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Local and Semilocal Convergence of Nourein's Iterative Method for Finding All Zeros of a Polynomial Simultaneously

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Abstract: In 1977, Nourein (Intern. J. Comput. Math. 6:3, 1977) constructed a fourth-order iterative method for finding all zeros of a polynomial simultaneously. This method is also known as Ehrlich's method with Newton's correction because it is obtained by combining Ehrlich's method (Commun. ACM 10:2, 1967) and the classical Newton's method. The paper provides a detailed local convergence analysis of a well-known but not well-studied generalization of Nourein's method for simultaneous finding of multiple polynomial zeros. As a consequence, we obtain two types of local convergence theorems as well as semilocal convergence theorems (with verifiable initial condition and a posteriori error bound) for the classical Nourein's method. Each of the new semilocal convergence results improves the result of Petković, Petković and Rančić (J. Comput. Appl. Math. 205:1, 2007) in several directions. The paper ends with several examples that show the applicability of our semilocal convergence theorems.

Keywords: iterative methods; Nourein's method; polynomial zeros; local convergence; semilocal convergence; error estimates

1. Introduction

This paper deals with the convergence of two iterative methods for finding all zeros of a polynomial simultaneously. The first one is due to Nourein [1] and it has quartic convergence when all zeros of the polynomial are simple and has linear convergence otherwise. The second one is a generalization of Nourein's method for simultaneously finding all zeros of an arbitrary polynomial that has at least one multiple zero. To our knowledge, Nourein's method for multiple zeros appears for the first time in the book of Sendov, Andreev and Kjurkchiev [2].

Throughout this paper, $(\mathbb{K}, |\cdot|)$ stands for a valued field with absolute value $|\cdot|$ and $\mathbb{K}[z]$ denotes the ring of polynomials over \mathbb{K} .

Weierstrass' method and elementary symmetric functions. Historically, the first iterative method for simultaneous finding all zeros of a polynomial was constructed by Weierstrass [3] in 1891. Let

$$f(z) = c_0 z^n + c_1 z^{n-1} + \dots + c_n \quad (1)$$

be a polynomial in $\mathbb{K}[z]$ of degree $n \geq 2$. Furthermore, let us define the elementary symmetric functions $\varphi_\nu: \mathbb{K}^n \rightarrow \mathbb{K}$, $\nu = 1, 2, \dots, n$ as follows:

$$\varphi_\nu(x_1, \dots, x_n) = (-1)^\nu \sum_{1 \leq j_1 < \dots < j_\nu \leq n} x_{j_1} \dots x_{j_\nu}.$$

It is well-known that a vector $\xi = (\xi_1, \dots, \xi_n)$ is a solution of the symmetric system

$$\varphi_\nu(x_1, \dots, x_n) = \frac{c_\nu}{c_0}, \quad \nu = 1, \dots, n, \quad (2)$$

if and only if ξ_1, \dots, ξ_n are all zeros of the polynomial f . Using this fact, Weierstrass [3] derived an iterative algorithm for solving the symmetric system (2). Weierstrass's method is defined by the following iteration:

$$x^{(k+1)} = x^{(k)} - W_f(x^{(k)}), \quad k = 0, 1, 2, \dots, \quad (3)$$

where the so-called *Weierstrass correction* $W_f: \mathcal{D} \subset \mathbb{K}^n \rightarrow \mathbb{K}^n$ is defined by

$$W_f(x) = (W_1(x), \dots, W_n(x)) \quad \text{with} \quad W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)} \quad (i = 1, \dots, n) \quad (4)$$

and \mathcal{D} is the set of all vectors in \mathbb{K}^n with pairwise distinct coordinates. Weierstrass's method (3) has quadratic convergence when all zeros of f are simple and has linear convergence otherwise. For historical notes and advanced results for Weierstrass's method, we refer to [4,5].

Ehrlich's method. In 1967, Ehrlich [6] introduced a third-order simultaneous method defined by the following iteration:

$$x^{(k+1)} = G(x^{(k)}), \quad k = 0, 1, 2, \dots, \quad (5)$$

where the iteration function $G: D_G \subset \mathbb{K}^n \rightarrow \mathbb{K}^n$ is defined by $G(x) = (G_1(x), \dots, G_n(x))$ with

$$G_i(x) = x_i - \frac{f(x_i)}{f'(x_i) - f(x_i) \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{x_i - x_j}} \quad (i = 1, \dots, n).$$

In 1975, Börsch-Supan [7] presented another third-order simultaneous method. In 1982, Werner [8] proved that both Ehrlich's and Börsch-Supan's methods are identical. For historical notes and advanced results for Ehrlich's method, we refer to [9–11].

Local and semilocal convergence analysis. Recently, a general convergence theory of iterative methods of the type $x^{(k+1)} = T(x^{(k)})$, where $T: D \subset X \rightarrow X$ is an iteration function of a metric space X , was developed in [12,13]. Central to this theory is the concept of the *function of initial approximations* (see ([13], Section 3)). Roughly speaking, this is a real-valued function $E: D \subset X \rightarrow \mathbb{R}_+$ that sets the initial conditions. The initial condition of any convergence theorem of an iterative method can be represented in the form

$$E(x^{(0)}) \in J, \quad (6)$$

where J is an interval on \mathbb{R} of the form $[0, R]$, $[0, R)$ or $[0, +\infty)$, where R is a positive number.

Convergence analysis of an iterative method always is done with respect to a function of the initial conditions E . The goal of convergence is to find initial guesses that guarantees convergence of the iteration sequence to $x^{(k+1)} = T(x^{(k)})$ to a fixed point $\xi \in D$ of the iteration function T .

Definition 1. A convergence analysis is called *semilocal* when the function of initial conditions E does not depend on the fixed point ξ and is *local* otherwise.

Semilocal convergence theorems have great practical applications because their initial conditions are computer-verifiable. In general, local convergence theorems have mainly theoretical significance. However, very recently in [14], it was proved that two kinds of local convergence theorems for iterative methods for simultaneous approximation of polynomial zeros can be transformed into semilocal convergence results.

Nourein's method for simple zeros. There are different ways to increase the convergence order of an iterative method for simultaneous computation of polynomial zeros. In 1977, Nourein [1,15] constructed three simultaneous methods that increase the convergence order of Weierstrass's, Ehrlich's and Börsch-Supan's methods. Each of these three methods was constructed as a combination of two already known iterative methods. In particular, combining Ehrlich's method and the classical Newton's method, Nourein [1] constructed in \mathbb{K}^n the following fourth-order iterative method (for simple zeros):

$$x^{(k+1)} = F(x^{(k)}), \quad k = 0, 1, 2, \dots, \quad (7)$$

where the iteration function $F: D_F \subset \mathbb{K}^n \rightarrow \mathbb{K}^n$ is defined by $F(x) = (F_1(x), \dots, F_n(x))$ with

$$F_i(x) = x_i - \frac{f(x_i)}{f'(x_i) - f(x_i) \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{x_i - x_j + f(x_j)/f'(x_j)}} \quad (i = 1, \dots, n).$$

Nourein's method (7) is also known as *Ehrlich's method with Newton's corrections*. As it is shown in Milovanović and Petković [16] and in Petković and Herceg [17], the iterative method (7) is one of the most efficient and powerful simultaneous methods with globally convergent performance.

In 1998, Petković, Herceg and Ilić [18] proved that the method (7) is convergent under the initial condition

$$\|W_f(x^{(0)})\|_\infty < \frac{\delta(x^{(0)})}{3n},$$

where the function $\delta: \mathbb{K}^n \rightarrow \mathbb{R}_+$ is defined by

$$\delta(x) = \min_{i \neq j} |x_i - x_j|.$$

In 2007, Petković, Petković and Rančić [19] (see also ([20], Theorem 3.10)) established the following improvement of this result.

Theorem 1 (Petković, Petković and Rančić [19]). *Suppose $f \in \mathbb{C}[z]$ is a polynomial of degree $n \geq 3$ with simple zeros. If an initial approximation $x^{(0)} \in \mathbb{C}^n$ with distinct coordinates satisfies the condition*

$$\|W_f(x^{(0)})\|_\infty < c_n \delta(x^{(0)}) \quad \text{with} \quad c_n = \begin{cases} 1/(2.2n + 1.9), & 3 \leq n \leq 21, \\ 1/(2.2n), & n \geq 22, \end{cases} \quad (8)$$

then Nourein's iteration (7) is convergent with an order of convergence of four.

Nourein's method for multiple zeros. Nourein's method (7) has a well-known but not well-studied generalization for the simultaneous finding of multiple polynomial zeros. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s (s is a positive integer such that $1 \leq s \leq n$) be all distinct zeros of f with multiplicity m_1, \dots, m_s ($m_1 + \dots + m_s = n$), respectively.

In what follows, we use a binary relation $\#$ on \mathbb{K}^s defined by

$$x \# y \Leftrightarrow x_i \neq y_j \text{ for all } i, j \in I_s \text{ with } i \neq j,$$

where $I_s = \{1, 2, \dots, s\}$.

Let us define Norein's method for polynomials that has at least one multiple zero. First, we define the Schröder's iteration function $N: D_N \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ by [21]:

$$N(x) = (N_1(x), \dots, N_s(x)) \quad \text{with} \quad N_i(x) = \begin{cases} x_i - m_i \frac{f(x_i)}{f'(x_i)} & \text{if } f(x_i) \neq 0, \\ x_i & \text{if } f(x_i) = 0, \end{cases} \quad (9)$$

where the domain of N is the set

$$D_N = \{x \in \mathbb{K}^s: f'(x_i) \neq 0 \text{ whenever } f(x_i) \neq 0\}. \quad (10)$$

Now the generalized Norein's method for simultaneously finding all the zeros of f is defined in \mathbb{K}^s by the following fixed-point iteration (see, e.g., ([2], Section 20) and ([22], Section 7.2)):

$$x^{(k+1)} = \Phi(x^{(k)}), \quad k = 0, 1, 2, \dots, \quad (11)$$

where the iteration function $\Phi: D_\Phi \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ is defined by

$$\Phi(x) = (\Phi_1(x), \dots, \Phi_s(x)) \quad \text{with} \quad \Phi_i(x) = \begin{cases} x_i - \frac{m_i}{\frac{f'(x_i)}{f(x_i)} - \sum_{\substack{j=1 \\ j \neq i}}^s \frac{m_j}{x_i - N_j(x)}} & \text{if } f(x_i) \neq 0, \\ x_i & \text{if } f(x_i) = 0, \end{cases} \quad (12)$$

and the domain of Φ is the set

$$D_\Phi = \{x \in D_N: x \# N(x) \text{ and } \frac{f'(x_i)}{f(x_i)} - \sum_{\substack{j=1 \\ j \neq i}}^s \frac{m_j}{x_i - N_j(x)} \neq 0 \text{ whenever } f(x_i) \neq 0\}. \quad (13)$$

Contributions. In this paper, we present a detailed local convergence analysis for generalized Norein's method (11) for multiple zeros. As a consequence of these results, we obtain two types of local convergence theorems as well as semilocal convergence theorems (with verifiable initial condition and a posteriori error bound) for the classical Norein's method (7). Each of the new semilocal convergence results improves Theorem 1 in several directions.

The paper is structured as follows: Section 2 gives some notations that are used throughout the paper without specific quoting. In Section 3, we study the local convergence of generalized Norein's iteration (11) with respect to a function of initial conditions of the first kind. In this section, the main new result is Theorem 3. In Section 4, we study the convergence of generalized Norein's iteration (11) with respect to a function of initial conditions of the second kind. The main new result of this section is Theorem 5, which plays an important role in Section 7 for obtaining new semilocal convergence results for the classical Norein's method (7).

In Section 5, we present two new local convergence results of the first kind (Theorem 6 and Corollary 1) for the classical Norein's method (7). In Section 6, we obtain three new local convergence results of the second kind (Theorems 7 and 8 and Corollary 3) for the classical Norein's method (7).

To the best of authors' knowledge, the theorems given in Sections 3–6 are the first local convergence results in the literature about both Norein's methods (for simple or multiple zeros).

In Section 7, we provide three new semilocal convergence results for polynomials with simple zeros (Theorems 11 and 12 and Corollary 5). Each of these semilocal convergence results improves the result of Petković, Petković and Rančić [19] in several directions. Note that these results are based on some results given in [14] and some local convergence results obtained in the previous sections. Section 8 provides several numerical examples that show the applicability of our semilocal convergence theorems. Finally, the paper ends with a conclusion section.

2. Notations

In this short section, we give some notations that are used throughout the paper without specific quoting. We denote by \mathbb{R} and \mathbb{R}_+ the real and the nonnegative numbers, respectively.

Let \mathbb{R}^s be equipped with partial coordinate-wise ordering defined by

$$x \preceq y \quad \text{if and only if} \quad x_i \leq y_i \quad \text{for each } i \in I_s,$$

and let the vector space \mathbb{K}^s be equipped with the norm

$$\|x\|_p = \left(\sum_{i=1}^s |x_i|^p \right)^{1/p} \quad \text{for some } 1 \leq p \leq \infty,$$

and with a vector norm $\|\cdot\|$ (with values in \mathbb{R}^s) defined by

$$\|x\| = (|x_1|, \dots, |x_s|).$$

Let $x \in \mathbb{K}^s$ and $y \in \mathbb{R}^s$ be two vectors. We denote by $\frac{x}{y}$ a vector in \mathbb{R}^s defined by

$$\frac{x}{y} = \left(\frac{|x_1|}{y_1}, \dots, \frac{|x_s|}{y_s} \right)$$

if y has only nonzero coordinates. We define a function $d: \mathbb{K}^s \rightarrow \mathbb{R}^s$ by

$$d(x) = (d_1(x), \dots, d_s(x)) \quad \text{with} \quad d_i(x) = \min_{j \neq i} |x_i - x_j| \quad (i = 1, \dots, s).$$

Also, we define a function $\delta: \mathbb{K}^n \rightarrow \mathbb{R}_+$ by

$$\delta(x) = \min_{i \neq j} |x_i - x_j|. \quad (14)$$

We assume by definition that $0^0 = 1$. For two integers $k \geq 0$ and $r \geq 0$, we define the quantity $S_k(r)$ by

$$S_k(r) = \sum_{j=0}^{k-1} r^j \quad \text{if } k \geq 1,$$

and $S_k(r) = 0$ if $k = 0$. In the short, we write

$$\sum_{j \neq i} \quad \text{instead of} \quad \sum_{\substack{j=1 \\ j \neq i}}^s.$$

Throughout the paper, J denotes an interval on \mathbb{R}_+ containing 0, that is, an interval of the form $[0, R]$, $[0, R)$ or $[0, +\infty)$, where R is a positive number.

Definition 2 ([13]). A function $\varphi: J \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be quasi-homogeneous of degree $u \geq 0$ if

$$\varphi(\lambda t) \leq \lambda^u \varphi(t) \quad \text{for all } \lambda \in [0, 1] \text{ and } t \in J.$$

3. Local Convergence Theorem of the First Kind for Multiple Zeros

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. In this section, we investigate the local convergence

of Nouredin's iteration (11) with respect to a function of initial conditions $E: \mathbb{K}^s \rightarrow \mathbb{R}_+$ defined as follows:

$$E(x) = \left\| \frac{x - \xi}{d(\xi)} \right\|_p \quad (1 \leq p \leq \infty), \quad (15)$$

where $\xi = (\xi_1, \dots, \xi_s)$. The function of initial conditions (15) has been used in [23–25] for studying the local convergence of the first kind of some iterative methods for simultaneous approximation of multiple polynomial zeros.

We define the quantities $m = m(m_1, \dots, m_s)$, $a = a(p, m_1, \dots, m_s)$ and $b = b(p)$ as follows:

$$m = \min_{1 \leq i \leq s} m_i \quad \text{and} \quad a = \max_{1 \leq i \leq s} \frac{1}{m_i} \left(\sum_{j \neq i} m_j^q \right)^{1/q} \quad \text{and} \quad b = 2^{1/q}, \quad (16)$$

where $1 \leq q \leq \infty$ is defined by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We begin this section with some useful inequalities in \mathbb{K}^s which play an important role in the proofs of our results.

Lemma 1 ([5], Lemma 6.1). *Let $x, \xi \in \mathbb{K}^s$ and ξ have pairwise distinct coordinates. Then for all $i \neq j$, the following inequalities hold:*

$$|x_i - x_j| \geq (1 - bE(x)) d_i(\xi) \quad \text{and} \quad |x_i - \xi_j| \geq (1 - E(x)) d_i(\xi),$$

where $E: \mathbb{K}^s \rightarrow \mathbb{R}_+$ is defined by (15) and b is defined by (16).

Lemma 2. *Let $\alpha \geq 0$ and $x, y, \xi \in \mathbb{K}^s$ be three vectors such that*

$$\|y - \xi\| \leq \alpha \|x - \xi\|. \quad (17)$$

If ξ is a vector with pairwise distinct coordinates, then for all $i \neq j$, we have

$$|x_i - y_j| \geq (1 - (1 + \alpha)E(x)) d_j(\xi),$$

where $E: \mathbb{K}^s \rightarrow \mathbb{R}_+$ is defined by (15).

Proof. From the triangle inequality in \mathbb{K} and the inequality (17), we obtain

$$\begin{aligned} |x_i - y_j| &\geq |\xi_i - \xi_j| - |x_i - \xi_i| - |y_j - \xi_j| \\ &\geq |\xi_i - \xi_j| - |x_i - \xi_i| - \alpha |x_j - \xi_j| \\ &\geq \left(1 - \frac{|x_i - \xi_i|}{d_i(\xi)} - \alpha \frac{|x_j - \xi_j|}{d_j(\xi)} \right) |\xi_i - \xi_j| \\ &\geq \left(1 - (1 + \alpha) \left\| \frac{x - \xi}{d(\xi)} \right\|_p \right) d_j(\xi), \end{aligned}$$

which completes the proof. \square

The following general convergence theorem plays a substantial role in our paper.

Theorem 2 (Proinov [26]). *Let $T: D \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ be an iteration function, and let $\xi \in \mathbb{K}^s$ be a vector with pairwise distinct coordinates, and let a function $E: \mathbb{K}^s \rightarrow \mathbb{R}_+$ be defined by (15). Suppose $\phi: J \rightarrow \mathbb{R}_+$ is a*

quasi-homogeneous function of degree $u \geq 0$ such that, for every vector $x \in \mathbb{K}^s$ with $E(x) \in J$, the following conditions hold:

$$x \in D \quad \text{and} \quad \|Tx - \xi\| \preceq \phi(E(x)) \|x - \xi\|.$$

Let $x^{(0)} \in \mathbb{K}^s$ be an initial guess such that

$$E(x^{(0)}) \in J \quad \text{and} \quad \phi(E(x^{(0)})) < 1.$$

Then the Picard iteration $x^{(k+1)} = T(x^{(k)})$ is well-defined and converges to ξ with order $r = u + 1$ and with error estimates

$$\|x^{(k+1)} - \xi\| \preceq \lambda^k \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \lambda^{S_k(r)} \|x^{(0)} - \xi\| \quad \text{for all } k \geq 0,$$

where $\lambda = \phi(E(x^{(0)}))$.

Before formulating the main result of this section, we need a few more lemmas.

Lemma 3 ([12], Lemma 4.4). Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. Suppose a vector $x \in \mathbb{K}^s$ satisfies

$$E(x) < \frac{m}{n},$$

where the function $E: \mathbb{K}^s \rightarrow \mathbb{R}_+$ is defined by (15) and m is given by (16). Then

$$x \in D_N \quad \text{and} \quad \|N(x) - \xi\| \preceq \frac{(n-m)E(x)}{m-nE(x)} \|x - \xi\|,$$

where the iteration function $N: D_N \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ is defined by (9).

Lemma 4. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. Suppose a vector $x \in D_\Phi$ is such that $f(x_i) \neq 0$ for some i . Then

$$\Phi_i(x) - \xi_i = -\frac{\sigma_i}{1 - \sigma_i}(x_i - \xi_i), \quad (18)$$

where the iteration function $\Phi: D_\Phi \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ is given by (12) and $\sigma_i \in \mathbb{K}$ is given by

$$\sigma_i = \frac{x_i - \xi_i}{m_i} \sum_{j \neq i} \frac{m_j(N_j(x) - \xi_j)}{(x_i - \xi_j)(x_i - N_j(x))}. \quad (19)$$

Proof. Taking into account that ξ_1, \dots, ξ_s are the zeros of f with multiplicity m_1, \dots, m_s , we get

$$\begin{aligned} \frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i} \frac{m_j}{x_i - N_j(x)} &= \frac{m_i}{x_i - \xi_i} + \sum_{j \neq i} \frac{m_j}{x_i - \xi_j} - \sum_{j \neq i} \frac{m_j}{x_i - N_j(x)} \\ &= \frac{m_i}{x_i - \xi_i} - \sum_{j \neq i} \frac{m_j(N_j(x) - \xi_j)}{(x_i - \xi_j)(x_i - N_j(x))} \\ &= \frac{m_i(1 - \sigma_i)}{x_i - \xi_i}, \end{aligned} \quad (20)$$

where σ_i is given by (19). From (12) and (20), we have

$$\Phi_i(x) - \xi_i = x_i - \xi_i - \frac{x_i - \xi_i}{1 - \sigma_i} = -\frac{\sigma_i}{1 - \sigma_i}(x_i - \xi_i),$$

which proves (18). \square

In this and the next section, we use the following polynomial functions:

$$A(t) = (1-t)(m - (n+m)t + mt^2) - a(n-m)t^3, \quad (21)$$

$$B(t) = (1-t)(m - (n+m)t + mt^2) - 2a(n-m)t^3, \quad (22)$$

$$Q(t) = (1-bt)(1-t)(m - (n+m)t + mt^2) - 2a(n-m)t^3, \quad (23)$$

where m , a and b are given by (16).

It is easy to show that each of the functions A and B strictly decreases on \mathbb{R} and has a unique zero which lies in the interval $(0, m/n]$. If $\eta \in (0, m/n)$ is the unique solution of the equation $A(t) = 0$, then it can be shown that

$$A(t) > Q(t) \quad \text{for every } t \in (0, \eta). \quad (24)$$

Lemma 5. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. Suppose $x \in \mathbb{K}^s$ is such that

$$A(E(x)) > 0, \quad (25)$$

where the function $E: \mathbb{K}^s \rightarrow \mathbb{R}_+$ is defined by (15) and the function A is defined by (21). Then $x \in D_\Phi$ and

$$\|\Phi(x) - \xi\| \preceq \phi(E(x)) \|x - \xi\|, \quad (26)$$

where the function ϕ is defined by

$$\phi(t) = \frac{a(n-m)t^3}{(1-t)(m - (n+m)t + mt^2) - a(n-m)t^3} \quad (27)$$

with a and m defined by (16).

Proof. First we prove that $x \in D_\Phi$. It follows from (25) that $E(x) < m/n$, which according to Lemma 3 shows that $x \in D_N$. We can apply Lemma 3 with $y = N(x)$ since the inequality (17) holds with

$$\alpha = \frac{(n-m)E(x)}{m - nE(x)}. \quad (28)$$

Then by Lemma 2 and (25), we obtain

$$|x_i - N_j(x)| \geq \left(1 - \frac{m(1-E(x))}{m - nE(x)} E(x)\right) d_j(\xi) = \frac{m - (n+m)E(x) + mE(x)^2}{m - nE(x)} d_j(\xi) > 0 \quad (29)$$

for all $i \neq j$. Consequently, $x \# N(x)$. Now suppose $f(x_i) \neq 0$. According to (13), it remains to prove that

$$\frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i} \frac{m_j}{x_i - N_j(x)} \neq 0. \quad (30)$$

Using (20), we get that (30) holds true if and only if $\sigma_i \neq 1$, where σ_i is given by (19). By the triangle inequality, Lemma 3, the second part of Lemma 1, the inequality (29), Hölder's inequality and condition (25), we obtain for σ_i the following estimate:

$$\begin{aligned} |\sigma_i| &\leq \frac{|x_i - \xi_i|}{d_i(\xi)} \frac{(n-m)E(x)}{(1-E(x))(m-(n+m)E(x)-mE(x)^2)} \frac{1}{m_i} \sum_{j \neq i} \frac{m_j |x_j - \xi_j|}{d_j(\xi)} \\ &\leq \frac{a(n-m)E(x)^3}{(1-E(x))(m-(n+m)E(x)+mE(x)^2)} < 1, \end{aligned} \quad (31)$$

which yields $\sigma_i \neq 1$. Hence, $x \in D_\Phi$. To prove (26), we have to show that

$$|\Phi_i(x) - \xi_i| \leq \phi(E(x)) |x_i - \xi_i| \quad \text{for every } i = 1, \dots, s. \quad (32)$$

If $x_i = \xi_i$, then $\Phi_i(x) = \xi_i$ and so (32) becomes an equality. Suppose $x_i \neq \xi_i$. From Lemma 4 and the triangle inequality, we obtain

$$\begin{aligned} |\Phi_i(x) - \xi_i| &\leq \frac{|\sigma_i|}{1 - |\sigma_i|} |x_i - \xi_i| \\ &\leq \frac{a(n-m)E(x)^3}{(1-E(x))(m-(n+m)E(x)+mE(x)^2) - a(n-m)E(x)^3} |x_i - \xi_i| \\ &= \phi(E(x)) |x_i - \xi_i|, \end{aligned}$$

which completes the proof. \square

We are ready now to state the first main result of this paper.

Theorem 3. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. Suppose $x^{(0)} \in \mathbb{K}^s$ is an initial guess satisfying the following condition:

$$B(E(x^{(0)})) > 0, \quad (33)$$

where the functions E and B are defined by (15) and (22), respectively. Then Nourin's iteration (11) is well-defined and converges with fourth-order to the root vector $\xi = (\xi_1, \dots, \xi_s)$ with the following error estimates:

$$\|x^{(k+1)} - \xi\| \leq \lambda^{4^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \leq \lambda^{(4^k-1)/3} \|x^{(0)} - \xi\| \quad \text{for all } k \geq 0, \quad (34)$$

where $\lambda = \phi(E(x^{(0)}))$ and the function ϕ is defined by (27).

Proof. We shall apply Theorem 2 to the iteration function $\Phi: D_\Phi \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ defined by (12). Let $\eta > 0$ be the unique solution of the equation $A(t) = 0$. The function ϕ is quasi-homogeneous of degree $m = 3$ on $[0, \eta)$. It follows from Lemma 5 that, for every vector $x \in \mathbb{K}^s$ with $E(x) < \eta$, we have $x \in D_\Phi$ and that the inequality (26) holds. Then it follows from Theorem 2 that, under the initial condition

$$E(x^{(0)}) < \eta \quad \text{and} \quad \phi(E(x^{(0)})) < 1,$$

the iteration (11) is well-defined and converges to ξ with order $r = 4$ and with error estimates (34). It is easy to see that the above initial condition is equivalent to (33). This completes the proof. \square

4. Local Convergence Theorem of the Second Kind for Multiple Zeros

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. In this section, we investigate the local convergence

of Nouredin's iteration (11) with respect to the function of initial conditions $E: \mathcal{D} \rightarrow \mathbb{R}_+$ defined as follows:

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p \quad (1 \leq p \leq \infty), \quad (35)$$

where $\xi = (\xi_1, \dots, \xi_s)$. Recall that \mathcal{D} denotes the set of all vectors in \mathbb{K}^s with pairwise distinct coordinates. We note that the function of initial conditions (35) has been used in [23–25] to study the local convergence of the second kind of some iterative methods for finding simultaneously multiple polynomial zeros.

Lemma 6 ([5], Lemma 7.1). *Let $x, \xi \in \mathbb{K}^s$. If x has pairwise distinct coordinates, then for all $i \neq j$, the following inequalities hold:*

$$|x_i - \xi_j| \geq (1 - E(x)) d_i(x) \quad \text{and} \quad |x_i - x_j| \geq d_j(x)$$

where $E: \mathcal{D} \rightarrow \mathbb{R}_+$ is defined by (35).

Lemma 7 ([27], Lemma 3.4). *Let $\alpha \geq 0$ and $x, y, \xi \in \mathbb{K}^s$ be three vectors satisfying (17). If x is a vector with pairwise distinct coordinates, then for all $i \neq j$, we have*

$$|x_i - y_j| \geq (1 - (1 + \alpha)E(x)) d_j(x),$$

where $E: \mathcal{D} \rightarrow \mathbb{R}_+$ is defined by (35).

Theorem 4 (Proinov [26]). *Let $T: D \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ be an iteration function, $\xi \in \mathbb{K}^s$ be a vector, and let a function $E: \mathcal{D} \subset \mathbb{K}^s \rightarrow \mathbb{R}_+$ be defined by (35). Suppose $\beta: J \rightarrow \mathbb{R}_+$ is a nonzero quasi-homogeneous function of degree $u \geq 0$ such that, for any $x \in \mathcal{D}$ with $E(x) \in J$, we have*

$$x \in D \quad \text{and} \quad \|Tx - \xi\| \preceq \beta(E(x)) \|x - \xi\|.$$

Let $x^{(0)} \in \mathcal{D}$ be an initial guess such that

$$E(x^{(0)}) \in J \quad \text{and} \quad \Psi(E(x^{(0)})) \geq 0, \quad (36)$$

where the function $\Psi: J \rightarrow \mathbb{R}$ is defined by

$$\Psi(t) = 1 - bt - \beta(t)(1 + bt).$$

Then the Picard iteration $x^{(k+1)} = T(x^{(k)})$ is well-defined and converges to ξ with error estimates

$$\|x^{(k+1)} - \xi\| \preceq \theta \lambda^{r^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \theta^k \lambda^{S_k(r)} \|x^{(0)} - \xi\| \quad \text{for all } k \geq 0,$$

where $r = u + 1$, $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$ and the functions ψ and ϕ are defined by

$$\psi(t) = 1 - bt(1 + \beta(t)) \quad \text{and} \quad \phi(t) = \beta(t)/\psi(t).$$

Besides, if the inequality in (36) is strict, then the order of convergence is at least r .

Now, we define the real functions β, Ψ, ψ , as follows:

$$\beta(t) = \frac{a(n-m)t^3}{(1-t)(m-(n+m)t+mt^2)-a(n-m)t^3}, \quad (37)$$

$$\Psi(t) = 1 - bt - \beta(t)(1 + bt) = \frac{(1 - bt)(1 - t)(m - (n + m)t + mt^2) - 2a(n - m)t^3}{(1 - t)(m - (n + m)t + mt^2) - a(n - m)t^3}, \quad (38)$$

$$\psi(t) = 1 - bt(1 + \beta(t)) = \frac{1 - bt(1 - t)(m - (n + m)t + mt^2)}{(1 - t)(m - (n + m)t + mt^2) - a(n - m)t^3}, \quad (39)$$

where a , b and m are defined by (16). Let $\eta > 0$ be the unique zero of the function A , where A is defined by (21). It is easy to show that β strictly increases on $[0, \eta]$ and that it is quasi-homogeneous of degree $u = 3$ on $[0, \eta]$. The function Ψ strictly decreases $[0, \eta]$ and it has a unique zero on $[0, \eta]$ because $\Psi(0) = 1$ and $\lim_{t \rightarrow \eta^-} \Psi(t) = +\infty$. On the other hand, $\Psi(t) = Q(t)/A(t)$ for every $t \in [0, \eta]$, where the functions A and Q are defined by (21) and (23). Thus, we conclude that the function Q has a unique zero on $[0, \eta]$ too.

Analogously to Lemma 3, we can prove the following lemma:

Lemma 8. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. Suppose $x \in \mathcal{D}$ is a vector satisfying

$$E(x) < \frac{m}{n},$$

where the function $E: \mathcal{D} \rightarrow \mathbb{R}_+$ is defined by (35) and m is defined by (16). Then,

$$x \in D_N \quad \text{and} \quad \|N(x) - \xi\| \leq \frac{(n - m)E(x)}{m - nE(x)} \|x - \xi\|, \quad (40)$$

where the iteration function $N: D_N \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ is defined by (9).

Proof. The first part of (40) follows from the first part of Lemma 3. The second part of (40) is equivalent to

$$|N_i(x) - \xi_i| \leq \frac{(n - m)E(x)}{m - nE(x)} |x_i - \xi_i| \quad (41)$$

for every $i = 1, \dots, s$. If $x_i = \xi_i$, then (41) holds trivially. Suppose that $x_i \neq \xi_i$. Then it is easy to show that

$$N_i(x) - \xi_i = \frac{\mu_i}{1 + \mu_i} (x_i - \xi_i), \quad (42)$$

where

$$\mu_i = \frac{x_i - \xi_i}{m_i} \sum_{j \neq i} \frac{m_j}{x_i - \xi_j}.$$

From the triangle inequality and Lemma 6, we obtain

$$|\mu_i| \leq \frac{|x_i - \xi_i|}{m_i} \sum_{j \neq i} \frac{m_j}{|x_i - \xi_j|} \leq \frac{|x_i - \xi_i|}{m_i(1 - E(x))d_i(x)} \sum_{j \neq i} m_j \leq \frac{(n - m)E(x)}{m(1 - E(x))} < 1. \quad (43)$$

From (43), we obtain the following estimate:

$$|1 + \mu_i| \geq 1 - |\mu_i| \geq \frac{m - nE(x)}{m(1 - E(x))} > 0. \quad (44)$$

From (42) and the estimates (43) and (44), we get (41), which completes the proof. \square

Lemma 9. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. Suppose a vector $x \in \mathbb{K}^s$ with distinct coordinates satisfies

$$A(E(x)) > 0, \quad (45)$$

where the functions E and A are defined by (35) and (21) respectively. Then $x \in D_\Phi$ and

$$\|\Phi(x) - \xi\| \preceq \beta(E(x)) \|x - \xi\|, \quad (46)$$

where the function β is defined by (37).

Proof. The proof is carried out in the same way as the proof of Lemma 5 using Lemma 6, Lemma 7 and Lemma 9 instead of Lemma 1, Lemma 2 and Lemma 3, respectively. \square

Now, we are ready to state and prove the main result in this section.

Theorem 5. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits in \mathbb{K} , and let ξ_1, \dots, ξ_s be all distinct zeros of f with multiplicity m_1, \dots, m_s , respectively. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial guess with distinct coordinates which satisfies the following conditions:

$$A(E(x^{(0)})) > 0 \quad \text{and} \quad Q(E(x^{(0)})) \geq 0, \quad (47)$$

where the functions E , A and Q are defined by (35), (21) and (23), respectively. Then the iteration (11) is well-defined and converges to ξ with error estimates

$$\|x^{(k+1)} - \xi\| \preceq \theta \lambda^{4^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \theta^k \lambda^{(4^k - 1)/3} \|x^{(0)} - \xi\| \quad \text{for all } k \geq 0, \quad (48)$$

where $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$, $\phi = \beta/\psi$ and ψ is defined by (39). Moreover, if the second inequality in (47) is strict, then the convergence order is at least four.

Proof. We shall apply Theorem 4 to the iteration function $\Phi: D_\Phi \subset \mathbb{K}^s \rightarrow \mathbb{K}^s$ defined by (12). Let η be the unique positive solution of the equation $A(t) = 0$. The function β is quasi-homogeneous of degree $m = 3$ on $[0, \eta)$. It follows from Lemma 9 that, for every vector $x \in \mathcal{D}$ with $E(x) < \eta$, we have $x \in D_\Phi$ and the inequality (46) holds. Then it follows from Theorem 4 that under the initial condition

$$E(x^{(0)}) < \eta \quad \text{and} \quad \Psi(E(x^{(0)})) \geq 0, \quad (49)$$

the iteration (11) is well-defined and converges to ξ with order $r = 4$ and with error estimates (48). Taking into account that $\Psi(t) = Q(t)/A(t)$, we can see that the initial conditions (47) and (49) are equivalent. This completes the proof. \square

5. Local Convergence Theorem of the First Kind for Simple Zeros

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in \mathbb{K} , and let $\xi \in \mathbb{K}^n$ be a root vector of the polynomial f . In this section, we study the local convergence of the classical Norein's method (7) with respect to the function of initial conditions $E: \mathbb{K}^n \rightarrow \mathbb{R}_+$ defined by

$$E(x) = \left\| \frac{x - \xi}{d(\xi)} \right\|_p \quad (1 \leq p \leq \infty). \quad (50)$$

Now the real functions A , B , Q and ϕ , considered in Section 3, take the following forms:

$$A(t) = (1 - t)(1 - (n + 1)t + t^2) - a(n - 1)t^3, \quad (51)$$

$$B(t) = (1 - t)(1 - (n + 1)t + t^2) - 2a(n - 1)t^3, \quad (52)$$

$$Q(t) = (1 - bt)(1 - t)(1 - (n + 1)t + t^2) - 2a(n - 1)t^3, \quad (53)$$

$$\phi(t) = \frac{a(n - 1)t^3}{(1 - t)(1 - (n + 1)t + t^2) - a(n - 1)t^3}, \quad (54)$$

where a and b are defined by

$$a = (n-1)^{1/q} \quad \text{and} \quad b = 2^{1/q}. \quad (55)$$

As a consequence of Theorem 3, we get the following two convergence results for simple zeros.

Theorem 6. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in \mathbb{K} , and let $\xi \in \mathbb{K}^n$ be a root vector of f . Suppose that $x^{(0)} \in \mathbb{K}^n$ is an initial guess satisfying the following condition:

$$B(E(x^{(0)})) > 0, \quad (56)$$

where the functions E and B are defined by (50) and (52), respectively. Then Nourein's iteration (7) is well-defined and converges with fourth-order to ξ with the following error estimates:

$$\|x^{(k+1)} - \xi\| \preceq \lambda^{4^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \lambda^{(4^k-1)/3} \|x^{(0)} - \xi\| \quad \text{for all } k \geq 0,$$

where $\lambda = \phi(E(x^{(0)}))$ and the function ϕ is defined by (54).

Corollary 1. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ that has n simple zeros in \mathbb{K} , and let $\xi \in \mathbb{K}^n$ be a root vector of f . If $x^{(0)} \in \mathbb{K}^n$ is an initial guess satisfying

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{d(\xi)} \right\|_{\infty} < \frac{4}{7n-2}, \quad (57)$$

then Nourein's iteration (7) is well-defined and converges with fourth-order to ξ with error estimates

$$\|x^{(k+1)} - \xi\| \preceq \lambda^{4^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \lambda^{(4^k-1)/3} \|x^{(0)} - \xi\| \quad \text{for all } k \geq 0,$$

where the real function ϕ is defined by

$$\phi(t) = \frac{(n-1)^2 t^3}{(1-t)(1-(n+1)t+t^2) - (n-1)^2 t^3}.$$

Proof. Let $x^{(0)} \in \mathbb{K}^n$ be an initial guess satisfying condition (57). In view of Theorem 6 ($p = \infty$), we have to prove that $x^{(0)}$ satisfies the initial condition (56). In the case $p = \infty$, the function B defined by (52) takes the form

$$B(t) = (1-t)(1-(n+1)t+t^2) - 2(n-1)^2 t^3.$$

Using condition (57) and the monotonicity of the function B , we obtain

$$B(E(x^{(0)})) > B\left(\frac{4}{7n-2}\right) = \frac{147n^3 - 590n^2 + 740n - 296}{(7n-2)^3} = \frac{(n-2)(147n^2 - 296n^2 + 148)}{(7n-2)^3} \geq 0$$

which completes the proof. \square

The following convergence result is an immediate consequence of Corollary 1.

Corollary 2. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ that has n simple zeros in \mathbb{K} , and let $\xi \in \mathbb{K}^n$ be a root vector of f . Suppose a vector $x^{(0)} \in \mathbb{K}^n$ satisfies an initial condition

$$\|x^{(0)} - \xi\|_{\infty} < \frac{4}{7n-2} \delta(\xi), \quad (58)$$

where the function δ is defined by (14). Then Nourein's iteration (7) is well-defined and converges with fourth-order to ξ .

Remark 1. For the first time, an initial condition of the type (58) was presented by Dochev [28]. He has proved that, if $f \in \mathbb{C}[z]$ is a polynomial of degree $n \geq 2$ and has only simple zeros, then Weierstrass's iteration (3) is well-defined and convergent quadratically to a root vector $\xi \in \mathbb{C}^n$ of f under the initial condition

$$\|x^{(0)} - \xi\|_{\infty} < \frac{\sqrt[n-1]{2} - 1}{2^{\sqrt[n-1]{2} - 1}} \delta(\xi),$$

where the function δ is defined by (14).

For other local convergence theorems of the first kind for other simultaneous iterative methods, we refer to [5,9,29–33].

6. Local Convergence Theorem of the Second Kind for Simple Zeros

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ that has n simple zeros in \mathbb{K} and let $\xi \in \mathbb{K}^n$ be a root vector of f . We study the convergence of the classical Nourein's method (7) with respect to a function of initial conditions $E: \mathcal{D} \rightarrow \mathbb{R}_+$ defined by

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p \quad (1 \leq p \leq \infty). \quad (59)$$

Define real functions β, Ψ, ψ and ϕ as follows:

$$\beta(t) = \frac{a(n-1)t^3}{(1-t)(1-(n+1)t+t^2) - a(n-1)t^3}, \quad (60)$$

$$\Psi(t) = \frac{(1-bt)(1-t)(1-(n+1)t+t^2) - 2a(n-1)t^3}{(1-t)(1-(n+1)t+t^2) - 2(n-1)t^3}, \quad (61)$$

$$\phi(t) = \frac{a(n-1)t^3}{1-2t(1-t)(1-(n+1)t+t^2)}, \quad \psi(t) = 1 - \frac{1-bt(1-t)(1-(n+1)t+t^2)}{(1-t)(1-(n+1)t+t^2) - a(n-1)t^3}, \quad (62)$$

where a and b are defined by (55).

Applying Theorem 5 to the polynomials with simple zeros, we get the following convergence theorem.

Theorem 7. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ that has n simple zeros in \mathbb{K} , and let $\xi \in \mathbb{K}^n$ be a root vector of f . Suppose that $x^{(0)} \in \mathbb{K}^n$ is an initial guess with distinct coordinates which satisfies the following condition:

$$A(E(x^{(0)})) > 0 \quad \text{and} \quad Q(E(x^{(0)})) \geq 0, \quad (63)$$

where the functions E and Q are defined by (59) and (53), respectively. Then Nourein's iteration (7) is well-defined and converges to ξ with error estimates

$$\|x^{(k+1)} - \xi\| \preceq \theta \lambda^{4^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \theta^k \lambda^{(4^k-1)/3} \|x^{(0)} - \xi\| \quad \text{for all } k \geq 0, \quad (64)$$

where $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$ and the functions ϕ and ψ are defined by (62). Moreover, if the second inequality in (63) is strict, then the rate of convergence is of order four.

In the case $p = \infty$, as a consequence of Theorem 5, we obtain the next two results. Now, we define the function of initial conditions $E: \mathcal{D} \rightarrow \mathbb{R}_+$ by

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_{\infty}. \quad (65)$$

Theorem 8. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ that has n simple zeros in \mathbb{K} , and let $\xi \in \mathbb{K}^n$ be a root vector of f . Suppose that $x^{(0)} \in \mathbb{K}^n$ is an initial guess with distinct coordinates which satisfies the following condition:

$$E(x^{(0)}) < \frac{1}{n} \quad \text{and} \quad Q(E(x^{(0)})) \geq 0, \quad (66)$$

where the function E is defined by (65) and the function Q is defined by

$$Q(t) = (1 - 2t)(1 - t)(1 - (n + 1)t + t^2) - 2(n - 1)^2 t^3. \quad (67)$$

Then Norein's iteration (7) is well-defined and converges to ξ with error estimates (64), where $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$ and ϕ and ψ are defined by

$$\phi(t) = \frac{(n - 1)^2 t^3}{1 - 2t(1 - t)(1 - (n + 1)t + t^2)} \quad \text{and} \quad \psi(t) = \frac{1 - 2t(1 - t)(1 - (n + 1)t + t^2)}{(1 - t)(1 - (n + 1)t + t^2) - (n - 1)^2 t^3}.$$

Moreover, if the second inequality in (66) is strict, then the convergence order is at least four.

Proof. According to Theorem 7, we have to prove that $A(E(x^{(0)})) > 0$. It can be proved that Q strictly decreases on $[0, 1/n]$, $Q(0) = 1$ and $Q(1/n) = -(n - 1)(3n^2 - 4n + 1)/n^4 < 0$. Hence, Q has a unique zero R on $(0, 1/n)$. Then it follows from (66) that

$$E(x^{(0)}) \leq R.$$

On the other hand, we have mentioned above that the function Q has a unique zero on $(0, \eta)$. Consequently, $R < \eta$. Then we deduce that $E(x^{(0)}) < \eta$, which yields $A(E(x^{(0)})) > 0$. This completes the proof. \square

Corollary 3. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ that has n simple zeros in \mathbb{K} , and let $\xi \in \mathbb{K}^n$ be a root vector of f . Suppose that $x^{(0)} \in \mathbb{K}^n$ is an initial guess with distinct coordinates which satisfies the following initial condition:

$$E(x^{(0)}) \leq \frac{23}{24n + 44}, \quad (68)$$

where the function E is defined by (65). Then Norein's iteration (7) is well-defined and converges to ξ with order of convergence four and with error estimates (64).

Proof. In view of Theorem 8, we have to show that $x^{(0)}$ satisfies the initial condition (66). We shall prove the second inequality in (66) because the first one is trivial. From condition (68) and the fact that Q defined by (67) strictly decreases on $(0, 1/n)$, we obtain

$$Q(E(x^{(0)})) \geq Q\left(\frac{23}{24n + 44}\right) = \frac{6912n^4 - 128712n^3 + 591572n^2 + 1098464n - 565861}{128(6n + 11)^4} > 0.$$

Hence, the condition (66) is satisfied, which completes the proof. \square

From Corollary 3, we immediately obtain the following convergence result.

Corollary 4. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ that has n simple zeros and let $\xi \in \mathbb{K}^n$ be a root vector of f . Suppose $x^{(0)} \in \mathbb{K}^n$ is a vector with distinct coordinates satisfying the initial condition

$$\|x^{(0)} - \xi\|_\infty < \frac{23}{24n + 44} \delta(\xi), \quad (69)$$

where the function δ is defined by (14). Then Norein's iteration (7) is well-defined and converges to ξ with order of convergence four.

Remark 2. For the first time, an initial condition of the type (69) was presented by Wang and Zhao [34]. They proved that, if $f \in \mathbb{C}[z]$ is a polynomial of degree $n \geq 2$ and has only simple zeros, then Ehrlich's iteration (5) is well-defined and convergent cubically to a root vector $\xi \in \mathbb{C}^n$ of f under the initial condition

$$\|x^{(0)} - \xi\|_\infty < \frac{2}{8 + \sqrt{8n-7}} \delta(x^{(0)}),$$

where the function δ is defined by (14).

For local convergence of the second kind for other iterative method, we refer to [5,9,27,29,31,32,35].

7. Semilocal Convergence Analysis for Simple Zeros

In this section, we establish three semilocal convergence theorems for the classical Nourdin's method (7). Each of these results improves Theorem 1 in several directions.

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. We study the convergence of the method (7) with respect to the function of initial conditions $E_f: \mathcal{D} \rightarrow \mathbb{R}_+$ defined by

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p \quad (1 \leq p \leq \infty), \quad (70)$$

where the operator W_f is defined by (4). We define a relation of equivalence \equiv on \mathbb{K}^n by $x \equiv y$ if there exists a permutation (i_1, \dots, i_n) of the indexes $(1, \dots, n)$ such that

$$(x_1, \dots, x_n) = (y_{i_1}, \dots, y_{i_n}).$$

Now, we can define a distance between two vectors $x, y \in \mathbb{C}^n$ as follows [32,33]:

$$\rho(x, y) = \min_{v \equiv y} \|x - v\|_p. \quad (71)$$

For proof of the main theorem of this section, we need two results from [14]. In order to make the paper self-contained, we include recall of these results.

Theorem 9 ([14], Theorem 5.1). Let \mathbb{K} be an algebraically closed field and let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. Suppose that $x \in \mathbb{K}^n$ is a vector with distinct coordinates satisfying

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p < \tau = \frac{1}{(1 + \sqrt{a})^2}, \quad (72)$$

where a is defined by (55). Then f has only simple zeros and there exists a root vector $\xi \in \mathbb{K}^n$ of f such that

$$\rho(x, \xi) \leq \alpha(E_f(x)) \|W_f(x)\|_p \quad \text{and} \quad \left\| \frac{x - \xi}{d(x)} \right\|_p < h(E_f(x)), \quad (73)$$

where the distance function ρ is defined by (71) and the real functions $\alpha, h: [0, \tau] \rightarrow \mathbb{R}_+$ are defined by

$$\alpha(t) = \frac{2}{1 - (a-1)t + \sqrt{(1 - (a-1)t)^2 - 4t}} \quad \text{and} \quad h(t) = t\alpha(t). \quad (74)$$

We note that the functions α and h strictly increase on $[0, \tau]$, where τ is defined by (72).

Theorem 10 ([14], Theorem 5.2). Let \mathbb{K} be algebraically closed field, and let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. Suppose that $x \in \mathbb{K}^n$ is a vector with distinct coordinates such that

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p < \frac{R(1-R)}{1+(a-1)R} \quad (75)$$

and $0 \leq R \leq 1/(1+\sqrt{a})$, where W_f is defined by (4) and a is defined by (55). Then polynomial f has only simple zeros in \mathbb{K} and there exists a root vector $\xi \in \mathbb{K}^n$ of f such that

$$\rho(x, \xi) \leq \alpha(E_f(x)) \|W_f(x)\| \quad \text{and} \quad \left\| \frac{x - \xi}{d(x)} \right\|_p < R, \quad (76)$$

where the function α is defined by (74).

Now, we can state and prove the main results of this paper.

Theorem 11. Let \mathbb{K} be an algebraically closed field. and let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. Suppose that $x^{(0)} \in \mathbb{K}^n$ is an initial guess with distinct coordinates satisfying

$$E_f(x^{(0)}) < \frac{n-1}{n(a+n+1)}, \quad A(h(E_f(x^{(0)}))) > 0 \quad \text{and} \quad \Psi(h(E_f(x^{(0)}))) > 0, \quad (77)$$

where a is defined by (55) and the functions E_f , A , Q and h are defined by (70), (51), (53) and (74), respectively. Then f has only simple zeros in \mathbb{K} and Nourine's iteration (7) is well-defined and converges to a root vector ξ of f with order of convergence four and with a posteriori error estimate

$$\rho(x^{(k)}, \xi) \leq \alpha(E_f(x^{(k)})) \|W_f(x^{(k)})\|_p \quad \text{for all } k \geq 0 \text{ such that } E_f(x^{(k)}) < \tau, \quad (78)$$

where the distance function ρ is defined by (71), τ is defined by (72) and the real function α is defined by (74).

Proof. First, we note that the function h is defined on $[0, \tau]$ by (74), where τ is defined by (72). It follows from the first inequality of (77) that

$$E_f(x^{(0)}) < \frac{n-1}{n(a+n+1)} \leq \tau.$$

Then by Theorem 9, it follows that f has n simple zeros in \mathbb{K} and there exists a root vector $\xi \in \mathbb{K}^n$ of f such that

$$E(x^{(0)}) < h(E_f(x^{(0)})), \quad (79)$$

where the function $E: \mathcal{D} \rightarrow \mathbb{R}_+$ is defined by (59). From the second inequality of (77), we conclude that $h(E_f(x^{(0)})) < \eta$, where η is the unique solution of the equation $A(t) = 0$. Now from (79), we conclude that $E_f(x^{(0)}) < \eta$, which yields the inequality

$$A(E_f(x^{(0)})) > 0.$$

Define the real function Ψ on $[0, \eta]$ by (61). From the inequality (79), taking into account that the functions Ψ strictly decreases on $[0, \eta]$, we obtain $\Psi(E_f(x^{(0)})) > \Psi(h(E_f(x^{(0)}))) > 0$, which implies

$$Q(E_f(x^{(0)})) > 0$$

since $E_f(x^{(0)}) < \eta$ and $\Psi(t) = Q(t)/A(t)$ for every $t \in [0, \eta]$. Hence, the initial guess $x^{(0)}$ satisfies the condition (63). Now, it follows from Theorem 7 that the iteration (7) is well-defined and converges to ξ with order of convergence four. The error estimate (78) follows from Theorem 9. \square

Furthermore, we shall consider two semilocal convergence results in the case $p = \infty$. In this case, the function of initial conditions $E_f: \mathcal{D} \rightarrow \mathbb{R}_+$ is defined by

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_{\infty} \quad (80)$$

and the functions α and h are defined on $[0, \tau]$ by

$$\alpha(t) = \frac{2}{1 - (n-2)t + \sqrt{(1 - (n-2)t)^2 - 4t}} \quad \text{and} \quad h(t) = t\alpha(t), \quad (81)$$

where $\tau > 0$ is defined by

$$\tau = \frac{1}{(1 + \sqrt{n-1})^2}. \quad (82)$$

The distance function ρ is defined on \mathbb{K}^n by

$$\rho(x, y) = \min_{u \equiv y} \|x - u\|_{\infty}. \quad (83)$$

Let us define a real function Ω as follows

$$\Omega(t) = Q(h(t)), \quad (84)$$

where Q and h are defined by (67) and (81), respectively.

Theorem 12. Let \mathbb{K} be an algebraically closed field, and let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. Suppose that $x^{(0)} \in \mathbb{K}^n$ is an initial guess with distinct coordinates satisfying

$$E_f(x^{(0)}) < \frac{1}{2n} \quad \text{and} \quad \Omega(E_f(x^{(0)})) > 0, \quad (85)$$

where the functions E_f and Ω are defined by (80) and (84), respectively. Then f has only simple zeros in \mathbb{K} and Nourine's iteration (7) is well-defined and converges to a root vector ξ of f with order of convergence four and with a posteriori error estimate

$$\rho(x^{(k)}, \xi) \leq \alpha(E_f(x^{(k)})) \|W_f(x^{(k)})\|_{\infty} \quad \text{for all } k \geq 0 \text{ such that } E_f(x^{(k)}) < \tau, \quad (86)$$

where the distance function ρ is defined by (83), τ is defined by (82) and the real function α is defined by (81).

Proof. It follows from the first inequality of (85) that $E_f(x^{(0)}) < 1/(2n) \leq \tau$, where τ is defined by (82). By Theorem 9, we conclude that f has only simple zeros and there exists a root vector $\xi \in \mathbb{K}^n$ of f such that $E(x^{(0)}) < h(E_f(x^{(0)}))$, where the function $E: \mathcal{D} \rightarrow \mathbb{R}_+$ is defined by (65). Then by monotonicity of h , we get

$$E(x^{(0)}) < h(E_f(x^{(0)})) < h\left(\frac{1}{2n}\right) = \frac{1}{n}.$$

From this and the second inequality in (77), taking into account that Q is strictly decreasing on $[0, 1/n)$, we obtain

$$Q(E(x^{(0)})) > Q(h(E_f(x^{(0)}))) = \Omega(E_f(x^{(0)})) > 0.$$

It follows from Theorem 8 that the iteration (7) is well-defined and converges to ξ with order of convergence four. The error estimate (86) follows from the Theorem 9. \square

Using Corollary 3 and Theorem 10, we obtain the next semilocal result.

Corollary 5. Let \mathbb{K} be an algebraically closed field and $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. Suppose that $x^{(0)} \in \mathbb{K}^n$ is an initial guess with distinct coordinates satisfying

$$E_f(x^{(0)}) \leq \frac{69(8n+7)}{1128n^2 + 2020n - 88}, \quad (87)$$

where the function E is defined by (80). Then f has only simple zeros in \mathbb{K} and Norein's iteration (7) is well-defined and converges to a root vector ξ of f with order of convergence four and with a posteriori error estimate (86).

Proof. The initial condition (87) can be represented in the form (75) with R defined by

$$R = \frac{23}{24n + 44}.$$

It is easy to check that $R < 1/(1 + \sqrt{n-1})$. Then it follows from Theorem 10 that f has only simple zeros in \mathbb{K} and there exists a root vector $\xi \in \mathbb{K}^n$ of f such that

$$E(x^{(0)}) < R,$$

where the function $E: \mathcal{D} \rightarrow \mathbb{R}_+$ is defined by (65). Now, Corollary 3 implies that Norein's iteration (7) converges to ξ with order of convergence four. As we have mentioned in the proof of the previous theorem, the error estimate (86) follows from the Theorem 9. This ends the proof. \square

Remark 3. We note that each of our semilocal convergence results (Theorems 11 and 12 and Corollary 5) improves and complements (with a posteriori error estimate) the result of Petković, Petković and Rančić [19] (see Theorem 1 above). In particular, they give larger convergence domains than Theorem 1 and they do not require in advance the simplicity of the zeros of f . For instance, let us prove that Corollary 5 is an improvement of Theorem 1. Let an initial vector satisfies the initial condition (8). Then,

$$E_f(x^{(0)}) = \left\| \frac{W_f(x^{(0)})}{d(x^{(0)})} \right\|_{\infty} \leq \frac{\|W_f(x^{(0)})\|_{\infty}}{\delta(x^{(0)})} < c_n < \frac{69(8n+7)}{1128n^2 + 2020n - 88},$$

which shows that the initial condition (87). Hence, it follows from Corollary 5 that the conclusion of Theorem 1 holds.

Semilocal convergence of the same kind as above results can be found in [5,27,31–33,35,36].

8. Numerical Examples

In this section, we present three numerical examples to show the applicability of Theorem 12. Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$, and let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. Starting from the vector $x^{(0)}$, we generate Norein's iterative sequence $(x^{(k)})_{k=0}^{\infty}$ for the polynomial f . Each of our semilocal convergence results (Theorem 11 and 12 and Corollary 5) gives the following two very useful criteria:

- Convergence criterion that determines whether Norein's method is convergent.
- Accuracy criterion that determines whether Norein's method has reached a preset accuracy $\varepsilon > 0$. It can be used as stopping criterion.

Below, we consider the criteria obtained from Theorem 12. As in the previous sections, we define the functions E_f, W_f, Ω and α by (4), (80), (81) and (84), respectively.

Convergence criterion. If there exists an integer $m \geq 0$ such that

$$E_f(x^{(m)}) \leq \mu = \frac{1}{2n} \quad \text{and} \quad \Omega(E_f(x^{(m)})) \geq 0, \quad (88)$$

then f has only simple zeros and Nourain's iteration (7) starting from $x^{(0)}$ is well-defined and converges to a root vector ξ of f with order of convergence four. In each example, we calculate the smallest m that satisfies convergence criterion (88).

Accuracy criterion (stopping criterion). Let $\varepsilon > 0$. If there exists an integer $k \geq 0$, such that

$$E_f(x^{(k)}) \leq \tau = \frac{1}{(1 + \sqrt{n-1})^2} \quad \text{and} \quad \varepsilon_k = \alpha(E_f(x^{(k)})) \|W_f(x^{(k)})\|_\infty \leq \varepsilon, \quad (89)$$

then the iterate $x^{(k)}$ approximates the vector of zeros of f with accuracy ε . Moreover, the guaranteed accuracy is ε_k . Indeed, according to Theorem 11, we have

$$\rho(x^{(k)}, \xi) < \varepsilon,$$

where the distance function ρ is defined by (71) and $\xi \in \mathbb{C}^n$ is a root vector of f . In each example, we calculate the smallest k that satisfies accuracy criterion (89) with

$$\varepsilon = 10^{-15}.$$

In the examples, we apply Nourain's method to three monic polynomials f of degree $16 \leq n \leq 21$ taken from [17]. In each example, we choose two types of very crude initial approximations $x^{(0)} \in \mathbb{C}^n$ as follows:

First type of initial approximations. For a monic polynomial

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$$

of degree $n \geq 2$, we choose the coordinates $x_1^{(0)}, \dots, x_n^{(0)}$ of the initial vector $x^{(0)} \in \mathbb{C}^n$ by Aberth's formula [37]:

$$x_j^{(0)} = -\frac{a_1}{n} + r_0 \exp(i\theta_j), \quad \theta_j = \frac{\pi}{n} \left(2j - \frac{3}{2}\right), \quad j = 1, \dots, n, \quad (90)$$

where $r_0 > 0$ is a parameter.

The Aberth's initial approximations $x_1^{(0)}, \dots, x_n^{(0)}$ are equidistantly spaced on the circle with radius with center a_1/n and radius r_0 . We take very crude Aberth's approximations on the circle with radius $r_0 = 200$ for Example 1 and $r_0 = 10$ for Examples 2 and 3.

Second type of initial approximations. We choose the coordinates $x_1^{(0)}, \dots, x_n^{(0)}$ of the initial vector $x^{(0)} \in \mathbb{C}^n$ randomly in the square

$$\{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq r_0 \text{ and } |\operatorname{Im}(z)| \leq r_0\},$$

where r_0 is the same as in the previous type of initial approximations.

We use CAS Wolfram Mathematica 11 to implement the corresponding algorithms and to present approximations of higher accuracy.

Example 1. Let us consider Wilkinson's ill-conditioned polynomial, often a hard nut to crack for most methods:

$$\begin{aligned} f_1(z) = & z^{18} - 76z^{17} + 2451z^{16} - 42636z^{15} + 405042z^{14} - 1480632z^{13} - 9162218z^{12} \\ & + 124928648z^{11} - 407525547z^{10} - 1153431708z^9 + 11555719383z^8 - 18182560188z^7 \\ & - 73778959736z^6 + 272611286816z^5 - 23388233616z^4 - 923526085824z^3 \\ & + 833270250240z^2 + 670127385600z - 747242496000 = \prod_{\substack{k=-5 \\ k \neq 0}}^{13} (z - k). \end{aligned} \quad (91)$$

Our random initial guess $x^{(0)}$ for Example 1 is

$$x^{(0)} = \{23.842 - 12.426i, 196.390 - 136.008i, 180.322 - 50.643i, 183.774 + 14.551i, \\ 113.220 + 159.627i, 159.800 + 22.698i, -27.676 - 167.497i, 195.857 + 18.631i, \\ 178.262 + 114.435i, -23.150 + 157.953i, 69.927 + 5.095i, -81.282 - 137.910i, \\ 154.174 - 68.812i, -57.413 - 151.660i, 10.138 + 98.167i, -115.399 - 33.393i, \\ 106.991 - 161.731i, -190.202 - 80.620i\}.$$

Example 2. Consider the following polynomial with clusters

$$f_2(z) = z^{16} + z^{10} - 10z^9 + 45z^8 - 120z^7 + 210z^6 - 252z^5 + 210z^4 - 120z^3 + 45z^2 - 10z + 1.$$

The random initial guess $x^{(0)}$ for Example 2 is

$$x^{(0)} = \{-2.631 + 3.166i, -2.600 + 9.323i, -6.468 - 7.619i, -7.365 + 6.063i, \\ 5.325 - 1.413i, 4.844 - 2.670i, -5.890 - 4.534i, 0.205 + 9.302i, \\ 9.232 - 9.540i, 9.672 + 7.694i, 4.235 - 8.552i, -0.012 + 1.758i, \\ -1.980 + 8.182i, 9.820 + 6.746i, 1.307 - 3.871i, -6.293 - 0.111i\}.$$

Example 3. Consider the polynomial with ring zeros:

$$f_3(z) = z^{21} + 7z^{20} - 9765626z^{11} - 68359382z^{10} + 9765625z + 68359375 = (z + 7)(z^{10} - 1)(z^{10} - 5^{10})$$

The random initial guess $x^{(0)}$ for Example 3 is

$$x^{(0)} = \{8.057 - 3.640i, 0.257 - 4.298i, -7.718 + 8.568i, 0.852 + 1.961i, \\ 9.167 - 5.264i, 4.866 - 5.726i, 0.552 - 9.817i, -7.496 - 2.718i, \\ -6.733 - 4.709i, -7.203 + 9.421i, 5.545 - 0.359i, 5.599 + 5.612i, \\ 7.335 + 4.054i, 6.545 - 0.471i, -4.882 - 9.617i, -2.488 - 5.469i, -7.712 + 6.404i\}.$$

Numerical results. In Table 1 are presented the results for the considered examples, we exhibit the values of m , $E_f(x^{(m)})$, $\Omega(E_f(x^{(m)}))$, k and ε_k . We recall that:

- m is the smallest nonnegative integer that satisfies the convergence criterion (88);
- ε_m is defined in (89) and denotes the guaranteed accuracy (by Theorem 12) for the approximation x_m of the zeros of f ;
- k is the smallest nonnegative integer that satisfies convergence accuracy criterion (89) with the preset accuracy $\varepsilon = 10^{-15}$;
- ε_k is the guaranteed accuracy (Theorem 12) for the approximation x_k of the zeros of f .

It can be seen from the table that, in all six experiments, Theorem 12 guarantees that Nourein's method (7) is convergent under the given very rough initial approximations. Also it shows on which iteration that the preset accuracy is reached.

For instance, for Example 1, under the first initial approximation, it is seen that the convergence criterion (88) is satisfied for $m = 31$ and that the accuracy criterion (89) is satisfied for $k = 33$, which means that the preset accuracy 10^{-15} is reached. Moreover, the table shows that, at 33 iterations, Theorem 12 guarantees an accuracy of 10^{-41} and at 34 iterations, it guarantees that each of the roots of the polynomial (91) is calculated with a guaranteed accuracy of 10^{-167} .

Table 1. Numerical results.

Example	m	$E_f(x^{(m)})$	$\Omega(E_f(x^{(m)}))$	ε_m	k	ε_k	ε_{k+1}
First type initial approximations							
Example 1	31	6.254×10^{-3}	0.848	6.988×10^{-3}	33	1.042×10^{-41}	1.442×10^{-167}
Example 2	20	2.845×10^{-5}	0.999	3.498×10^{-6}	22	5.275×10^{-20}	5.711×10^{-76}
Example 3	14	6.688×10^{-5}	0.998	4.139×10^{-5}	15	2.719×10^{-17}	5.946×10^{-66}
Second type initial approximations							
Example 1	31	2.774×10^{-5}	0.999	2.775×10^{-5}	32	5.366×10^{-20}	1.170×10^{-78}
Example 2	18	1.945×10^{-3}	0.960	2.460×10^{-4}	20	2.684×10^{-51}	6.697×10^{-202}
Example 3	14	1.045×10^{-6}	0.999	6.463×10^{-7}	15	1.622×10^{-24}	7.769×10^{-95}

In Figures 1–3, we present the trajectories of approximations $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$ in the complex plane with different colors. For the random initial guess (second type), all initial approximations are numbered and colored by blue. From these figures, one can observe that all trajectories starting with Aberth's initial approximations have regular paths. However, it is not necessary that the initial approximations lie in an inclusion disk with a minimum radius to have this nice and regular form. About the second type of initial approximations, one can see that some initial points during iterating are not going to the nearest zero of the polynomial.

9. Conclusions

In this paper, we studied the convergence of two well-known iterative methods for finding all zeros of a polynomial simultaneously. The first one is due to Norein [1] and it has quartic convergence when all zeros of the polynomial are simple. It is also known as Ehrlich's method with Newton's corrections because it is obtained by combining Ehrlich's method [6] and the classical Newton's method. The second one is a generalization of Norein's method for simultaneously finding all zeros of a polynomial that has at least one multiple zero. To our knowledge, it appears for the first time in the book of Sendov, Andreev and Kjurkchiev [2].

We have proved several new local and semilocal convergence theorems (Theorems 6–8, 11 and 12) for the classical Norein's method (7) under different initial conditions. The initial conditions and error bounds of the semilocal convergence results (Theorems 11 and 12) are computationally verifiable, which is of practical importance. Each of our semilocal convergence results improves the previous result due to Petković, Petković and Rančić [19] in several directions. We note that our approach to semilocal convergence analysis is different from those of the previous authors [18,19]. In Section 8, we present several numerical examples that show the applicability of our semilocal convergence results.

We have obtained two new local convergence theorems (Theorems 3 and 5) for the generalized Norein's method (11) under different initial conditions. To the best of authors' knowledge, our local convergence theorems are the first local convergence results in the literature about Norein's method (for simple or multiple zeros).

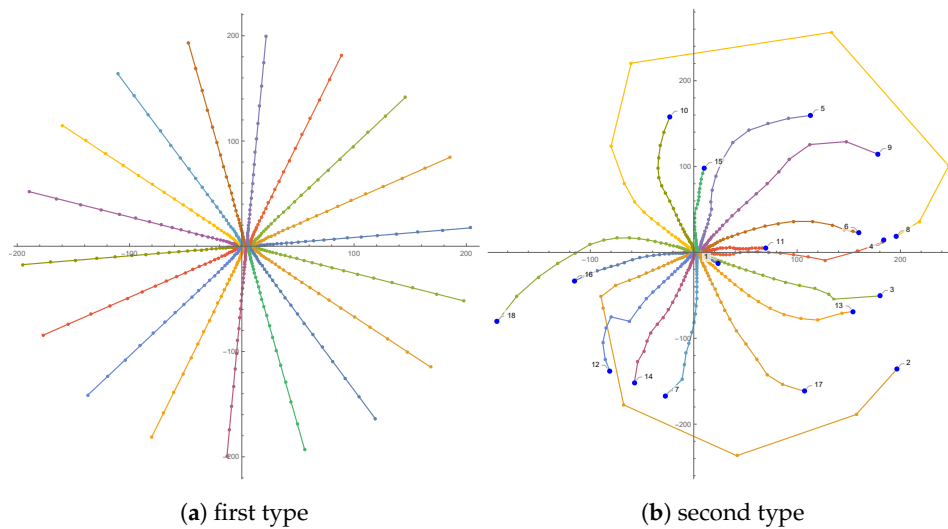


Figure 1. Trajectories of approximations for Example 1.

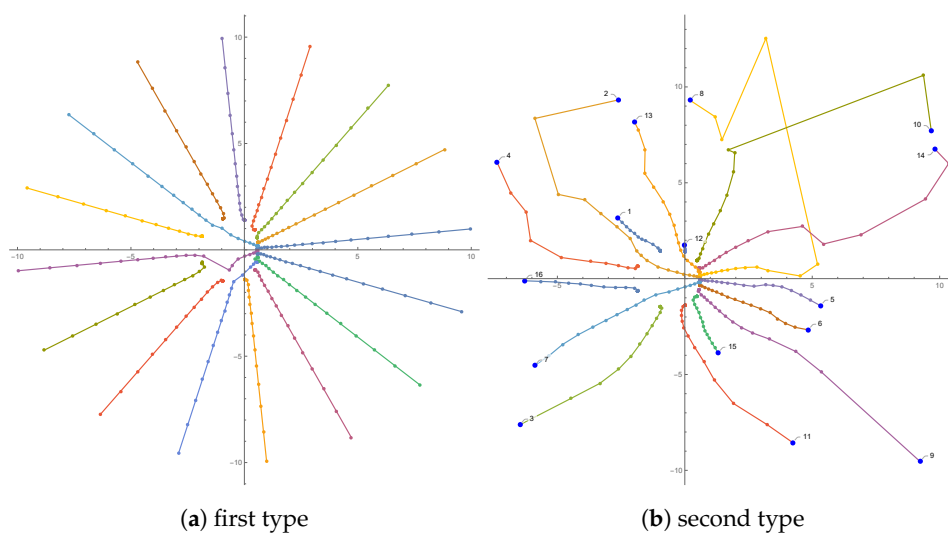


Figure 2. Trajectories of approximations for Example 2.

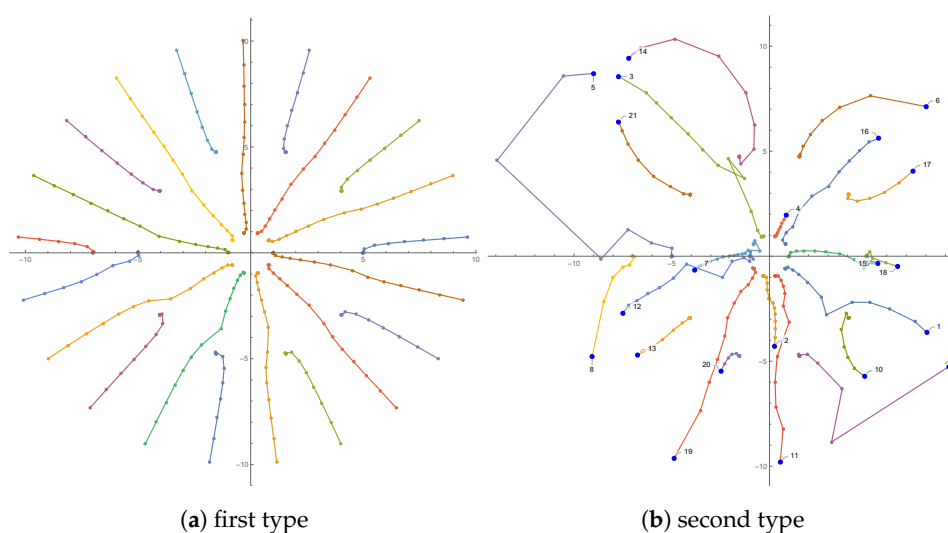


Figure 3. Trajectories of approximations for Example 3.

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