolic balancing numbers) Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then:

$$\mathbb{DHB}_{n-r} \cdot \mathbb{DHB}_{n+r} - (\mathbb{DHB}_n)^2 = \frac{2 - \left(\frac{\beta}{\alpha}\right)^r - \left(\frac{\alpha}{\beta}\right)^r}{32} \hat{\alpha} \hat{\beta}.$$

where α , β , $\hat{\alpha}$, and $\hat{\beta}$ are given by (10).

Corollary 4. (Cassini-type identity for dual-hyperbolic balancing numbers) Let $n \geq 1$ be an integer. Then:

$$\mathbb{DHB}_{n-1} \cdot \mathbb{DHB}_{n+1} - (\mathbb{DHB}_n)^2 = -\hat{\alpha} \hat{\beta}.$$

where $\hat{\alpha}$ and $\hat{\beta}$ are given by (10).

Corollary 5. (d'Ocagne-type identity for dual-hyperbolic balancing numbers) Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then:

$$\mathbb{DHB}_m \cdot \mathbb{DHB}_{n+1} - \mathbb{DHB}_{m+1} \cdot \mathbb{DHB}_n = \frac{(\alpha^{m-n} - \beta^{m-n})}{2\sqrt{8}} \hat{\alpha} \hat{\beta},$$

where α , β , $\hat{\alpha}$, and $\hat{\beta}$ are given by (10).

Moreover, by simple calculations, we get:

$$\alpha_k + \beta_k = 6k,$$

$$\alpha_k - \beta_k = 2\sqrt{9k^2 - 1},$$

$$\alpha_k \beta_k = 1,$$

$$\alpha_k^2 + \beta_k^2 = (\alpha_k + \beta_k)^2 - 2\alpha_k \beta_k = 36k^2 - 2,$$

$$\alpha_k^3 + \beta_k^3 = (\alpha_k + \beta_k)^3 - 3\alpha_k \beta_k (\alpha_k + \beta_k) = 216k^3 - 18k$$

and:

$$\begin{aligned} \hat{\alpha}_k \hat{\beta}_k &= \left(1 + \alpha_k j + \alpha_k^2 \varepsilon + \alpha_k^3 j \varepsilon\right) \left(1 + \beta_k j + \beta_k^2 \varepsilon + \beta_k^3 j \varepsilon\right) \\ &= 1 + \alpha_k \beta_k + (\alpha_k + \beta_k) j + (\alpha_k^2 + \beta_k^2)(1 + \alpha_k \beta_k) \varepsilon \\ &\quad + \left(\alpha_k^3 + \beta_k^3 + \alpha_k \beta_k (\alpha_k + \beta_k)\right) j \varepsilon \\ &= 2 + 6kj + (72k^2 - 4) \varepsilon + (216k^3 - 12k) j \varepsilon. \end{aligned}$$

In particular, for $k = 1$, we have:

$$\hat{\alpha} \hat{\beta} = 2 + 6j + 68\varepsilon + 204j\varepsilon.$$

In the same way, using Formulas (ii)–(v) of Theorem 3, one can obtain properties of other classes of dual-hyperbolic numbers defined in this paper.

Let us return to the first of the generalizations of the dual-hyperbolic balancing numbers. For integer $n \geq 0$, the n th dual-hyperbolic Horadam number $\mathbb{DHW}_n$ is defined as:

$$\mathbb{DHW}_n = W_n + W_{n+1}j + W_{n+2}\varepsilon + W_{n+3}j\varepsilon, \quad (11)$$

where W_n is the n th Horadam number and $\varepsilon, j, j\varepsilon$ are dual-hyperbolic units, which satisfy (7).

We give the Binet-type formula for dual-hyperbolic Horadam numbers and Catalan-, Cassini-, and d'Ocagne-type identities for these numbers. The proofs of these theorems can be made analogous to those proven earlier, so we omit them.

Theorem 6. (Binet-type formula for dual-hyperbolic Horadam numbers) Let $n \geq 0$ be an integer. Then:

$$\mathbb{DHW}_n = At_1^n (1 + t_1j + t_1^2\varepsilon + t_1^3j\varepsilon) + Bt_2^n (1 + t_2j + t_2^2\varepsilon + t_2^3j\varepsilon),$$

where t_1, t_2, A , and B are given by (2).

For the simplicity of notation, let:

$$\begin{aligned}\hat{t}_1 &= 1 + t_1j + t_1^2\varepsilon + t_1^3j\varepsilon, \\ \hat{t}_2 &= 1 + t_2j + t_2^2\varepsilon + t_2^3j\varepsilon.\end{aligned}\tag{12}$$

Theorem 7. (Catalan-type identity for dual-hyperbolic Horadam numbers) Let $n \geq 0, r \geq 0$ be integers such that $n \geq r$. Then:

$$\begin{aligned}\mathbb{DHW}_{n-r} \cdot \mathbb{DHW}_{n+r} - (\mathbb{DHW}_n)^2 \\ = ABt_1^n t_2^r \left(\left(\frac{t_1}{t_2} \right)^r + \left(\frac{t_2}{t_1} \right)^r - 2 \right) \hat{t}_1 \hat{t}_2,\end{aligned}$$

where t_1, t_2, A, B and $\hat{t}_1, \hat{t}_2$ are given by (2) and (12), respectively.

Theorem 8. (Cassini-type identity for dual-hyperbolic Horadam numbers) Let $n \geq 1$ be an integer. Then:

$$\begin{aligned}\mathbb{DHW}_{n+1} \cdot \mathbb{DHW}_{n-1} - (\mathbb{DHW}_n)^2 \\ = ABt_1^n t_2^n \left(\frac{t_1}{t_2} + \frac{t_2}{t_1} - 2 \right) \hat{t}_1 \hat{t}_2,\end{aligned}$$

where t_1, t_2, A, B and $\hat{t}_1, \hat{t}_2$ are given by (2) and (12), respectively.

Theorem 9. (d'Ocagne-type identity for dual-hyperbolic Horadam numbers) Let $m \geq 0, n \geq 0$ be integers such that $m \geq n$. Then:

$$\begin{aligned}\mathbb{DHW}_m \cdot \mathbb{DHW}_{n+1} - \mathbb{DHW}_{m+1} \cdot \mathbb{DHW}_n \\ = ABt_1^n t_2^n (t_1 - t_2) (t_2^{m-n} - t_1^{m-n}) \hat{t}_1 \hat{t}_2,\end{aligned}$$

where t_1, t_2, A, B and $\hat{t}_1, \hat{t}_2$ are given by (2) and (12), respectively.

For special values of W_n , we obtain the dual-hyperbolic Fibonacci numbers, the dual-hyperbolic Pell numbers, the dual-hyperbolic Jacobsthal numbers, the dual-hyperbolic balancing numbers, etc. Theorems 6–9 generalize the previously obtained properties of the dual-hyperbolic balancing numbers, some results of [17–19], and more specifically, the Binet-type formula, Catalan's identity, Cassini's identity, and d'Ocagne's identity for the dual-hyperbolic Fibonacci-type numbers.

3. Concluding Remarks

In this paper, we discussed an extended version of dual-hyperbolic balancing numbers, dual-hyperbolic Horadam numbers, and dual-hyperbolic k -balancing numbers. It seems interesting to complement these results by matrix representation, which is another way to generate the considered generalizations of balancing numbers and their related number sequences through matrices. Then, we can give many known and new formulas for these numbers. Matrix generators also help to obtain new identities for the considered numbers by different methods.

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