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Fixed Circle and Fixed Disc Results for New Types of Θ_c -Contractive Mappings in Metric Spaces

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Abstract: In this manuscript, we introduce the notions of various types of Θ_c -contractions for which we establish some fixed circle and fixed disc theorems in the setting of metric spaces. Some illustrative examples are also provided to support our results. Moreover, we present some fixed circle and fixed disc results of integral type contractive self-mappings, which generalize many results of invariance and transformations in the literature.

Keywords: fixed point; fixed circle; fixed disc; Θ_c -contractions

1. Introduction and Preliminaries

Metric fixed point theory is a very natural and well-developed branch of mathematical analysis which deals with the existence and uniqueness results for fixed points of the mappings defined on metric spaces. The source of this theory is the classical result due to Banach [1] which is known as the Banach contraction principle and states that every contraction mapping defined on a complete metric space (Y, ξ) admits a unique fixed point. Several authors extended and generalized this principle in various directions (for instance, see [2–10]). In this regard, Jleli and Samet [11] introduced an attractive class of auxiliary functions $\Theta: (0, \infty) \rightarrow (1, \infty)$ satisfying some suitable properties and utilized the same functions to prove some fixed point theorems concerning a new type of contractions called Θ -contractions (or JS-contractions).

The Banach contraction principle and its generalizations have been applied in various disciplines of mathematics, economics and engineering. One of the interesting applications of Banach contraction principle is to study the graph neural network model [12]. On the other hand, in the case that the mapping S defined on a metric space (Y, ξ) has more than one fixed point, the number of fixed points of an activation function used in a neural network is important (see [13] and the references therein). There exist some mappings that fix all points of a circle or a disc. These kind of mappings have some applications to neural networks (for instance, see [14]). For example, let $\mathbb{C}$ be the set of all complex numbers with the usual metric $\xi(z, w) = |z - w|$, for all $z, w \in \mathbb{C}$ and consider $S : \mathbb{C} \rightarrow \mathbb{C}$ to be defined by

$$Sz = \begin{cases} \frac{1}{\bar{z}}, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0, \end{cases}$$

where $\bar{z}$ is the conjugate of z . Then, S fixes all points of the circle $C_{0,1} = \{z \in \mathbb{C} : |z| = 1\}$. It is worth pointing out that there exist some mappings which map the circle $C_{z_0,r}$ to itself but do not fix all points of the circle $C_{z_0,r}$. For instance, let T be a self-mapping on $\mathbb{C}$ defined by

$$Tz = \begin{cases} \frac{1}{z}, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases}$$

Then, $T(C_{0,1}) = C_{0,1}$, but T does not fix all points of $C_{0,1}$. In fact, the mapping T fixes only two points of the unit circle.

Natural and interesting problems were found through the investigations of the geometric properties of fixed points. In this context, the fixed circle and fixed disc problem have been studied in metric and generalized metric spaces via different contractive conditions (see [15–24]). For example, in [15], some fixed circle results were proved using the Caristi type contraction on a metric space. Using Wardowski's technique and some classical contractive conditions, some fixed circle theorems were obtained in [19]. In [20,21], the fixed circle problem was studied in the setting of S -metric spaces. In [16], some fixed disc theorems were obtained using several types of contractive conditions on rectangular metric space.

Motivated by the above, we aim to present in this paper some fixed circle (disc) results for some types of contraction self-mappings, namely Θ_c -contractions, Ćirić type Θ_c -contractions, Hardy–Rogers type Θ_c -contractions and Khan type Θ_c -contractions in the setting of metric spaces by using JS technique. Furthermore, we establish some fixed circle (disc) results of integral type contractive self-mappings. To do this, we use the notion of Θ -function introduced by Jleli and Samet [11]. The function $\Theta: (0, \infty) \rightarrow (1, \infty)$ is said to be a Θ -function, if the following conditions are satisfied:

- (JS1) Θ is nondecreasing;
- (JS2) for every sequence $\{\alpha_n\} \subset (0, \infty)$,

$$\lim_{n \rightarrow \infty} \Theta(\alpha_n) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} \alpha_n = 0^+;$$

- (JS3) there exist $\lambda \in (0, 1)$ and $k \in (0, \infty)$ such that $\lim_{\alpha \rightarrow 0^+} \frac{\Theta(\alpha) - 1}{\alpha^\lambda} = k$.

In the sequel, we adopt the following notations:

- Ω_1 the class of all functions Θ which satisfy (JS1).
- $\Omega_{1,2,3}$ the class of all functions Θ which satisfy (JS1)–(JS3).

Remark 1. Observe that the auxiliary function Θ used in [11] satisfies the conditions (JS1)–(JS3), that is, $\Theta \in \Omega_{1,2,3}$ while the auxiliary function Θ used in the present paper need not to satisfy (JS2) and (JS3), that is, $\Theta \in \Omega_1$.

For a metric space (Y, ξ) , the two sets $C_{z_0,r} = \{z \in Y : \xi(z_0, z) = r\}$ and $D_{z_0,r} = \{z \in Y : \xi(z_0, z) \leq r\}$ are called circle and disc, respectively, with center z_0 and radius r . The circle $C_{z_0,r}$ (disc $D_{z_0,r}$) is said to be a fixed circle (fixed disc) of a mapping $S : Y \rightarrow Y$ if $Sz = z$, for all $z \in C_{z_0,r}$ ($D_{z_0,r}$).

Before starting our main results, we would like to bring to the reader's attention that, in the next three sections, we are working on transformations that not only fix one element but also fix a well-defined set of "fixed points" which is either a circle or a disc.

2. Fixed Circle (Disc) Results for Θ_c -Contraction and Ćirić Type Θ_c -Contraction

Let (Y, ξ) be a metric space. From now on, S is a self-mapping defined on a non-empty set Y and

$$r = \inf\{\xi(z, Sz) : z \neq Sz\}.$$

At first, we introduce the notion of Θ_c -contraction as follows:

Definition 1. Let (Y, ξ) be a metric space and $\Theta \in \Omega_1$. A mapping S is said to be a Θ_c -contraction if there exist $\lambda \in (0, 1)$ and $z_0 \in Y$ such that

$$\xi(z, Sz) > 0 \Rightarrow \Theta(\xi(z, Sz)) \leq [\Theta(\xi(z, z_0))]^\lambda, \forall z \in Y.$$

Using the Θ_c -contraction condition, we present the following fixed circle result.

Theorem 1. Let (Y, ξ) be a metric space. Assume that S is a Θ_c -contraction mapping defined on Y with $z_0 \in Y$ and $\xi(z_0, Sz) = r$, for all $z \in C_{z_0, r}$. Then, $C_{z_0, r}$ is a fixed circle of S .

Proof. Let $z \in C_{z_0, r}$ and assume on contrary that $\xi(Sz, z) > 0$, that is, $Sz \neq z$. From the definition of r , we have $\xi(z, Sz) \geq r$. As S is a Θ_c -contraction, so in view of the fact that Θ is nondecreasing, we get

$$\Theta(r) \leq \Theta(\xi(z, Sz)) \leq [\Theta(M(z, z_0))]^\lambda < \Theta(M(z, z_0)) = \Theta(r),$$

which is a contradiction. Therefore, $Sz = z$, for all $z \in C_{z_0, r}$, that is, $C_{z_0, r}$ is a fixed circle of S . This concludes the proof. $\square$

Next, we prove the following fixed disc result.

Theorem 2. Let (Y, ξ) be a metric space. Assume that S is a Ćirić type Θ_c -contraction mapping with $z_0 \in Y$ and $\xi(z_0, Sz) = r$ for all $z \in D_{z_0, r}$. Then, $D_{z_0, r}$ is a fixed disc of S .

Proof. In view of Theorem 1, the mapping S fixes the circle $C_{z_0, r}$. In order to show that $D_{z_0, r}$ is a fixed disc of S , it is sufficient to show that S fixes any circle $C_{z_0, \rho}$ with $\rho < r$. Let $z \in C_{z_0, \rho}$, and, contrarily, let us assume that $z \neq Sz$ for some $z \in C_{z_0, \rho}$. Since S is a Θ_c -contraction, we have

$$\Theta(\xi(z, Sz)) \leq [\Theta(M(z, z_0))]^\lambda < \Theta(M(z, z_0)) = \Theta(\rho).$$

Since Θ is nondecreasing, we then obtain

$$\xi(z, Sz) < \rho < r,$$

which leads to a contradiction as $r = \inf\{\xi(Sz, z) : z \neq Sz\}$. Therefore, $Sz = z$ for all $z \in D_{z_0, r}$. Consequently, $D_{z_0, r}$ is a fixed disc of S . $\square$

Next, we generalize Θ_c -contractions to Ćirić type Θ_c -contractions and prove the following new results in this section.

Definition 2. Let (Y, ξ) be a metric space and $\Theta \in \Omega_1$. A mapping S is said to be a Ćirić type Θ_c -contraction if there exist $\lambda \in (0, 1)$ and $z_0 \in Y$ such that

$$\xi(z, Sz) > 0 \Rightarrow \Theta(\xi(z, Sz)) \leq [\Theta(M(z, z_0))]^\lambda, \forall z \in Y,$$

where

$$M(z, z_0) = \max \left\{ \begin{array}{l} \xi(z, z_0), a\xi(z, Sz) + (1-a)\xi(z_0, Sz_0), \\ (1-a)\xi(z, Sz) + a\xi(z_0, Sz_0), \frac{1}{2}[\xi(z, Sz_0) + \xi(z_0, Sz)] \end{array} \right\}, \quad 0 \leq a < 1. \quad (1)$$

The following proposition follows from Definition 2.

Proposition 1. Let (Y, ξ) be a metric space. If S is a Ćirić type Θ_c -contraction mapping with $z_0 \in Y$, then $Sz_0 = z_0$.

Proof. Assume that $Sz_0 \neq z_0$. From Definition 2, we have

$$\begin{aligned} \Theta(\xi(z_0, Sz_0)) &\leq [\Theta(M(z_0, z_0))]^\lambda \\ &= \left[\Theta \left(\max \left\{ \begin{array}{l} \xi(z_0, z_0), a\xi(z_0, Sz_0) + (1-a)\xi(z_0, Sz_0), \\ (1-a)\xi(z_0, Sz_0) + a\xi(z_0, Sz_0), \\ \frac{1}{2}[\xi(z_0, Sz_0) + \xi(z_0, Sz_0)] \end{array} \right\} \right) \right]^\lambda \\ &= [\Theta(\max\{0, \xi(z_0, Sz_0)\})]^\lambda \\ &= [\Theta(\xi(z_0, Sz_0))]^\lambda, \end{aligned}$$

a contradiction as $\lambda \in (0, 1)$. Therefore, we must have $Sz_0 = z_0$. $\square$

Using Ćirić type Θ_c -contraction condition, we present the following fixed circle result.

Theorem 3. Let (Y, ξ) be a metric space. Assume that S is a Ćirić type Θ_c -contraction mapping with $z_0 \in Y$ and $\xi(z_0, Sz) = r$ for all $z \in C_{z_0, r}$. Then, $C_{z_0, r}$ is a fixed circle of S .

Proof. Let $z \in C_{z_0, r}$ and assume contrarily that $Sz \neq z$. From the definition of r , we have $\xi(z, Sz) \geq r$. As S is a Ćirić type Θ_c -contraction, using Proposition 1, we have

$$\begin{aligned} \Theta(\xi(z, Sz)) &\leq [\Theta(M(z, z_0))]^\lambda \\ &= \left[\Theta \left(\max \left\{ \begin{array}{l} \xi(z, z_0), a\xi(z, Sz) + (1-a)\xi(z_0, Sz_0), \\ (1-a)\xi(z, Sz) + a\xi(z_0, Sz_0), \\ \frac{1}{2}[\xi(z, Sz_0) + \xi(z_0, Sz)] \end{array} \right\} \right) \right]^\lambda \\ &= [\Theta(\max\{r, a\xi(z, Sz), (1-a)\xi(z, Sz)\})]^\lambda \\ &< \Theta(\max\{r, a\xi(z, Sz), (1-a)\xi(z, Sz)\}). \end{aligned} \quad (2)$$

Now, we have the following possibilities:

Case 1: If $\max\{r, a\xi(z, Sz), (1-a)\xi(z, Sz)\} = r$, then from Equation (2), the definition of r and the fact that the function Θ is nondecreasing, we have

$$\Theta(r) \leq \Theta(\xi(z, Sz)) < \Theta(r),$$

a contradiction.

Case 2: If $\max\{r, a\xi(z, Sz), (1-a)\xi(z, Sz)\} = a\xi(z, Sz)$, then we have two possibilities: either $a = 0$ or $0 < a < 1$. Assuming that $0 < a < 1$ from Equation (2) and the fact that the function Θ is nondecreasing, we have

$$\Theta(\xi(z, Sz)) < \Theta(a\xi(z, Sz)) \leq \Theta(\xi(z, Sz)),$$

a contradiction. If $a = 0$, then from Equation (2), we get

$$\Theta(\xi(z, Sz)) < \Theta(0);$$

this inequality contradicts with the definition of Θ (as $\Theta : (0, \infty) \rightarrow (1, \infty)$).

Case 3: If $\max\{r, a\xi(z, Sz), (1-a)\xi(z, Sz)\} = (1-a)\xi(z, Sz)$, then from Equation (2) and the fact that the function Θ is nondecreasing, we have

$$\Theta(\xi(z, Sz)) < \Theta((1-a)\xi(z, Sz)) \leq \Theta(\xi(z, Sz)),$$

a contradiction. Therefore, $Sz = z$ for all $z \in C_{z_0, r}$. Consequently, $C_{z_0, r}$ is a fixed circle of S . $\square$

Next, we prove the following fixed disc result.

Theorem 4. Let (Y, ξ) be a metric space. Assume that S is a Ćirić type Θ_c -contraction mapping with $z_0 \in Y$ and $\xi(z_0, Sz) = r$ for all $z \in D_{z_0, r}$. Then, $D_{z_0, r}$ is a fixed disc of S .

Proof. The mapping S fixes the circle $C_{z_0, r}$ (in view of Theorem 3). Now, in order to show that $D_{z_0, r}$ is a fixed disc of S , it is sufficient to show that S fixes any circle $C_{z_0, \rho}$ with $\rho < r$. Let $z \in C_{z_0, \rho}$ and, contrarily, let us assume that $z \neq Sz$ for some $z \in C_{z_0, \rho}$. Since S is a Ćirić type Θ_c -contraction, using Proposition 1, we have

$$\begin{aligned} \Theta(\xi(z, Sz)) &\leq [\Theta(M(z, z_0))]^\lambda \\ &= \left[\Theta \left(\max \left\{ \rho, a\xi(z, Sz), (1-a)\xi(z, Sz), \frac{\rho+r}{2} \right\} \right) \right]^\lambda \\ &< \Theta \left(\max \left\{ \rho, a\xi(z, Sz), (1-a)\xi(z, Sz), \frac{\rho+r}{2} \right\} \right). \end{aligned} \quad (3)$$

Now, we have the following possibilities:

Case 1: If $\max\left\{\rho, a\xi(z, Sz), (1-a)\xi(z, Sz), \frac{\rho+r}{2}\right\} = \rho$, then from Equation (3), the definition of r and the fact that the function Θ is nondecreasing, we have

$$\Theta(\rho) < \Theta(r) \leq \Theta(\xi(z, Sz)) < \Theta(\rho),$$

a contradiction.

Case 2: If $\max\left\{\rho, a\xi(z, Sz), (1-a)\xi(z, Sz), \frac{\rho+r}{2}\right\} = a\xi(z, Sz)$, then we have two possibilities: either $a = 0$ or $0 < a < 1$. Assuming that $0 < a < 1$ from Equation (3) and the fact that the function Θ is nondecreasing, we have

$$\Theta(\xi(z, Sz)) < \Theta(a\xi(z, Sz)) \leq \Theta(\xi(z, Sz))$$

a contradiction. If $a = 0$, then from Equation (2), we get

$$\Theta(\xi(z, Sz)) < \Theta(0).$$

This inequality contradicts with the definition of Θ (as $\Theta : (0, \infty) \rightarrow (1, \infty)$).

Case 3: If $\max\left\{\rho, a\xi(z, Sz), (1-a)\xi(z, Sz), \frac{\rho+r}{2}\right\} = (1-a)\xi(z, Sz)$, then from Equation (3) and the fact that the function Θ is nondecreasing, we have

$$\Theta(\xi(z, Sz)) < \Theta((1-a)\xi(z, Sz)) \leq \Theta(\xi(z, Sz)),$$

a contradiction.

Case 4: If $\max \left\{ \rho, a\zeta(z, Sz), (1-a)\zeta(z, Sz), \frac{\rho+r}{2} \right\} = \frac{\rho+r}{2}$, then by the definition of r , the inequality Equation (3) and the fact that the function Θ is nondecreasing, we have

$$\Theta(r) \leq \Theta(\zeta(z, Sz)) < \Theta\left(\frac{\rho+r}{2}\right) \leq \Theta(r),$$

a contradiction. Therefore, $Sz = z$ for all $z \in D_{z_0, r}$. Consequently, $D_{z_0, r}$ is a fixed disc of S . $\square$

Corollary 1. Let (Y, ζ) be a metric space and S a mapping defined on Y . If there exist $\lambda \in (0, 1)$ and $z_0 \in Y$ with $\zeta(z_0, Sz) = r$ for all $z \in C_{z_0, r}$ (or $D_{z_0, r}$) such that

$$\zeta(z, Sz) > 0 \Rightarrow \Theta(\zeta(z, Sz)) \leq [\Theta(m(z, z_0))]^\lambda, \forall z \in Y,$$

where $m(z, z_0) = \max\{\zeta(z, z_0), \zeta(z, Sz), \zeta(z_0, Sz_0), \frac{1}{2}[\zeta(z, Sz_0) + \zeta(z_0, Sz)]\}$, then $C_{z_0, r}$ (or $D_{z_0, r}$) is a fixed circle (or disc) of S .

Proof. The result follows from Theorems 3 and 4 by taking $a = 0$ in the contractive condition (1). $\square$

Example 1. Let $Y = [-4, \infty)$ be a metric space endowed with the usual metric ζ . Define $S : Y \rightarrow Y$ as

$$Sz = \begin{cases} z, & \text{if } -4 \leq z < 4 \\ z+3, & \text{if } z \geq 4. \end{cases}$$

The self-mapping S is a Θ_c -contraction. In fact, let $\Theta(t) = e^t$, $z_0 = 0$ and $\lambda = \frac{3}{4}$. For all $z \in [-4, \infty)$ such that $z \geq 4$, we have

$$\zeta(z, Sz) = 3 > 0,$$

and

$$\begin{aligned} \Theta(\zeta(z, Sz)) &= e^3 \leq e^{\frac{3}{4}(z)} \\ &= [e^{\zeta(z, z_0)}]^{\frac{3}{4}} = [\Theta(\zeta(z, z_0))]^\lambda. \end{aligned}$$

The self-mapping S is a Ćirić type Θ_c -contraction. To show this, let $\Theta(t) = e^t$, $z_0 = 0$, $a = \frac{1}{2}$ and $\lambda = \frac{6}{11}$. For all $z \in [-4, \infty)$ such that $z \geq 4$, we have

$$\zeta(z, Sz) = 3 > 0,$$

$$\begin{aligned} M(z, z_0) &= \max \left\{ \zeta(z, 0), \frac{1}{2}\zeta(z, Sz), \frac{1}{2}\zeta(z, Sz), \frac{\zeta(z, 0) + \zeta(0, Sz)}{2} \right\} \\ &= \max \left\{ |z|, \frac{3}{2}, \frac{3}{2}, \frac{|z| + |z+3|}{2} \right\} \\ &= \frac{|z| + |z+3|}{2} \end{aligned}$$

and

$$\begin{aligned} \Theta(\zeta(z, Sz)) &= e^3 \leq e^{\frac{3}{11}(|z| + |z+3|)} \\ &= [e^{\frac{|z| + |z+3|}{2}}]^{\frac{6}{11}} = [\Theta(M(z, z_0))]^\lambda. \end{aligned}$$

We also have

$$r = \inf\{\xi(z, Sz) : z \neq Sz\} = 3.$$

Therefore, all the conditions of Theorems 1–4 are satisfied. Observe that $C_{0,3} = \{-3, 3\}$ is a fixed circle and $D_{0,3} = [-3, 3]$ is a fixed disc of S .

3. Fixed Circle (Disc) Results for Hardy–Rogers Θ_c -Contraction and Khan Type Θ_c -Contraction

We start this section by introducing the concept Hardy–Rogers type Θ_c -contractions as follows:

Definition 3. Let (Y, ξ) be a metric space and $\Theta \in \Omega_1$. A mapping S is called a Hardy–Rogers type Θ_c -contraction if there exist $\lambda \in (0, 1)$ and $z_0 \in Y$ such that

$$\xi(z, Sz) > 0 \Rightarrow \Theta(\xi(z, Sz)) \leq [\Theta(M^*(z, z_0))]^\lambda, \forall z \in Y, \quad (4)$$

where $M^*(z, z_0) = \alpha\xi(z, z_0) + \beta\xi(z, Sz) + \gamma\xi(z_0, Sz_0) + \delta\xi(z, Sz_0) + \eta\xi(z_0, Sz)$ with $\alpha + \beta + \gamma + \delta + \eta < 1$ and $\alpha, \beta, \gamma, \delta, \eta \geq 0$.

The following proposition follows from Definition 3.

Proposition 2. Let (Y, ξ) be a metric space. If S is a Hardy–Rogers type Θ_c -contraction mapping with $z_0 \in Y$, then $Sz_0 = z_0$.

Proof. Assume that $Sz_0 \neq z_0$. From Definition 3 and the fact that Θ is nondecreasing, we obtain

$$\begin{aligned} \Theta(\xi(z_0, Sz_0)) &\leq [\Theta(M^*(z_0, z_0))]^\lambda \\ &< \Theta(M^*(z_0, z_0)) \\ &= \Theta(\alpha\xi(z_0, z_0) + \beta\xi(z_0, Sz_0) + \gamma\xi(z_0, Sz_0) + \delta\xi(z_0, Sz_0) + \eta\xi(z_0, Sz_0)) \\ &= \Theta((\beta + \gamma + \delta + \eta)\xi(z_0, Sz_0)) \leq \Theta(\xi(z_0, Sz_0)), \end{aligned}$$

a contradiction. Therefore, we must have $Sz_0 = z_0$. $\square$

Using a Hardy–Rogers type Θ_c -contraction, we present the following fixed circle results.

Theorem 5. Let (Y, ξ) be a metric space. Assume that S is a Hardy–Rogers type Θ_c -contraction with $z_0 \in Y$ and $\xi(z_0, Sz) = r$ for all $z \in C_{z_0, r}$. Then, $C_{z_0, r}$ is a fixed circle of S .

Proof. Let $z \in C_{z_0, r}$ and assume contrarily that $Sz \neq z$. By the definition of r , we have $\xi(z, Sz) \geq r$. Since S is a Hardy–Rogers type Θ_c -contraction, using Proposition 2 and the fact that Θ is nondecreasing, we have

$$\begin{aligned} \Theta(\xi(z, Sz)) &\leq [\Theta(M^*(z, z_0))]^\lambda \\ &< \Theta(\alpha\xi(z, z_0) + \beta\xi(z, Sz) + \gamma\xi(z_0, Sz_0) + \delta\xi(z, Sz_0) + \eta\xi(z_0, Sz)) \\ &= \Theta(\alpha r + \beta\xi(z, Sz) + \delta r + \eta r) \\ &\leq \Theta((\alpha + \beta + \delta + \eta)\xi(z, Sz)) \leq \Theta(\xi(z, Sz)), \end{aligned}$$

a contradiction. Therefore, $Sz = z$ for all $z \in C_{z_0, r}$. Consequently, $C_{z_0, r}$ is a fixed circle of S . $\square$

Next, we prove the following fixed disc result.

Theorem 6. Let (Y, ξ) be a metric space. Assume that S is a Hardy–Rogers type Θ_c -contraction with $z_0 \in Y$ and $\xi(z_0, Sz) = r$, for all $z \in D_{z_0, r}$. Then, $D_{z_0, r}$ is a fixed disc of S .

Proof. In view of Theorem 5, S fixes the circle $C_{z_0,r}$. Now, in order to show that $D_{z_0,r}$ is a fixed disc of the mapping S , it is sufficient to show that S fixes any circle $C_{z_0,\rho}$ with $\rho < r$. Let $z \in C_{z_0,\rho}$ and assume that $\xi(z, Sz) > 0$. Since S is a Hardy–Rogers type Θ_c -contraction, using Proposition 2 and the fact that Θ is nondecreasing, we have

$$\begin{aligned}\Theta(\xi(z, Sz)) &\leq [\Theta(M^*(z, z_0))]^\lambda \\ &< \Theta(\alpha\xi(z, z_0) + \beta\xi(z, Sz) + \gamma\xi(z_0, Sz_0) + \delta\xi(z, Sz_0) + \eta\xi(z_0, Sz)) \\ &= \Theta(\alpha\rho + \beta\xi(z, Sz) + \delta\rho + \eta\rho) \\ &\leq \Theta((\alpha + \beta + \delta + \eta)\xi(z, Sz)) \leq \Theta(\xi(z, Sz)),\end{aligned}$$

a contradiction. Thus, we obtain $Sz = z$, and $D_{z_0,\rho}$ is a fixed disc of S . $\square$

From Theorems 5 and 6, we deduce the following corollaries.

Corollary 2. Let (Y, ξ) be a metric space and S a mapping. If there exists $\lambda \in (0, 1)$ and $z_0 \in Y$ with $\xi(z_0, Sz) = r$, for all $z \in C_{z_0,r}$ (or $D_{z_0,r}$) such that

$$\xi(z, Sz) > 0 \Rightarrow \Theta(\xi(z, Sz)) \leq [\Theta(\alpha\xi(z, z_0) + \beta\xi(z, Sz) + \gamma\xi(z_0, Sz_0))]^\lambda, \forall z \in Y,$$

where $\alpha + \beta + \gamma < 1$ and $\alpha, \beta, \gamma \geq 0$, then $C_{z_0,r}$ (or $D_{z_0,r}$) is a fixed circle (or disc) of S .

Proof. The claim follows from Theorems 5 and 6 if we take $\alpha = \beta = \gamma = 0$ and $\delta = \eta$ in the contractive condition (4). $\square$

Corollary 3. Let (Y, ξ) be a metric space and S a mapping. If there exists $\lambda \in (0, 1)$ and $z_0 \in Y$ with $\xi(z_0, Sz) = r$, for all $z \in C_{z_0,r}$ (or $D_{z_0,r}$) such that

$$\xi(z, Sz) > 0 \Rightarrow \Theta(\xi(z, Sz)) \leq [\Theta(\eta(\xi(z, Sz_0) + \xi(z_0, Sz)))]^\lambda, \forall z \in Y,$$

where $\eta \in (0, \frac{1}{2})$, then $C_{z_0,r}$ (or $D_{z_0,r}$) is a fixed circle (or disc) of S .

Proof. The claim follows from Theorems 5 and 6 if we put $\eta = \delta = 0$ in the contractive condition (4). $\square$

The following example exhibits the utility of Theorems 5 and 6.

Example 2. Let $Y = \{3, 4, \ln(\frac{3}{e}), \ln(3), \ln(3e)\}$ be endowed with the usual metric ξ . Define $S : Y \rightarrow Y$ as

$$Sz = \begin{cases} 4, & \text{if } z = 3 \\ z, & \text{otherwise.} \end{cases}$$

Let $\Theta(t) = e^t$, $z_0 = \ln 3$, $k = \frac{3}{10-3\ln 3}$, $\alpha = \delta = \eta = \frac{1}{3}$ and $\beta = 0$. Then, S is a Hardy–Rogers type Θ_c -contraction. Indeed, for $z = 3$

$$\xi(z, Sz) = \xi(3, T3) = 1 > 0,$$

$$\begin{aligned}M(z, z_0) &= \alpha\xi(z, z_0) + \beta\xi(z, Sz) + \delta\xi(z, Sz_0) + \eta\xi(z_0, Sz) \\ &= \frac{1}{3}[\xi(3, \ln 3) + \xi(3, \ln 3) + \xi(\ln 3, 4)] \\ &= \frac{10}{3} - \ln 3\end{aligned}$$

and

$$\begin{aligned}\Theta(\xi(z, Sz)) &= \Theta(\xi(3, 4)) = e \leq [e^{\frac{10}{3} - \ln 3}]^{\frac{3}{10 - 3 \ln 3}} \\ &= [\Theta(M(z, z_0))]^\lambda.\end{aligned}$$

We also have

$$r = \inf\{\xi(z, Sz) : z \neq Sz\} = \{\xi(3, 4)\} = 1.$$

Hence, all the conditions of Theorems 5 and 6 are satisfied. Observe that S fixes the circle $C_{\ln 3, 1} = \{\ln(\frac{3}{e}), \ln(3e)\}$ and the disc $D_{\ln 3, 1} = \{\ln(\frac{3}{e}), \ln 3, \ln(3e)\}$.

We close this section by introducing the concept of the Khan type Θ_c -contraction followed by the related fixed circle (disc) results.

Definition 4. Let (Y, ξ) be a metric space and $\Theta \in \Omega_1$. A mapping S is called a Khan type Θ_c -contraction if there exist $\lambda \in (0, 1)$ and $z_0 \in Y$ such that for all $z \in Y$, if $\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} \neq 0$, then $Sz \neq z$ and

$$\Theta(\xi(Sz, z)) \leq \left[\Theta \left(h \frac{\xi(Sz, z)\xi(Sz_0, z) + \xi(Sz_0, z_0)\xi(Sz, z_0)}{\max\{\xi(Sz_0, z_0), \xi(Sz, z)\}} \right) \right]^\lambda,$$

where $h \in (0, \frac{1}{2})$ and if $\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} = 0$, then $Sz = z$.

The following proposition is a direct consequence of Definition 4.

Proposition 3. Let (Y, ξ) be a metric space. If S is a Khan type Θ_c -contraction with $z_0 \in Y$, then $Sz_0 = z_0$.

Proof. Assume that $Sz_0 \neq z_0$; then, $\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} \neq 0$. As S is a Khan type Θ_c -contraction, we have

$$\begin{aligned}\Theta(\xi(Sz_0, z_0)) &\leq \left[\Theta \left(h \frac{\xi(Sz_0, z_0)\xi(Sz_0, z_0) + \xi(Sz_0, z_0)\xi(Sz_0, z_0)}{\xi(Sz_0, z_0)} \right) \right]^\lambda \\ &= [\Theta(2h\xi(Sz_0, z_0))]^\lambda \\ &< \Theta(2h\xi(Sz_0, z_0)) \leq \Theta(\xi(Sz_0, z_0)),\end{aligned}$$

a contradiction. Therefore, we must have $Sz_0 = z_0$. $\square$

Now, utilizing the definition of the Khan type Θ_c -contraction, we prove the following fixed circle and fixed disc results.

Theorem 7. Let (Y, ξ) be a metric space and let $S : Y \rightarrow Y$ be a Khan type Θ_c -contraction with $z_0 \in Y$. Then, $C_{z_0, r}$ is a fixed circle of S .

Proof. Let $z \in C_{z_0, r}$ and assume contrarily that $Sz \neq z$; then, $\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} \neq 0$. From the definition of r , we have $\xi(z, Sz) \geq r$. As S is a Khan type Θ_c -contraction, using Proposition 3 and the fact that Θ is nondecreasing, we have

$$\begin{aligned}\Theta(\xi(Sz, z)) &\leq \left[\Theta \left(h \frac{\xi(Sz, z)\xi(Sz_0, z) + \xi(Sz_0, z_0)\xi(Sz, z_0)}{\xi(Sz, z)} \right) \right]^\lambda \\ &= [\Theta(hr)]^\lambda < \Theta(hr) \leq \Theta(h\xi(Sz, z)) \leq \Theta(\xi(Sz, z)),\end{aligned}$$

a contradiction. Therefore, $Sz = z$ for all $z \in C_{z_0, r}$. Consequently, $C_{z_0, r}$ is a fixed circle of S . $\square$

Theorem 8. Let (Y, ξ) be a metric space and let $S : Y \rightarrow Y$ be a Khan type Θ_c -contraction with $z_0 \in Y$. Then, $D_{z_0, r}$ is a fixed disc of S .

Proof. In view of Theorem 7, S fixes the circle $C_{z_0, r}$. Now, in order to show that $D_{z_0, r}$ is a fixed disc of the mapping S , it is sufficient to show that S fixes any circle $C_{z_0, \rho}$ with $\rho < r$. Let $z \in C_{z_0, \rho}$ and assume that $\xi(z, Sz) > 0$. By the Khan type Θ_c -contractive condition, we have

$$\Theta(\xi(z, Sz)) \leq [\Theta(h\rho)]^\lambda < \Theta(h\rho) \leq \Theta(\rho).$$

As Θ is a nondecreasing function, we get

$$\xi(z, Sz) < \rho < r,$$

a contradiction (as $r \leq \xi(z, Sz)$). Thus, we obtain $Sz = z$ for all $z \in C_{z_0, \rho}$ with $\rho < r$. Therefore, $D_{z_0, r}$ is a fixed disc of S . $\square$

The following example shows the utility of Theorems 7 and 8.

Example 3. Let $X = \mathbb{R}$ be endowed with the usual metric ξ . Define $S : \mathbb{R} \rightarrow \mathbb{R}$ by

$$Sz = \begin{cases} z, & \text{if } |z| < 4.5, \\ z + 1, & \text{if } |z| \geq 4.5. \end{cases}$$

Then, S is a Khan type Θ -contraction with $\Theta(t) = e^t$, $z_0 = 0$, $k = \frac{2}{3}$ and $h = \frac{1}{3}$. In fact,

$$\max\{\xi(Sz_0, z_0), \xi(Sz, z)\} = 1 \neq 0,$$

for all $z \in \mathbb{R}$ such that $|z| \geq 4.5$. Now, we have

$$h \frac{\xi(Sz, z)\xi(Sz_0, z) + \xi(Sz_0, z_0)\xi(Sz, z_0)}{\xi(Sz, z)} = h\xi(0, z) = \frac{1}{3}|z|$$

and

$$\begin{aligned} \Theta(\xi(Sz, z)) &= e \leq e^{\frac{2}{3}|z|} \\ &= [e^{\frac{1}{3}|z|}]^{\frac{2}{3}} = \left[\Theta\left(h \frac{\xi(Sz, z)\xi(Sz_0, z) + \xi(Sz_0, z_0)\xi(Sz, z_0)}{\xi(Sz, z)}\right) \right]^\lambda. \end{aligned}$$

We also have

$$r = \inf\{\xi(z, Sz) : z \neq Sz\} = 1.$$

Therefore, all the conditions of Theorems 7 and 8 are satisfied. Observe that S fixes the circle $C_{0,1} = \{-1, 1\}$ and the disc $D_{0,1} = [-1, 1]$.

4. Fixed Circle (Disc) Results of Integral Type

In this section, we establish some fixed circle and disc results of the integral type.

Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a locally integrable function such that

$$\int_0^t \varphi(s) ds > 0, \quad \text{for all } t > 0. \quad (5)$$

Definition 5. Let $S : Y \rightarrow Y$ and $\Theta \in \Omega_1$. The mapping S is called an integral type Θ_c -contraction if there exist $\lambda \in (0, 1)$ and $z_0 \in Y$ such that for all $z \in Y$

$$\xi(z, Sz) > 0 \Rightarrow \int_0^{\Theta(\xi(z, Sz))} \varphi(t) dt \leq \int_0^{[\Theta(\xi(z, z_0))]^\lambda} \varphi(t) dt.$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a function defined as in Equation (5).

The following proposition is useful in the proof of the main results of this section.

Proposition 4. Let (Y, ξ) be a metric space and $S : Y \rightarrow Y$. If S is an integral type Θ_c -contraction with $z_0 \in Y$, then $Sz_0 = z_0$.

Proof. Assume that $Sz_0 \neq z_0$. From Definition 5, we have

$$\int_0^{\Theta(\xi(z_0, Sz_0))} \varphi(t) dt \leq \int_0^{[\Theta(\xi(z_0, z_0))]^\lambda} \varphi(t) dt,$$

which contradicts the definition of Θ , as $\Theta : (0, \infty) \rightarrow (1, \infty)$ and $\xi(z_0, z_0) = 0$. Hence, we must have $Sz_0 = z_0$. $\square$

In the following theorem, we present a fixed circle result for integral type Θ_c -contraction.

Theorem 9. Let (Y, ξ) be a metric space and $S : Y \rightarrow Y$. If S is an integral type Θ_c -contraction with $z_0 \in Y$, then $C_{z_0, r}$ is a fixed circle of S .

Proof. Assume that $z \neq Sz$ for some $z \in C_{z_0, r}$. Making use of the definition of r , we have

$$r \leq \xi(z, Sz).$$

Since Θ is a nondecreasing function, we have

$$\Theta(r) \leq \Theta(\xi(z, Sz))$$

and

$$\int_0^{\Theta(r)} \varphi(t) dt \leq \int_0^{\Theta(\xi(z, Sz))} \varphi(t) dt. \quad (6)$$

As S is an integral type Θ_c -contraction, using Equation (6), we obtain

$$\begin{aligned} \int_0^{\Theta(r)} \varphi(t) dt &\leq \int_0^{\Theta(\xi(z, Sz))} \varphi(t) dt \leq \int_0^{[\Theta(\xi(z, z_0))]^\lambda} \varphi(t) dt \\ &< \int_0^{\Theta(\xi(z, z_0))} \varphi(t) dt = \int_0^{\Theta(r)} \varphi(t) dt, \end{aligned}$$

a contradiction. Therefore, we have $Sz = z$. Consequently, $C_{z_0, r}$ is a fixed circle of S . $\square$

Next, we prove the following fixed disc results.

Theorem 10. Let (Y, ξ) be a metric space, let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be defined as in Equation (5) and let S be an integral type Θ_c -contraction with $z_0 \in Y$. Then, $D_{z_0, r}$ is a fixed disc of S .

Proof. In view of Theorem 9, the mapping S fixes the circle $C_{z_0, r}$. Now, in order to show that $D_{z_0, r}$ is a fixed disc of the mapping S , it is sufficient to show that S fixes any circle $C_{z_0, \rho}$ with $\rho < r$. Let $z \in C_{z_0, \rho}$ and assume that $z \neq Sz$. Making use of the definition of r , we have

$$\rho < r \leq \xi(z, Sz).$$

Since Θ is a nondecreasing function, we have

$$\Theta(\rho) \leq \Theta(\xi(z, Sz))$$

and

$$\int_0^{\Theta(\rho)} \varphi(t) dt \leq \int_0^{\Theta(\xi(z, Sz))} \varphi(t) dt. \quad (7)$$

As S is an integral type Θ_c -contraction, using Equation (7), we obtain

$$\begin{aligned} \int_0^{\Theta(\rho)} \varphi(t) dt &\leq \int_0^{\Theta(\xi(z, Sz))} \varphi(t) dt \leq \int_0^{[\Theta(\xi(z, z_0))]^\lambda} \varphi(t) dt \\ &< \int_0^{\Theta(\xi(z, z_0))} \varphi(t) dt = \int_0^{\Theta(\rho)} \varphi(t) dt, \end{aligned}$$

a contradiction. Therefore, we must have $Sz = z, \forall z \in C_{z_0, \rho}$. Consequently, $D_{z_0, r}$ is a fixed disc of S . $\square$

Remark 2. Using similar arguments as in Definition 5, we can define the notions of an integral type Ćirić type Θ_c -contraction, an integral Hardy–Rogers type Θ_c -contraction and an integral Khan type Θ_c -contraction, and we can obtain corresponding fixed circle and fixed disc theorems.

5. Conclusions

We have obtained some new generalized fixed circle and fixed disc results using our newly introduced contractive conditions in the setting of metric spaces. Precisely, by means of some known techniques which usually are used to obtain some fixed point theorems, we have generated some useful fixed circle and fixed disc results.

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