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The n-Pythagorean Fuzzy Sets

Anna Bryniarska

Department of Computer Science, Opole University of Technology, Proszkowska 76, 45-758 Opole, Poland; a.bryniarska@po.edu.pl

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Abstract: The following paper presents deductive theories of n-Pythagorean fuzzy sets (n-PFS). N-PFS objects are a generalization of the intuitionistic fuzzy sets (IFSs) and the Yager Pythagorean fuzzy sets (PFSs). Until now, the values of membership and non-membership functions have been described on a one-to-one scale and a quadratic function scale. There is a symmetry between the values of this membership and non-membership functions. The scales of any power functions are used here in order to increase the scope of the decision-making problems. The theory of n-PFS introduces a conceptual apparatus analogous to the classic theory of Zadeh fuzzy sets, consistently striving to correctly define the n-PFS algebra.

Keywords: fuzzy set; n-Pythagorean; n-PFS algebra; triangular norms

1. Introduction

Zadeh [1] introduced the fuzzy set idea, which generalizes the theory of classical sets. In fuzzy sets, there is a membership function μ, which assigns a number from the set $[0,1]$ to each element of the universe. This determines how much this element belongs to this universe, where 0 means no belonging and 1 means full belonging to the set that is under consideration. Other values between 0 and 1 mean the degree of belonging to this set. This membership function is defined to describe the degree of belonging of an element to some class. The membership function in the fuzzy sets replace the characteristic function that is used in crisp sets. Since the work of Zadeh, the fuzzy set theory has been used in different disciplines such as management sciences, engineering, mathematics, social sciences, statistics, signal processing, artificial intelligence, automata theory, and medical and life sciences.

Atanassov studied the intuitionistic fuzzy sets (IFSs) [2,3]. IFSs have values for two functions: the membership function μ and the non-membership function v. Additionally, there is a constraint $0 \leq \mu + v \leq 1$. This is a symmetric relationship between the values and the membership function. In order to create model for imprecise information, the model of Pythagorean fuzzy sets (PFSs) was proposed by Yager [4,5]. This model is different than the IFSs model because it uses the condition $0 \leq \mu^2 + v^2 \leq 1$. Moreover, there is also the Pythagorean fuzzy number (PFN) idea established by Zhang and Xu [6]. In decision-making problems, there are also applications of PFSs proposed by Garg [7,8].

The decision-making problems in the model of Pythagorean fuzzy sets will significantly increase the application range of solving these problems than in the model of intuitionistic fuzzy sets. It is because more pairs (μ, v) satisfy the condition $0 \leq \mu^2 + v^2 \leq 1$ than the condition $0 \leq \mu + v \leq 1$. Is there any other data scale in decision-making problems that will help to extend its applicability even further? Yes, with the condition $0 \leq \mu^n + v^n \leq 1$, for any natural number $n > 2$.

In articles using local deduction regarding IFSs [3,9–12] and concerning PFSs [4,5,13–17], it was noted that there is a series of mathematical and logical inaccuracies. That is why the conceptual apparatus should be generalized and refined by formulating the deductive theory of n-Pythagorean fuzzy sets with the condition $0 \leq \mu^n + v^n \leq 1$, for any natural number $n\rangle 0$. The theory is presented below.

2. Triangular Norms

Definition 1. *The operation $\bullet_t : [0,1] \times [0,1] \to [0,1]$ is called a t-norm in the set $[0,1]$, or a triangular norm, when it meets the following conditions (for any numbers $x, y, z \in [0,1]$):*

1. *boundary conditions*
$$0 \bullet_t y = 0, y \bullet_t 1 = y, \tag{1}$$
2. *monotonicity*
$$x \bullet_t y \leq z \bullet_t y, \text{ when } x \leq z, \tag{2}$$
3. *commutativity*
$$x \bullet_t y = y \bullet_t x, \tag{3}$$
4. *associativity*
$$x \bullet_t (y \bullet_t z) = (x \bullet_t y) \bullet_t z, \tag{4}$$

Since there is $\sup\{t \in [0,1] : x \bullet_t t \leq y\}$, there can be specified the operation $\to_t$: $[0,1] \times [0,1] \to [0,1]$, such that for any numbers $x, y \in [0,1]$:

$$x \to_t y = \sup\{t \in [0,1] : x \bullet_t t \leq y\}. \tag{5}$$

This operation is called *t-residuum* in the set $[0,1]$.

The operation $\bullet_s : [0,1] \times [0,1] \to [0,1]$, described by formula for any numbers $x, y \in [0,1]$:

$$x \bullet_s y = 1 - (1 - x) \bullet_t (1 - y), \tag{6}$$

is called a *s-norm* or *triangular conorm*.

Using the definitions of t-norm and s-norm, after simple calculations we get (where the names of conditions are given analogously to the definition of t-norm):

Theorem 1. *For any numbers $x, y, z \in [0,1]$:*

- *boundary conditions*
$$0 \bullet_s y = y, y \bullet_s 1 = 1, \tag{7}$$
- *monotonicity*
$$x \bullet_s y \leq z \bullet_s y, \text{ when } x \leq z, \tag{8}$$
- *commutativity*
$$x \bullet_s y = y \bullet_s x, \tag{9}$$
- *associativity*
$$x \bullet_s (y \bullet_s z) = (x \bullet_s y) \bullet_s z, \tag{10}$$

Further, only continuous t-norms and s-norms are considered. The general discussion on the construction of triangular norms, using the results of functional equations, leads to the theorem from paper [18]:

Theorem 2.

1. *There is a continuous and strictly decreasing function for each continuous t-norm $f_t : [0,1] \to [0, +\infty)$ such that $f_t(1) = 0, f_t(0) = 1$ and for any $x, y \in [0,1]$:*
$$x \bullet_t y = \begin{cases} f_t^{-1}[f_t(x) + f_t(y)], & f_t(x) + f_t(y) \in [0,1] \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

2. *There is a continuous and strictly increasing function for each continuous s-norm* $f_s : [0,1] \to [0,+\infty)$ *such that* $f_s(0) = 0, f_s(1) = 1$ *and for any* $x, y \in [0,1]$*:*

$$x \bullet_s y = \begin{cases} f_s^{-1}[f_s(x) + f_s(y)], & f_s(x) + f_s(y) \in [0,1] \\ 1, & \text{otherwise.} \end{cases} \tag{12}$$

3. *For any* $x \in [0,1]$

$$f_s(x) = f_t(1-x). \tag{13}$$

Functions f_t, f_s are called *generators* of t-norm and s-norm, respectively.

Example 1. *For the t-norm* $x \bullet_t y = \min\{x, y\}$ *and any* $x, y \in [0,1]$*, the generator is* $f_t(x) = 1 - x$*.*
For the s-norm $x \bullet_s y = \max\{x, y\}$ *and any* $x, y \in [0,1]$*, the generator is* $f_s(x) = x$*.*

Example 2. *For the t-norm* $x \bullet_t y = 1 - \min\{1, ((1-x)^p + (1-y)^p)^{1/p}\}, p \geq 1$ *and any* $x, y \in [0,1]$*, the generator is* $f_t(x) = 1 - x^p$*.*
For the s-norm $x \bullet_s y = \min\{1, (x^p + y^p)^{1/p}\}, p \geq 1$ *and any* $x, y \in [0,1]$*, the generator is* $f_s(x) = x^p$*.*

Theorem 3. *Let* f_t, f_s *be generators of the triangular norms* $\bullet_t, \bullet_s$*. Then there exist operations* $\bullet_p : [0,1] \times [0,1] \to [0,1], \bullet_l : [0,1] \times [0,1] \to [0,1]$ *defined by formulas:*

$$\lambda \bullet_p x = \begin{cases} f_t^{-1}[\lambda f_t(x)], & \text{for } \lambda f_t(x) \in [0,1] \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

$$\lambda \bullet_l x = \begin{cases} f_s^{-1}[\lambda f_s(x)], & \text{for } \lambda f_s(x) \in [0,1] \\ 1, & \text{otherwise.} \end{cases} \tag{15}$$

Proof of Theorem 3. Because f_t is a strictly decreasing function, there is only one value $f_t^{-1}[\lambda f_t(x)] \in [0,1]$, when $\lambda f_t(x) \in [0, f_s(0)]$.
It is noted that $\lambda \bullet_p x =_{df} f_t^{-1}[\lambda f_t(x)]$, for $\lambda f_t(x) \in [0, f_t(0)]$, and $\lambda \bullet_p x =_{df} 0$ otherwise.
Similarly, we define the operation $\bullet_l$. □

Definition 2. *Operations* $\bullet_p, \bullet_l$ *specified in Theorem 3 are called* ***p-norm*** *(with properties similar to the power functions) and the* ***l-norm*** *(with properties similar to the linear function), respectively, and the system* $\mathbf{S_{Yager}} = \langle [0,1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1 \rangle$ *is called the Yager system of the triangular norms.*

The following notation agreement is accepted:

$$\lambda \bullet_p x =_{df} x^\lambda, \tag{16}$$

$$\lambda \bullet_l x =_{df} \lambda x. \tag{17}$$

Fact 1. *If* $x^\lambda = f_t^{-1}[\lambda f_t(x)]$*, for* $\lambda f_t(x) \in [0, f_t(0)]$*, then* $f_t(x^\lambda) = \lambda f_t(x)$*.*
If $\lambda x = f_s^{-1}[\lambda f_s(x)]$*, for* $\lambda f_s(x) \in [0, f_s(1)]$*, then* $f_s(\lambda x) = \lambda f_s(x)$*.*

Theorem 4. *In the system* $\mathbf{S_{Yager}} = \langle [0,1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1 \rangle$ *operations* $\bullet_p, \bullet_l$ *satisfy the following conditions (for any* $x, y, \lambda, \lambda_1, \lambda_2 \in [0,1]$*):*

$$\lambda(x \bullet_s y) = \lambda x \bullet_s \lambda y, \tag{18}$$

$$(x \bullet_t y)^\lambda = x^\lambda \bullet_t y^\lambda, \tag{19}$$

$$(\lambda_1 + \lambda_2)x = \lambda_1 x \bullet_s \lambda_2 x, \tag{20}$$

$$x^{\lambda_1+\lambda_2} = x^{\lambda_1} \bullet_t x^{\lambda_2}. \tag{21}$$

Proof of Theorem 4.

1. For Equation (18): $\lambda(x \bullet_s y) = f_s^{-1}[\lambda f_s(x \bullet_s y)] = f_s^{-1}[\lambda f_s(x) + \lambda f_s(y)] = f_s^{-1}[f_s(\lambda x) + f_s(\lambda y)] = \lambda x \bullet_s \lambda y$, for $f_s(x \bullet_s y) \in [0,1]$, and since $\lambda f_s(x \bullet_s y) \leq f_s(x \bullet_s y)$, so $\lambda f_s(x \bullet_s y) \in [0,1]$.
2. For Equation (19): $(x \bullet_t y)^\lambda = f_t^{-1}[\lambda f_t(x \bullet_t y)] = f_t^{-1}[\lambda f_t(x) + \lambda f_t(y)] = f_t^{-1}[f_t(x^\lambda) + f_t(y^\lambda)] = x^\lambda \bullet_t y^\lambda$, for $f_t(x \bullet_t y) \in [0,1]$, and since $\lambda f_t(x \bullet_t y) \leq f_t(x \bullet_t y)$, so $\lambda f_t(x \bullet_t y) \in [0,1]$.
3. For Equation (20): $(\lambda_1 + \lambda_2)x = f_s^{-1}[(\lambda_1 + \lambda_2) f_s(x)] = f_s^{-1}[\lambda_1 f_s(x) + \lambda_2 f_s(x)] = f_s^{-1}[f_s(\lambda_1 x) + f_s(\lambda_2 x)] = \lambda_1 x \bullet_s \lambda_2 x$, for $(\lambda_1 + \lambda_2) f_s(x) = f_s(\lambda_1 x) + f_s(\lambda_2 x) \in [0,1]$.
4. For Equation (21): $x^{\lambda_1+\lambda_2} = f_t^{-1}[(\lambda_1 + \lambda_2) f_t(x)] = f_t^{-1}[\lambda_1 f_t(x) + \lambda_2 f_t(x)] = f_t^{-1}[f_t(x^{\lambda_1}) + f_t(x^{\lambda_2})] = x^{\lambda_1} \bullet_t x^{\lambda_2}$, for $(\lambda_1 + \lambda_2) f_t(x) = f_t(x^{\lambda_1}) + f_t(x^{\lambda_2}) \in [0,1]$.

□

3. n-Pythagorean Fuzzy Set and Yager Aggregation Operators

Definition 3. *Let F be a set of all fuzzy sets for the nonempty space X. Any function $p : X \to [0,1] \times [0,1]$ defined for any $\mu_p, v_p \in F$ is:*

$$p = \{\langle x, \langle \mu_p(x), v_p(x) \rangle \rangle : x \in X\}. \tag{22}$$

*It is called the **n-Pythagorean fuzzy set** (n-PFS) if the following condition is satisfied (for any natural number $n > 0$):*

$$0 \leq (\mu_p(x))^n + (v_p(x))^n \leq 1, \text{ for any } x \in X. \tag{23}$$

Let **n-PFS** mean set of all n-PFS.

The fuzzy sets μ_p, v_p indicate the membership and non-membership functions. Zhang and Xu [6] considered $p(x) = \langle \mu_p(x), v_p(x) \rangle$ as n-Pythagorean fuzzy number (n-PFN) represented by $p = \langle \mu_p, v_p \rangle$. The notation is used:

$$\textbf{n-PFN} =_{df} \{\langle \mu, v \rangle \in [0,1] \times [0,1] : 0 \leq \mu^n + v^n \leq 1\}. \tag{24}$$

Fact 2.

$$\textbf{\textit{n-PFN}} = \{p(x) : x \in X, p \in \textbf{\textit{n-PFS}}\}. \tag{25}$$

When $n = 1$, then the 1-Pythagorean fuzzy sets are the intuitionistic fuzzy sets (IFS), which were studied by Atanassow [2]. Moreover, when $n = 2$, then the 2-Pythagorean fuzzy sets are the PFS of Yager [4].

Simple arithmetic properties of inequalities $0 \leq \mu^{n+1} + v^{n+1} \leq \mu^n + v^n \leq 1$ result from:

Theorem 5. *For any natural number $n > 1$*

$$\textbf{\textit{n-PFN}} \subseteq \textbf{\textit{(n+1)-PFN}} \subseteq [0,1] \times [0,1]. \tag{26}$$

Thus, entering a power scale for a value of the membership and non-membership functions allows to replace $\langle \mu, v \rangle \in [0,1] \times [0,1]$ such that $\mu + v > 1$, by $\langle \mu^n, v^n \rangle \in$ **1-PFN**, for some n. As a result, the aggregation operations on the *IFS* can be extended to the aggregation operations on the *n-PFS*.

Theorem 6. *In any system $\mathbf{S_{Yager}} = \langle [0,1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1 \rangle$, for any $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in$ **1-PFN** and a number $\lambda \in [0,1]$, the following conditions are satisfied:*

$$\langle \mu_1 \bullet_s \mu_2, v_1 \bullet_t v_2 \rangle \in \textbf{1-PFN}, \tag{27}$$

$$\langle \mu_1 \bullet_t \mu_2, v_1 \bullet_s v_2 \rangle \in \textbf{1-PFN}, \tag{28}$$

Furthermore, in some systems $\mathbf{S_{Yager}}$ *(not in all - see Proof of Theorem 3,4 and Remark 1) there are additional conditions:*

$$\langle \lambda \bullet_l \mu_1, \lambda \bullet_p v_1 \rangle \in \textbf{1-}\boldsymbol{PFN}, \tag{29}$$

$$\langle \lambda \bullet_p \mu_1, \lambda \bullet_l v_1 \rangle \in \textbf{1-}\boldsymbol{PFN}. \tag{30}$$

Proof of Theorem 6.

1. For Equation (27): $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in$ **1-PFN** iff $\mu_1 + v_1 \leq 1, \mu_2 + v_2 \leq 1$ if and only if $\mu_1 \leq 1 - v_1, \mu_2 \leq 1 - v_2$.

   Hence and from the monotonicity of the s-norm and its determination by t-norm:

   $\mu_1 \bullet_s \mu_2 \leq (1 - v_1) \bullet_s (1 - v_2) = 1 - (1 - (1 - v_1) \bullet_s (1 - v_2) = 1 - v_1 \bullet_t v_2$ iff $\mu_1 \bullet_s \mu_2 + v_1 \bullet_t v_2 \leq 1$ iff $\langle \mu_1 \bullet_s \mu_2, v_1 \bullet_t v_2 \rangle \in$ **1-PFN**.
2. For Equation (28): $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in$ **1-PFN** iff $\langle v_1, \mu_1 \rangle, \langle v_2, \mu_2 \rangle \in$ **1-PFN**.

   Hence, and from point 1, there is $\langle v_1 \bullet_s v_2, \mu_1 \bullet_t \mu_2 \rangle \in$ **1-PFN**, which is equivalent to $\langle \mu_1 \bullet_t \mu_2, v_1 \bullet_s v_2 \rangle \in$ **1-PFN**.
3. For Equation (29): let $\mu \bullet_s v = \max\{\mu, v\}, \mu \bullet_t v = \min\{\mu, v\}$, then $f_s(x) = x, f_t(x) = 1^{\smile} x$ (see Example 1);

   $\langle \mu_1, v_1 \rangle \in$ **1-PFN** iff $\mu_1 + v_1 \leq 1$;

   $\lambda \bullet_l \mu_1 = f_s^{-1}[\lambda f_s(\mu_1)] = \lambda \mu_1$ and $\lambda \bullet_p v_1 = f_s^{-1}[\lambda f_s(v_1)] = 1 - \lambda(1 - v_1)$;

   $\lambda \bullet_l \mu_1 + \lambda \bullet_p v_1 = \lambda \mu_1 + 1 - \lambda(1 - v_1) = 1 - \lambda + \lambda(\mu_1 + v_1)$;

   Since $\lambda(\mu_1 + v_1) \leq \lambda$, so $0 \leq 1 - \lambda + \lambda(\mu_1 + v_1) \leq 1$.
4. For Equation (30): let $\mu \bullet_s v = \max\{\mu, v\}, \mu \bullet_t v = \min\{\mu, v\}$, then $f_s(x) = x, f_t(x) = 1^{\smile} x$;

   $\langle \mu_1, v_1 \rangle \in$ **1-PFN** iff $\mu_1 + v_1 \leq 1$ iff $\langle v_1, \mu_1 \rangle \in$ **1-PFN**.

   Hence, and from point 3, there is $\langle \lambda \bullet_l v_1, \lambda \bullet_p \mu_1 \rangle \in$ **1-PFN**, which is equivalent to $\langle \lambda \bullet_p \mu_1, \lambda \bullet_l v_1 \rangle \in$ **1-PFN**.

□

Remark 1. *Assuming generators of triangular norms from the Example 2* $\lambda \bullet_l \mu_1 = \lambda^{1/p} \mu_1, \lambda \bullet_p v_1 = 1 - \lambda^{1/p}(1 - v_1)$*, for* $\mu_1 = 1/2, v_1 = 7/8, \lambda = 1/4$ *and* $p = 2$*, it is obtained that* $\langle \lambda \bullet_l \mu_1, \lambda \bullet_p v_1 \rangle = \langle 1/8, 15/16 \rangle \notin$ ***1-PFN***$, (1 < 1/8 + 15/16)$.

Theorem 7. *Let the system* $\mathbf{S_{Yager}} = \langle [0,1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1 \rangle$ *conditions from Equations (27)–(30) of the Theorem 6 apply. Then, for any natural number* $n > 1$*, for any* $\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in$ ***n-PFN****, and number* $\lambda \in [0,1]$ *the following conditions are satisfied:*

$$\langle (\mu_1^n \bullet_s \mu_2^n)^{1/n}, (v_1^n \bullet_t v_2^n)^{1/n} \rangle \in \textbf{\textit{n-PFN}}, \tag{31}$$

$$\langle (\mu_1^n \bullet_t \mu_2^n)^{1/n}, (v_1^n \bullet_s v_2^n)^{1/n} \rangle \in \textbf{\textit{n-PFN}}, \tag{32}$$

$$\langle (\lambda \bullet_l \mu_1^n)^{1/n}, (\lambda \bullet_p v_1^n)^{1/n} \rangle \in \textbf{\textit{n-PFN}}, \tag{33}$$

$$\langle (\lambda \bullet_p \mu_1^n)^{1/n}, (\lambda \bullet_l v_1^n)^{1/n} \rangle \in \textbf{\textit{n-PFN}}. \tag{34}$$

Proof of Theorem 7.

$$\langle \mu_1, v_1 \rangle, \langle \mu_2, v_2 \rangle \in \textbf{n-PFN} \text{ iff } \langle \mu_1^n, v_1^n \rangle, \langle \mu_2^n, v_2^n \rangle \in \textbf{1-PFN}. \tag{35}$$

Then, the conditions of Theorem 6 are satisfied, which are equivalent to the above conditions (31)–(34). □

Definition 4. *In the system* $\mathbf{S_{Yager}} = \langle [0,1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1\rangle$*, the following aggregation operators are defined: the* ***Yager operators on n-PFN****: for any* $\langle \mu_1, v_1\rangle, \langle \mu_2, v_2\rangle \in$ ***n-PFN*** *and number* $\lambda \in [0,1]$.

When conditions (27)–(30) of the Theorem 6 are satisfied:

$$\langle \mu_1, v_1\rangle \oplus \langle \mu_2, v_2\rangle = \langle (\mu_1^n \bullet_s \mu_2^n)^{1/n}, (v_1^n \bullet_t v_2^n)^{1/n}\rangle, \tag{36}$$

$$\langle \mu_1, v_1\rangle \oplus \langle \mu_2, v_2\rangle = \langle (\mu_1^n \bullet_t \mu_2^n)^{1/n}, (v_1^n \bullet_s v_2^n)^{1/n}\rangle, \tag{37}$$

$$\lambda\langle \mu_1, v_1\rangle = \langle (\lambda \bullet_l \mu_1^n)^{1/n}, (\lambda \bullet_p v_1^n)^{1/n}\rangle, \tag{38}$$

$$\langle \mu_1, v_1\rangle\lambda = \langle (\lambda \bullet_p \mu_1^n)^{1/n}, (\lambda \bullet_l v_1^n)^{1/n}\rangle. \tag{39}$$

when $\langle (\lambda \bullet_l \mu_1^n)^{1/n}, (\lambda \bullet_p v_1^n)^{1/n}\rangle \notin$ ***n-PFN****, then:*

$$\lambda\langle \mu_1, v_1\rangle = \langle 1, 0\rangle, \tag{40}$$

or when $\langle (\lambda \bullet_p \mu_1^n)^{1/n}, (\lambda \bullet_l v_1^n)^{1/n}\rangle \notin$ ***n-PFN****, then:*

$$\langle \mu_1, v_1\rangle^\lambda = \langle 0, 1\rangle. \tag{41}$$

Hence, in any system $\mathbf{S_{Yager}} = \langle [0,1], \bullet_t, \bullet_s, \bullet_p, \bullet_l, 0, 1\rangle$, using the Theorem 4, there is:

Theorem 8. *For any* $\langle \mu_1, v_1\rangle, \langle \mu_2, v_2\rangle, \langle \mu_3, v_3\rangle \in$ ***n-PFN*** *and number* $\lambda \in [0,1]$*:*

$$\langle \mu_1, v_1\rangle \oplus \langle 0, 1\rangle = \langle \mu_1, v_1\rangle, \langle \mu_1, v_1\rangle \oplus \langle 1, 0\rangle = \langle 1, 0\rangle, \tag{42}$$

$$\langle \mu_1, v_1\rangle \oplus \langle \mu_2, v_2\rangle = \langle \mu_2, v_2\rangle \oplus \langle \mu_1, v_1\rangle, \tag{43}$$

$$(\langle \mu_1, v_1\rangle \oplus \langle \mu_2, v_2\rangle) \oplus \langle \mu_3, v_3\rangle = \langle \mu_1, v_1\rangle \oplus (\langle \mu_2, v_2\rangle \oplus \langle \mu_3, v_3\rangle), \tag{44}$$

$$\langle \mu_1, v_1\rangle \oplus \langle 1, 0\rangle = \langle \mu_1, v_1\rangle, \langle \mu_1, v_1\rangle \otimes \langle 0, 1\rangle = \langle 0, 1\rangle, \tag{45}$$

$$\langle \mu_1, v_1\rangle \oplus \langle \mu_2, v_2\rangle = \langle \mu_2, v_2\rangle \oplus \langle \mu_1, v_1\rangle, \tag{46}$$

$$(\langle \mu_1, v_1\rangle \oplus \langle \mu_2, v_2\rangle) \oplus \langle \mu_3, v_3\rangle = \langle \mu_1, v_1\rangle \oplus (\langle \mu_2, v_2\rangle \oplus \langle \mu_3, v_3\rangle), \tag{47}$$

$$\lambda(\langle \mu_1, v_1\rangle \oplus \langle \mu_2, v_2\rangle) = \lambda\langle \mu_1, v_1\rangle \oplus \lambda\langle \mu_2, v_2\rangle, \tag{48}$$

$$(\langle \mu_1, v_1\rangle \oplus \langle \mu_2, v_2\rangle)^\lambda = \langle \mu_1, v_1\rangle^\lambda \oplus \langle \mu_2, v_2\rangle^\lambda, \tag{49}$$

$$(\lambda_1 + \lambda_2)\langle \mu_1, v_1\rangle = \lambda_1\langle \mu_1, v_1\rangle \oplus \lambda_2\langle \mu_1, v_1\rangle, \tag{50}$$

$$\langle \mu_1, v_1\rangle^{\lambda_1+\lambda_2} = \langle \mu_1, v_1\rangle^{\lambda_1} \oplus \langle \mu_1, v_1\rangle^{\lambda_2}. \tag{51}$$

4. Triangular Norms in the n-PFN and the n-PFS Algebra

Definition 5. *For any* $\langle \mu_1, v_1\rangle, \langle \mu_2, v_2\rangle \in$ ***n-PFN****:*

$$\langle \mu_1, v_1\rangle \leq_n \langle \mu_2, v_2\rangle \text{ iff } \mu_1 \leq \mu_2, v_1 \geq v_2. \tag{52}$$

The results of operations maximum and minimum for any $A \subseteq$ ***n-PFN*** *are described for the relation* $\leq_n$ *and are denoted by:* $\max_n A, \min_n A$.

Fact 3. *For any* $x, y \in [0,1]$*:*

1. $\langle 0,1\rangle \leq_n \langle 0,x\rangle \leq_n \langle 0,0\rangle \leq_n \langle y,0\rangle \leq_n \langle 1,0\rangle$,
2. $\langle 0,1\rangle \leq_n \langle x,y\rangle \leq_n \langle 1,0\rangle$, *when* $\langle x,y\rangle \in$ ***n-PFN***.
3. $\langle 0,1\rangle = \min_n$ ***n-PFN***, $\langle 1,0\rangle = \max_n$ ***n-PFN***.

Fact 4. *There is:*

$$\textbf{\textit{n-PFN}} = \{\langle x, y\rangle \in [0,1]\times[0,1] : \langle 0,1\rangle \leq_n \langle x, y\rangle \leq_n \langle 1,0\rangle\}. \tag{53}$$

Definition 6. *There are:*

1. *The operation* $\circ_t$: ***n-PFN*** $\times$ ***n-PFN*** $\rightarrow$ ***n-PFN*** *is called a* ***t-norm in the set n-PFN*** *ordered by the relations* $\leq_n$*, when for any* $\langle \mu_1, v_1\rangle, \langle \mu_2, v_2\rangle, \langle \mu_3, v_3\rangle \in$ ***n-PFN****:*

   (a) *boundary conditions*

   $$\langle 0,1\rangle \circ_t \langle \mu_1, v_1\rangle = \langle 0,1\rangle, \langle \mu_1, v_1\rangle \circ_t \langle 1,0\rangle = \langle \mu_1, v_1\rangle, \tag{54}$$

   (b) *monotonicity*

   $$\langle \mu_1, v_1\rangle \circ_t \langle \mu_2, v_2\rangle \leq_n \langle \mu_3, v_3\rangle \circ_t \langle \mu_2, v_2\rangle, \text{ when } \langle \mu_1, v_1\rangle \leq_n \langle \mu_3, v_3\rangle, \tag{55}$$

   (c) *commutativity*

   $$\langle \mu_1, v_1\rangle \circ_t \langle \mu_2, v_2\rangle = \langle \mu_2, v_2\rangle \circ_t \langle \mu_1, v_1\rangle, \tag{56}$$

   (d) *associativity*

   $$\langle \mu_1, v_1\rangle \circ_t (\langle \mu_2, v_2\rangle \circ_t \langle \mu_3, v_3\rangle) = (\langle \mu_1, v_1\rangle \circ_t \langle \mu_2, v_2\rangle) \circ_t \langle \mu_3, v_3\rangle. \tag{57}$$

2. *The operation* $\circ_s$: ***n-PFN*** $\times$ ***n-PFN*** $\rightarrow$ ***n-PFN*** *is called the* ***s-norm in the set n-PFN*** *ordered by the relations* $\leq_n$*, when for any* $\langle \mu_1, v_1\rangle, \langle \mu_2, v_2\rangle, \langle \mu_3, v_3\rangle \in$ ***n-PFN****:*

   (a) *boundary conditions*

   $$\langle 1,0\rangle \circ_s \langle \mu_1, v_1\rangle = \langle 1,0\rangle, \langle \mu_1, v_1\rangle \circ_s \langle 0,1\rangle = \langle \mu_1, v_1\rangle, \tag{58}$$

   (b) *monotonicity*

   $$\langle \mu_1, v_1\rangle \circ_s \langle \mu_2, v_2\rangle \leq_n \langle \mu_3, v_3\rangle \circ_s \langle \mu_2, v_2\rangle, \text{ when } \langle \mu_1, v_1\rangle \leq_n \langle \mu_3, v_3\rangle, \tag{59}$$

   (c) *commutativity*

   $$\langle \mu_1, v_1\rangle \circ_s \langle \mu_2, v_2\rangle = \langle \mu_2, v_2\rangle \circ_s \langle \mu_1, v_1\rangle, \tag{60}$$

   (d) *associativity*

   $$\langle \mu_1, v_1\rangle \circ_s (\langle \mu_2, v_2\rangle \circ_s \langle \mu_3, v_3\rangle) = (\langle \mu_1, v_1\rangle \circ_s \langle \mu_2, v_2\rangle) \circ_s \langle \mu_3, v_3\rangle. \tag{61}$$

Theorem 9. *The Yager operator* $\otimes$ *on the* ***n-PFN*** *is a t-norm in the set* ***n-PFN*** *and the operator* $\oplus$ *is a s-norm in the set* ***n-PFN****.*

Proof of Theorem 9. Conditions (45)–(47) of the Theorem 8 proof that the operator $\otimes$ satisfies conditions (a),(c), and (d) of the Definition 6 (1) of the t-norm in the set **n-PFN**. It is enough to prove that this operation is monotonous.

For any $\langle \mu_1, v_1\rangle, \langle \mu_2, v_2\rangle, \langle \mu_3, v_3\rangle \in$ **n-PFN**
$\langle \mu_1, v_1\rangle \otimes \langle \mu_2, v_2\rangle = \langle (\mu_1^n \bullet_t \mu_2^n)^{1/n}, (v_1^n \bullet_s v_2^n)^{1/n}\rangle$,
$\langle \mu_3, v_3\rangle \otimes \langle \mu_2, v_2\rangle = \langle (\mu_3^n \bullet_t \mu_2^n)^{1/n}, (v_3^n \bullet_s v_2^n)^{1/n}\rangle$.

Let $\langle \mu_1, v_1\rangle \leq_n \langle \mu_3, v_3\rangle$. Then $\mu_1 \leq \mu_3, v_1 \geq v_3$. Hence, and from the monotonicity of the t-norm and s-norm, it is obtained that:

$\mu_1^n \bullet_t \mu_2^n \leq \mu_3^n \bullet_t \mu_2^n$ and $v_3^n \bullet_s v_2^n \leq v_1^n \bullet_s v_2^n$ iff
$(\mu_1^n \bullet_t \mu_2^n)^{1/n} \leq (\mu_3^n \bullet_t \mu_2^n)^{1/n}$ and $(v_1^n \bullet_s v_2^n)^{1/n} \geq (v_3^n \bullet_s v_2^n)^{1/n}$ iff
$\langle (\mu_1^n \bullet_t \mu_2^n)^{1/n}, (v_1^n \bullet_s v_2^n)^{1/n}\rangle \leq_n \langle (\mu_3^n \bullet_t \mu_2^n)^{1/n}, (v_3^n \bullet_s v_2^n)^{1/n}\rangle$ iff
$\langle \mu_1, v_1\rangle \otimes \langle \mu_2, v_2\rangle \leq_n \langle \mu_3, v_3\rangle \otimes \langle \mu_2, v_2\rangle$.

Proof that the operator $\oplus$ satisfies conditions of the Definition 6 (2) about the s-norm in the set **n-PFN** is analogical. □

Summarizing, for the knowledge of operations and relationships introduced in the **n-PFN**, the following operations and conclusion relationships can be determined for the **n-PFS**:

Definition 7. *For any $p_1, p_2 \in$ **n-PFS**, and number $\lambda \in [0,1]$:*

$$p_1 \oplus p_2 = \{\langle x, \langle \mu_{p_1}(x), v_{p_1}(x)\rangle \oplus \langle \mu_{p_2}(x), v_{p_2}(x)\rangle\rangle : x \in X\}, \tag{62}$$

$$p_1 \otimes p_2 = \{\langle x, \langle \mu_{p_1}(x), v_{p_1}(x)\rangle \otimes \langle \mu_{p_2}(x), v_{p_2}(x)\rangle\rangle : x \in X\}, \tag{63}$$

$$\lambda p_1 = \{\langle x, \lambda\langle \mu_{p_1}(x), v_{p_1}(x)\rangle\rangle : x \in X\}, \tag{64}$$

$$p_1^{\lambda} = \{\langle x, \langle \mu_{p_1}(x), v_{p_1}(x)\rangle^{\lambda}\rangle : x \in X\}, \tag{65}$$

$$\text{supp}(p_1) = \{x \in X : \mu_{p_1}(x) > 0, v_{p_1}(x) > 0\}, \tag{66}$$

$$p_1 \subseteq_n p_2 \text{ iff for any } x \in X, \langle \mu_{p_1}(x), v_{p_1}(x)\rangle \leq_n \langle \mu_{p_2}(x), v_{p_2}(x)\rangle. \tag{67}$$

The system n-PFS, with defined in the Definition 7 operations (t-norm, s-norm, p-norm, l-norm, and support) and inclusion relations, is called the **n-PFS algebra**.

Let $\mathbf{1} =_{df} \langle 0,1\rangle, \mathbf{0} =_{df} \langle 1,0\rangle$. Then from the Definition 7 and the Theorem 8 there are:

Theorem 10. *In the algebra n-PFS, for any $p_1, p_2, p_3 \in$ **n-PFN** and the number $\lambda \in [0,1]$:*

$$p_1 \oplus \mathbf{0} = p_1, p_1 \oplus \mathbf{1} = \mathbf{1}, \tag{68}$$

$$p_1 \oplus p_2 = p_2 \oplus p_1, \tag{69}$$

$$(p_1 \oplus p_2) \oplus p_3 = p_1 \oplus (p_2 \oplus p_3), \tag{70}$$

$$p_1 \otimes \mathbf{1} = p_1, p_1 \otimes \mathbf{0} = \mathbf{0}, \tag{71}$$

$$p_1 \otimes p_2 = p_2 \otimes p_1, \tag{72}$$

$$(p_1 \otimes p_2) \otimes p_3 = p_1 \otimes (p_2 \otimes p_3), \tag{73}$$

$$\lambda(p_1 \oplus p_2) = \lambda p_1 \oplus \lambda p_2, \tag{74}$$

$$(p_1 \otimes p_2)^{\lambda} = p_1^{\lambda} \otimes p_2^{\lambda}, \tag{75}$$

$$(\lambda_1 + \lambda_2)p_1 = \lambda_1 p_1 \oplus \lambda_2 p_1, \tag{76}$$

$$p_1^{\lambda_1+\lambda_2} = p_1^{\lambda_1} \otimes p_1^{\lambda_2}. \tag{77}$$

5. Conclusions

This paper presents the elements of deductive theory of the n-PFS, where the membership degree μ and the non-membership degree v determine not only in the square scale, but in any power scale, i.e., $0 \leq \mu^n + v^n \leq 1$. As a result, any local deductions in the n-PFS range can be formulated. It may be interesting to use the described model to create similar models but based on other functional scales, for example for the function of scale $f_n(x) = x/n : 0 \leq \mu/n + v/n \leq 1$ or $f_n(x) = 1 + \log_n(x+1)$, for $x \in [0,1]$, where $f(x) < x : 0 \leq (l + \log_n(\mu+1)) + (1 + \log_n(v+1)) \leq 1$. In the research, the n-PFS theory can be used to describe n-PFS as a system of information granules [19].

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