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Nonlocal Conservation Laws of PDEs Possessing Differential Coverings †

Iosif Krasil′shchik

V.A. Trapeznikov Institute of Control Sciences RAS, Profsoyuznaya 65, 117342 Moscow, Russia; josephkra@gmail.com

† To the memory of Alexandre Vinogradov, my teacher.

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Abstract: In his 1892 paper, L. Bianchi noticed, among other things, that quite simple transformations of the formulas that describe the Bäcklund transformation of the sine-Gordon equation lead to what is called a nonlocal conservation law in modern language. Using the techniques of differential coverings, we show that this observation is of a quite general nature. We describe the procedures to construct such conservation laws and present a number of illustrative examples.

Keywords: nonlocal conservation laws; differential coverings

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1. Introduction

In [1], L. Bianchi, dealing with the celebrated Bäcklund auto-transformation (I changed the original notation slightly)

$$\frac{\partial(u-w)}{\partial x} = \sin(u+w), \quad \frac{\partial(u+w)}{\partial y} = \sin(u-w) \tag{1}$$

for the sine-Gordon equation

$$\frac{\partial^2(2u)}{\partial x \partial y} = \sin(2u) \tag{2}$$

in the course of intermediate computations (see ([1], p. 10)) notices that the function

$$\psi = \ln \frac{\partial u}{\partial C},$$

where C is an arbitrary constant on which the solution u may depend, enjoys the relations

$$\frac{\partial \psi}{\partial x} = \cos(u+w), \quad \frac{\partial \psi}{\partial y} = \cos(u-w).$$

Reformulated in modern language, this means that the 1-form

$$\omega = \cos(u+w)\,dx + \cos(u-w)\,dy$$

is a nonlocal conservation law for Equation (1).

It became clear much later, some 100 years after the publication of [1], that nonlocal conservation laws are important invariants of PDEs and are used in numerous applications, e.g.,: numerical methods [2,3], sociological models [4,5], integrable systems [6], electrodynamics [7,8], mechanics [9–11], etc.

Actually, Bianchi's observation is of a very general nature and this is shown below.

In Section 2, I shortly introduce the basic constructions in nonlocal geometry of PDEs, i.e., the theory of differential coverings, [12]. Section 3 contains an interpretation of the result by L. Bianchi in the most general setting. In Section 4, a number of examples is discussed.

Everywhere below we use the notation $\mathscr{F}(\cdot)$ for the $\mathbb{R}$-algebra of smooth functions, $D(\cdot)$ for the Lie algebra of vector fields, and $\Lambda^*(\cdot) = \oplus_{k\geq 0}\Lambda^k(\cdot)$ for the exterior algebra of differential forms.

2. Preliminaries

Following [13], we deal with infinite prolongations $\mathscr{E} \subset J^\infty(\pi)$ of smooth submanifolds in $J^k(\pi)$, where $\pi\colon E \to M$ is a smooth locally trivial vector bundle over a smooth manifold M, $\dim M = n$, $\operatorname{rank}\pi = m$. These $\mathscr{E}$ are differential equations for us. Solutions of $\mathscr{E}$ are graphs of infinite jets that lie in $\mathscr{E}$. In particular, $\mathscr{E} = J^\infty(\pi)$ is the tautological equation $0 = 0$.

The bundle $\pi_\infty\colon \mathscr{E} \to M$ is endowed with a natural flat connection $\mathscr{C}\colon D(M) \to D(\mathscr{E})$ called the *Cartan connection*. Flatness of $\mathscr{C}$ means that $\mathscr{C}_{[X,Y]} = [\mathscr{C}_X, \mathscr{C}_Y]$ for all $X, Y \in D(M)$. The distribution on $\mathscr{E}$ spanned by the fields of the form $\mathscr{C}_X$ (the *Cartan distribution*) is Frobenius integrable. We denote it by $\mathscr{C} \subset D(\mathscr{E})$ as well.

A (higher infinitesimal) *symmetry* of $\mathscr{E}$ is a π_∞-vertical vector field $S \in D(\mathscr{E})$ such that $[X, \mathscr{C}] \subset \mathscr{C}$.

Consider the submodule $\Lambda_h^k(\mathscr{E})$ generated by the forms $\pi_\infty^*(\theta)$, $\theta \in \Lambda^k(M)$. Elements $\omega \in \Lambda_h^k(\mathscr{E})$ are called horizontal k-forms. Generalizing slightly the action of the Cartan connection, one can apply it to the de Rham differential $d\colon \Lambda^k(M) \to \Lambda^{k+1}(M)$ and obtain the *horizontal de Rham* complex

$$0 \longrightarrow \mathscr{F}(\mathscr{E}) \longrightarrow \dots \longrightarrow \Lambda_h^k(\mathscr{E}) \xrightarrow{d_h} \Lambda_h^{k+1}(\mathscr{E}) \longrightarrow \dots \longrightarrow \Lambda_h^n(\mathscr{E}) \longrightarrow 0$$

on $\mathscr{E}$. Elements of its $(n-1)$st cohomology group $H_h^{n-1}(\mathscr{E})$ are called *conservation laws* of $\mathscr{E}$. We always assume $\mathscr{E}$ to be *differentially connected* which means that $H_h^0(\mathscr{E}) = \mathbb{R}$.

Remark 1. *The concept of a differentially connected equation reflects Vinogradov's* correspondence principle *[14], (p. 195): when 'secondary dimension' (dimension of the Cartan distribution)* Dim $\to 0$, *the objects of PDE geometry degenerate to their counterparts in geometry of finite-dimensional manifolds. Following this principle, we informally have*

$$\lim_{\mathrm{Dim}\to 0} H_h^i(\mathscr{E}) = H_{\mathrm{dR}}^i(M).$$

Since $H_{\mathrm{dR}}^0(M)$ is responsible for topological connectedness of M, the group $H_h^0(\mathscr{E})$ stands for differential one.

Coordinates. Consider a trivialization of π with local coordinates $x^1, \dots, x^n$ in $\mathscr{U} \subset M$ and $u^1, \dots, u^m$ in the fibers of $\pi|_{\mathscr{U}}$. Then in $\pi_\infty^{-1}(\mathscr{U}) \subset J^\infty(\pi)$ the adapted coordinates u_σ^i arise and the Cartan connection is determined by the total derivatives

$$\mathscr{C}\colon \frac{\partial}{\partial x^i} \mapsto D_i = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u_{\sigma i}^j \frac{\partial}{\partial u_\sigma^j}.$$

Let $F = (F^1, \dots, F^r)$, where F^j are smooth functions on $J^k(\pi)$. The the infinite prolongation of the locus

$$\{\, z \in J^k(\pi) \mid F^1(z) = \dots = F^r(z) = 0 \,\} \subset J^k(\pi)$$

is defined by the system

$$\mathscr{E} = \mathscr{E}_F = \{\, z \in J^\infty(\pi) \mid D_\sigma(F^j)(z) = 0,\ j = 1, \dots, r,\ |\sigma| \geq 0 \,\},$$

where D_σ denotes the composition of the total derivatives corresponding to the multi-index σ. The total derivatives, as well as all differential operators in total derivatives, can be restricted to infinite prolongations and we preserve the same notation for these restrictions. Given an $\mathscr{E}$, we always choose internal local coordinates in it for subsequent computations. To restrict an operator to $\mathscr{E}$ is to express this operator in terms of internal coordinates.

Any symmetry of $\mathscr{E}$ is an evolutionary vector field

$$\mathbf{E}_\varphi = \sum D_\sigma(\varphi^j)\frac{\partial}{\partial u^j_\sigma}$$

(summation on internal coordinates), where the functions $\varphi^1, \dots, \varphi^m \in \mathscr{F}(\mathscr{E})$ satisfy the system

$$\sum_{\sigma,\alpha} \frac{\partial F^j}{\partial u^\alpha_\sigma} D_\sigma(\varphi^\alpha) = 0, \quad j = 1, \dots, r.$$

A horizontal $(n-1)$-form

$$\omega = \sum_i a_i\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$$

defines a conservation law of $\mathscr{E}$ if

$$\sum_i (-1)^{i+1} D_i(a_i) = 0.$$

We are interested in nontrivial conservation laws, i.e., such that ω is not exact.

Finally, $\mathscr{E}$ is differentially connected if the only solutions of the system

$$D_1(f) = \cdots = D_n(f) = 0, \quad f \in \mathscr{F}(\mathscr{E}),$$

are constants.

Consider now a locally trivial bundle $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ such that there exists a flat connection $\tilde{\mathscr{C}}$ in $\pi_\infty \circ \tau\colon \tilde{\mathscr{E}} \to M$. Following [12], we say that τ is a (*differential*) *covering* over $\mathscr{E}$ if one has

$$\tau_*(\tilde{\mathscr{C}}_X) = \mathscr{C}_X$$

for any vector field $X \in D(M)$. Objects existing on $\tilde{\mathscr{E}}$ are nonlocal for $\mathscr{E}$: e.g., symmetries of $\tilde{\mathscr{E}}$ are *nonlocal symmetries* of $\mathscr{E}$, conservation laws of $\tilde{\mathscr{E}}$ are *nonlocal conservation* laws of $\mathscr{E}$, etc. A derivation $S\colon \mathscr{F}(\mathscr{E}) \to \mathscr{F}(\tilde{\mathscr{E}})$ is called a *nonlocal shadow* if the diagram

$$\begin{tikzcd} \mathscr{F}(\mathscr{E}) \arrow[r, "\mathscr{C}_X"] \arrow[d, "S"'] & \mathscr{F}(\mathscr{E}) \arrow[d, "S"] \\ \mathscr{F}(\tilde{\mathscr{E}}) \arrow[r, "\tilde{\mathscr{C}}_X"] & \mathscr{F}(\tilde{\mathscr{E}}) \end{tikzcd}$$

is commutative for any $X \in D(M)$. In particular, any symmetry of the equation $\mathscr{E}$, as well as restrictions $\tilde{S}\big|_{\mathscr{F}(\mathscr{E})}$ of nonlocal symmetries may be considered as shadows. A nonlocal symmetry is said to be *invisible* if its shadow $\tilde{S}\big|_{\mathscr{F}(\mathscr{E})}$ vanishes.

A covering τ is said to be *irreducible* if $\tilde{\mathscr{E}}$ is differentially connected. Two coverings are equivalent if there exists a diffeomorphism $g\colon \tilde{\mathscr{E}}_1 \to \tilde{\mathscr{E}}_2$ such that the diagrams

$$
\begin{tikzcd}
\tilde{\mathscr{E}}_1 \arrow[rr, "g"] \arrow[dr, "\tau_1"'] & & \tilde{\mathscr{E}}_2 \arrow[dl, "\tau_2"] \\
& \mathscr{E}, &
\end{tikzcd}
\qquad
\begin{tikzcd}
D(\tilde{\mathscr{E}}_1) \arrow[rr, "g_*"] & & D(\tilde{\mathscr{E}}_2) \\
& D(M) \arrow[ul, "\tilde{\mathscr{C}}_1"] \arrow[ur, "\tilde{\mathscr{C}}_2"'] &
\end{tikzcd}
$$

are commutative. Note also that for any two coverings their *Whitney product* is naturally defined. A covering is called *linear* if τ is a vector bundle and the action of vector fields $\tilde{\mathscr{C}}_X$ preserves the subspace of fiber-wise linear functions in $\mathscr{F}(\tilde{\mathscr{E}})$.

In the case of 2D equations, there exists a fundamental relation between special type of coverings over $\mathscr{E}$ and conservation laws of the latter. Let τ be a covering of rank $l < \infty$. We say that τ is an *Abelian covering* if there exist l independent conservation laws $[\omega_i] \in H_h^1(\mathscr{E})$, $i = 1, \dots, l$, such that the forms $\tau^*(\omega_i)$ are exact. Then equivalence classes of such coverings are in one-to-one correspondence with l-dimensional $\mathbb{R}$-subspaces in $H_h^1(\mathscr{E})$.

Coordinates. Choose a trivialization of the covering τ and let $w^1, \dots, w^l, \dots$ be coordinates in fibers (the are called nonlocal variables). Then the covering structure is given by the extended total derivatives

$$\tilde{D}_i = D_i + X_i, \quad i = 1, \dots, n,$$

where

$$X_i = \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}$$

are τ-vertical vector fields (nonlocal tails) enjoying the condition

$$D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \quad i < j. \tag{3}$$

Here $D_i(X_j)$ denotes the action of D_i on coefficients of X_j. Relations (3) (flatness of $\tilde{\mathscr{C}}$) amount to the fact that the manifold $\tilde{\mathscr{E}}$ endowed with the distribution $\tilde{\mathscr{C}}$ coincides with the infinite prolongation of the overdetermined system

$$\frac{\partial w^\alpha}{\partial x^i} = X_i^\alpha,$$

which is compatible modulo $\mathscr{E}$.

Irreducible coverings are those for which the system of vector fields $\tilde{D}_1, \dots, \tilde{D}_n$ has no nontrivial integrals. If $\bar{\tau}$ is another covering with the nonlocal tails $\bar{X}_i = \sum \bar{X}_i^\beta \partial/\partial \bar{w}^\beta$, then the Whitney product $\tau \oplus \bar{\tau}$ of τ and $\bar{\tau}$ is given by

$$\tilde{D}_i = D_i + \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha} + \sum_\beta \bar{X}_i^\beta \frac{\partial}{\partial \bar{w}^\beta}.$$

A covering is Abelian if the coefficients X_i^α are independent of nonlocal variables w^j. If $n = 2$ and $\omega_\alpha = X_1^\alpha\, dx^1 + X_2^\alpha\, dx^2$, $\alpha = 1, \dots, l$, are conservation laws of $\mathscr{E}$ then the corresponding Abelian covering is given by the system

$$\frac{\partial w^\alpha}{\partial x^i} = X_i^\alpha, \qquad i = 1, 2, \quad \alpha = 1, \dots, l,$$

or

$$\tilde{D}_i = D_i + \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}.$$

Vice versa, if such a covering is given, then one can construct the corresponding conservation law.

The horizontal de Rham differential on $\tilde{\mathscr{E}}$ is $\tilde{d}_h = \sum_i dx^i \wedge \tilde{D}_i$. A covering is linear if

$$X_i^\alpha = \sum_\beta X_{i,\beta}^\alpha w^\beta, \tag{4}$$

where $X_{i,\beta}^\alpha \in \mathscr{F}(\mathscr{E})$.

Remark 2. *Denote by* $\mathbf{X}_i$ *the* $\mathscr{F}(\mathscr{E})$*-valued matrix* $(X_{i,\beta}^\alpha)$ *that appears in* (4)*. Then Equation* (3) *may be rewritten as*

$$D_i(\mathbf{X_j}) - D_j(\mathbf{X_i}) + [\mathbf{X_i}, \mathbf{X_j}] = 0.$$

for linear coverings. Thus, a linear covering defines a zero-curvature representation *for* $\mathscr{E}$ *and vice versa.*

A nonlocal symmetry in τ is a vector field

$$S_{\varphi,\psi} = \sum \tilde{D}_\sigma(\varphi^j)\frac{\partial}{\partial u_\sigma^j} + \sum \psi^\alpha \frac{\partial}{\partial w^\alpha},$$

where the vector functions $\varphi = (\varphi^1, \dots, \varphi^m)$ and $\psi = (\psi^1, \dots, \psi^\alpha, \dots)$ on $\tilde{\mathscr{E}}$ satisfy the system of equations

$$\sum \frac{\partial F^j}{\partial u_\sigma^j}\tilde{D}_\sigma(\varphi^j) = 0, \tag{5}$$

$$\tilde{D}_i(\psi^\alpha) = \sum \frac{\partial X_i^\alpha}{\partial u_\sigma^j}\tilde{D}_\sigma(\varphi^j) + \sum \frac{\partial X_i^\alpha}{\partial w^\beta}\psi^\beta. \tag{6}$$

Nonlocal shadows are the derivations

$$\tilde{\mathbf{E}}_\varphi = \sum \tilde{D}_\sigma(\varphi^j)\frac{\partial}{\partial u_\sigma^j},$$

where φ satisfies Equation (5), invisible symmetries are

$$S_{0,\psi} = \sum \psi^\alpha \frac{\partial}{\partial w^\alpha},$$

where ψ satisfies

$$\tilde{D}_i(\psi^\alpha) = \sum \frac{\partial X_i^\alpha}{\partial w^\beta}\psi^\beta. \tag{7}$$

In what follows, we use the notation $\tau^{\mathbf{I}} \colon \tilde{\mathscr{E}}^{\mathbf{I}} \to \tilde{\mathscr{E}}$ for the covering defined by Equation (7).

Remark 3. *Equation* (7) *defines a linear covering over* $\tilde{\mathscr{E}}$*. Due to Remark 2, we see that for any non-Abelian covering we obtain in such a way a nonlocal zero-curvature representation with the matrices* $\mathbf{X}_i = (\partial X_i^\alpha / \partial w^\beta)$*.*

Remark 4. *The covering* $\tau^{\mathbf{I}} \colon \tilde{\mathscr{E}}^{\mathbf{I}} \to \tilde{\mathscr{E}}$ *is the vertical part of the* tangent covering $\mathbf{t} \colon \mathscr{T}\tilde{\mathscr{E}} \to \tilde{\mathscr{E}}$*, see the definition in [15].*

3. The Main Result

From now on we consider two-dimensional scalar equations with the independent variables x and y. We shall show that any such an equation that admits an irreducible covering possesses a (nonlocal) conservation law.

Example 1. *Let us revisit the Bianchi example discussed in the beginning of the paper. Equation (1) define a one-dimensional non-Abelian covering* $\tau\colon \tilde{\mathcal{E}} = \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ *over the sine-Gordon Equation (2) with the nonlocal variable* w*. Then the defining Equation (7) for invisible symmetries in this covering are*

$$\frac{\partial\psi}{\partial x} = -\cos(u+w)\psi, \quad \frac{\partial\psi}{\partial y} = -\cos(u-w)\psi.$$

This is a one-dimensional linear covering over $\tilde{\mathcal{E}}$ *which is equivalent to the Abelian covering*

$$\frac{\partial\bar{\psi}}{\partial x} = -\cos(u+w), \quad \frac{\partial\bar{\psi}}{\partial y} = -\cos(u-w),$$

where $\bar{\psi} = \ln\psi$*. Thus, we obtain the nonlocal conservation law*

$$\omega = -\cos(u+w)\,dx - \cos(u-w)\,dy$$

of the sine-Gordon equation.

The next result shows that Bianchi's observation is of a quite general nature.

Proposition 1. *Let* $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ *be a one-dimensional non-Abelian covering over* $\mathcal{E}$*. Then, if* τ *is irreducible,* $\tau^{\mathbf{I}}\colon \tilde{\mathcal{E}}^{\mathbf{I}} \to \tilde{\mathcal{E}}$ *defines a nontrivial conservation law of the equation* $\tilde{\mathcal{E}}$ *(and, consequently, of* $\mathcal{E}$ *too).*

Proof. Consider the total derivatives

$$D_x^{\mathbf{I}} = \tilde{D}_x + \frac{\partial X}{\partial w}\psi\frac{\partial}{\partial\psi} = D_x + X\frac{\partial}{\partial w} + \frac{\partial X}{\partial w}\psi\frac{\partial}{\partial\psi}$$
$$D_y^{\mathbf{I}} = \tilde{D}_y + \frac{\partial Y}{\partial w}\psi\frac{\partial}{\partial\psi} = D_y + Y\frac{\partial}{\partial w} + \frac{\partial Y}{\partial w}\psi\frac{\partial}{\partial\psi}$$

on $\mathcal{E}^{\mathbf{I}}$ and assume that $a \in \mathcal{F}(\tilde{\mathcal{E}})$ is a common nontrivial integral of these fields:

$$D_x^{\mathbf{I}}(a) = D_y^{\mathbf{I}}(a) = 0, \quad a \neq \text{const}. \tag{8}$$

Choose a point in $\mathcal{E}^{\mathbf{I}}$ and assume that the formal series

$$a_0 + a_1\psi + \dots + a_j\psi^j + \dots, \quad a_j \in \mathcal{F}(\tilde{\mathcal{E}}), \tag{9}$$

converges to a in a neighborhood of this point. Substituting relations (9) to (8) and equating coefficients at the same powers of ψ, we get

$$\tilde{D}_x(a_j) + j\frac{\partial X}{\partial w}a_j = 0, \quad \tilde{D}_y(a_j) + j\frac{\partial Y}{\partial w}a_j = 0, \qquad j = 0, 1, \dots,$$

and, since τ is irreducible, this implies that $a_0 = k_0 = \text{const}$ and

$$\frac{\tilde{D}_x(a_j)}{a_j} = j\frac{\tilde{D}_x(a_1)}{a_1}, \quad \frac{\tilde{D}_y(a_j)}{a_j} = j\frac{\tilde{D}_y(a_1)}{a_1}.$$

Hence, $a_j = k_j(a_1)^j$, $j > 0$. Substituting these relations to (9), we see that $a = a(\theta)$, where $\theta = a_1\psi$, $a_1 \in \mathcal{F}(\mathcal{E})$. Then Equation (8) take the form

$$\dot{a}\psi\left(\tilde{D}_x(a_1) + \frac{\partial X}{\partial w}\right) = 0, \quad \dot{a}\psi\left(\tilde{D}_y(a_1) + \frac{\partial Y}{\partial w}\right) = 0, \qquad \dot{a} = \frac{da}{d\theta}.$$

Thus

$$\frac{\partial X}{\partial w} = -\tilde{D}_x(a_1), \quad \frac{\partial Y}{\partial w} = -\tilde{D}_y(a_1)$$

and the function $w + a_1$ is a nontrivial integral of $\tilde{D}_x$ and $\tilde{D}_y$. Contradiction.

Finally, repeating the scheme of Example 1, we pass to the equivalent covering by setting $\bar{\psi} = \ln \psi$ and obtain the nontrivial conservation law

$$\omega = \frac{\partial X}{\partial w}\, dx + \frac{\partial Y}{\partial w}\, dy$$

on $\mathscr{E}^{\mathbf{I}}$. $\square$

Indeed, Bianchi's result has a further generalization. To formulate the latter, let us say that a covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is *strongly non-Abelian* if for any nontrivial conservation law ω of the equation $\mathscr{E}$ its lift $\tau^*(\omega)$ to the manifold $\tilde{\mathscr{E}}$ is nontrivial as well. Now, a straightforward generalization of Proposition 1 is

Proposition 2. *Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be an irreducible covering over a differentially connected equation. Then τ is a strongly non-Abelian covering if and only if the covering $\tau^{\mathbf{I}}$ is irreducible.*

We shall now need the following construction. Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a linear covering. Consider the fiber-wise *projectivization* $\tau^{\mathbf{P}}\colon \tilde{\mathscr{E}}^{\mathbf{P}} \to \mathscr{E}$ of the vector bundle τ. Denote by $\mathbf{p}\colon \tilde{\mathscr{E}} \to \mathscr{E}^{\mathbf{P}}$ the natural projection. Then, obviously, the projection $\mathbf{p}_*(\tilde{\mathscr{C}})$ is well defined and is an n-dimensional integrable distribution on $\mathscr{E}^{\mathbf{P}}$. Thus, we obtain the following commutative diagram of coverings

$$\begin{tikzcd} \tilde{\mathscr{E}} \arrow[rr, "\mathbf{p}"] \arrow[dr, "\tau"'] & & \mathscr{E}^{\mathbf{P}} \arrow[dl, "\tau^{\mathbf{P}}"] \\ & \mathscr{E}, & \end{tikzcd}$$

where $\operatorname{rank}(\mathbf{p}) = 1$ and $\operatorname{rank}(\tau^{\mathbf{P}}) = \operatorname{rank}(\tau) - 1$.

Proposition 3. *Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be an irredicible covering. Then the covering $\tau^{\mathbf{P}}$ is irreducible as well.*

Coordinates. Let $\operatorname{rank}(\tau) = l > 1$ and

$$w^\alpha_{x^i} = \sum_{\beta=1}^{l} X^\alpha_{i,\beta} w^\beta, \qquad i = 1, \dots, n, \quad \alpha = 1, \dots, l, \tag{10}$$

be the defining equations of the covering τ, see Equation (4). Choose an affine chart in the fibers of $\tau^{\mathbf{P}}$. To this end, assume for example that $w^l \neq 0$ and set

$$\bar{w}^\alpha = \frac{w^\alpha}{w^l}, \qquad l = 1, \dots, l-1,$$

in the domain under consideration. Then from Equation (10) it follows that the system

$$\bar{w}^\alpha_{x^i} = X^\alpha_{i,l} - X^l_{i,l}\bar{w}^\alpha + \sum_{\beta=1}^{l-1} X^\alpha_{i,\beta}\bar{w}^\beta - \bar{w}^\alpha \sum_{\beta=1}^{l-1} X^l_{i,\beta}\bar{w}^\beta, \qquad i = 1, \dots, n, \quad \alpha = 1, \dots, l-1.$$

locally provides the defining equation for the covering $\tau^{\mathbf{P}}$.

We are now ready to state and prove the main result.

Theorem 1. *Assume that a differentially connected two-dimensional equation $\mathscr{E}$ admits a nontrivial covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ of finite rank. Then it possesses at least one nontrivial (nonlocal) conservation law.*

Proof. Actually, the proof is a description of a procedure that allows one to construct the desired conservation law.

Note first that we may assume the covering τ to be irreducible. Indeed, otherwise the space $\tilde{\mathscr{E}}$ is foliated by maximal integral manifolds of the distribution $\tilde{\mathscr{C}}$. Let l_0 denote the codimension of the generic leaf and $l = \operatorname{rank}(\tau)$. Then

- $l > l_0$, because τ is a nontrivial covering;
- the integral leaves project to $\mathscr{E}$ surjectively, because $\mathscr{E}$ is a differentially connected equation.

This means that in vicinity of a generic point we can consider τ as an l_0-parametric family of irreducible coverings whose rank is $r = l - l_0 > 0$. Let us choose one of them and denote it by $\tau_0\colon \mathscr{E}_0 \to \mathscr{E}$.

If τ_0 is not strongly non-Abelian, then this would mean that $\mathscr{E}$ possesses at least one nontrivial conservation law and we have nothing to prove further. Assume now that the covering τ_0 is strongly non-Abelian. Then due to Proposition 2 the linear covering $\tau_0^{\mathbf{I}}$ is irreducible and by Proposition 3 its projectivization $\tau_1 = (\tau_0^{\mathbf{I}})^{\mathbf{P}}$ possesses the same property and $\operatorname{rank}(\tau_1) = r - 1$. Repeating the construction, we arrive to the diagram

$$
\begin{tikzcd}
 & & \mathscr{E}_0^{\mathbf{I}} \arrow[dl, "\tau_0^{\mathbf{I}}"'] \arrow[d, "\mathbf{P}"] & \cdots & \mathscr{E}_{r-2}^{\mathbf{I}} \arrow[dl, "\tau_{r-2}^{\mathbf{I}}"'] \arrow[d, "\mathbf{P}"] \\
\mathscr{E} & \mathscr{E}_0 \arrow[l, "\tau_0"'] & (\mathscr{E}_0^{\mathbf{I}})^{\mathbf{P}} = \mathscr{E}_1 \arrow[l, "\tau_1=(\tau_0^{\mathbf{I}})^{\mathbf{P}}"] & \ldots \arrow[l] & (\mathscr{E}_{r-2}^{\mathbf{I}})^{\mathbf{P}} = \mathscr{E}_{r-1}, \arrow[l, "\tau_{r-1}=(\tau_{r-2}^{\mathbf{I}})^{\mathbf{P}}"]
\end{tikzcd}
$$

where $\operatorname{rank}(\tau_i) = l - i$. Thus, in $r - 1$ steps at most we shall arrive to a one-dimensional irreducible covering and find ourselves in the situation of Proposition 1 and this finishes the proof. $\square$

4. Examples

Let us discuss several illustrative examples.

Example 2. *Consider the Korteweg-de Vries equation in the form*

$$u_t = uu_x + u_{xxx} \tag{11}$$

and the well known Miura transformation [16]

$$u = w_x - \frac{1}{6}w^2.$$

The last formula is a part of the defining equations for the non-Abelian covering

$$
\begin{aligned}
w_x &= u + \frac{1}{6}w^2, \\
w_t &= u_{xx} + \frac{1}{3}wu_x + \frac{1}{3}u^2 + \frac{1}{18}w^2u,
\end{aligned}
$$

the covering equation being

$$w_t = w_{xxx} - \frac{1}{6}w^2w_x,$$

i.e., the modified KdV equation. Then the corresponding covering $\tau^{\mathbf{I}}$ is defined by the system

$$\begin{aligned}\psi_x &= \frac{1}{3}w\psi,\\ \psi_t &= \frac{1}{3}\left(u_x + \frac{1}{3}wu\right)\psi\end{aligned}$$

that, after relabeling $\psi \mapsto 3\ln\psi$ gives us the nonlocal conservation law

$$\omega = w\,dx + \left(u_x + \frac{1}{3}wu\right)dt$$

of the KdV equation.

Example 3. *The well known Lax pair, see [17], for the KdV equation may be rewritten in terms of zero-curvature representation*

$$D_x(\mathbf{T}) - D_t(\mathbf{X}) + [\mathbf{X}, \mathbf{T}] = 0.$$

The (2×2) matrices $\mathbf{X}$ and $\mathbf{T}$ become much simpler if we present the equation in the form

$$u_t = 6uu_x - u_{xxx}.$$

In this case, they are

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ u-\lambda & 0 \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} -u_x & 2(u+2\lambda) \\ 2u^2 - u_{xx} + 2\lambda u - 4\lambda^2 & u_x \end{pmatrix},$$

$\lambda \in \mathbb{R}$ being a real parameter. As it follows from Remark 2, this amounts to existence of the two-dimensional linear covering τ given by the system

$$\begin{aligned}w_{1,x} &= w_2,\\ w_{1,t} &= -u_xw_1 + 2(u+2\lambda)w_2,\\ w_{2,x} &= (u-\lambda)w_1,\\ w_{2,t} &= (2u^2 - u_{xx} + 2\lambda u - 4\lambda^2)w_1 + u_xw_2.\end{aligned}$$

Let us choose for the affine chart the domain $w_2 \neq 0$ and set $\psi = w_1/w_2$. Then the covering $\tau^{\mathbf{P}}$ is described by the system

$$\begin{aligned}\psi_x &= 1 - (u-\lambda)\psi,\\ \psi_t &= 2(u+2\lambda) - 2u_x\psi - (2u^2 - u_{xx} + 2\lambda u - 4\lambda^2)\psi^2,\end{aligned}$$

while $\tau_1 = (\tau^{\mathbf{P}})^{\mathbf{I}}$ is given by

$$\begin{aligned}\tilde{\psi}_x &= (\lambda - u)\tilde{\psi},\\ \tilde{\psi}_t &= -2\big(u_x + (2u^2 - u_{xx} + 2\lambda u - 4\lambda^2)\psi\big)\tilde{\psi}.\end{aligned}$$

Thus, we obtain the conservation law

$$\omega = (\lambda - u)\,dx - 2\big(u_x + (2u^2 - u_{xx} + 2\lambda u - 4\lambda^2)\psi\big)\,dt$$

that depends on the nonlocal variable ψ.

Example 4. *Consider the potential KdV equation in the form*

$$u_t = 3u_x^2 + u_{xxx}$$

Its Bäcklund auto-transformation is associated to the covering τ

$$\begin{aligned} w_x &= \lambda - u_x - \frac{1}{2}(w-u)^2, \\ w_t &= 2\lambda^2 - 2\lambda u_x - u_x^2 - u_{xxx} + 2u_{xx}(w-u) - (\lambda + u_x)(w-u)^2, \end{aligned}$$

where $\lambda \in \mathbb{R}$*, see [18]. Then the covering* $\tau^{\mathbf{I}}$ *is*

$$\begin{aligned} \psi_x &= -(w-u)\psi, \\ \psi_t &= 2\big(u_{xx}\psi - (\lambda + u_x)(w-u)\big)\psi, \end{aligned}$$

which leads to the nonlocal conservation law

$$\omega = -(w-u)\,dx + 2\big(u_{xx}\psi - (\lambda + u_x)(w-u)\big)\,dt$$

of the potential KdV equation.

Example 5. *The Gauss-Mainardi-Codazzi equations read*

$$u_{xy} = \frac{g - fh}{\sin u}, \qquad f_y = g_x + \frac{h - g\cos u}{\sin u}u_x, \qquad g_y = h_x - \frac{f - g\cos u}{\sin u}u_y, \tag{12}$$

see [19]. This is an under-determined system, and imposing additional conditions on the unknown functions u*,* f*,* g*, and* h *one obtains equations that describe various types of surfaces in* $\mathbb{R}^2$*, cf. [20]. System (12) always admits the following* $\mathbb{C}$*-valued zero-curvature representation*

$$D_x(\mathbf{Y}) - D_y(\mathbf{X}) + [\mathbf{X}, \mathbf{Y}] = 0$$

with the matrices

$$\mathbf{X} = \frac{\mathrm{i}}{2}\begin{pmatrix} u_x & \dfrac{\mathrm{e}^{\mathrm{i}u} f - g}{\sin u} \\ \dfrac{\mathrm{e}^{-\mathrm{i}u} f - g}{\sin u} & -u_x \end{pmatrix}, \qquad \mathbf{Y} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \dfrac{\mathrm{e}^{\mathrm{i}u} g - h}{\sin u} \\ \dfrac{\mathrm{e}^{-\mathrm{i}u} g - h}{\sin u} & 0 \end{pmatrix}$$

The corresponding two-dimensional linear covering τ *is defined by the system*

$$\begin{aligned} w_x^1 &= u_x w^1 + \frac{\mathrm{e}^{\mathrm{i}u} f - g}{\sin u} w^2, & w_x^2 &= \frac{\mathrm{e}^{-\mathrm{i}u} f - g}{\sin u} w^1 - u_x w^2, \\ w_y^1 &= \frac{\mathrm{e}^{iu} g - h}{\sin u} w^2, & w_y^2 &= \frac{\mathrm{e}^{-\mathrm{i}u} g - h}{\sin u} w^1. \end{aligned}$$

Hence, the covering $\tau^{\mathbf{P}}$ *in the domain* $w^2 \neq 0$ *is*

$$\psi_x = \frac{\mathrm{e}^{\mathrm{i}u} f - g}{\sin u} + 2u_x\psi - \frac{\mathrm{e}^{-\mathrm{i}u} f - g}{\sin u}\psi^2, \qquad \psi_y = \frac{\mathrm{e}^{\mathrm{i}u} g - h}{\sin u} - \frac{\mathrm{e}^{-\mathrm{i}u} g - h}{\sin u}\psi^2.$$

Thus, the covering $(\tau^{\mathbf{P}})^{\mathbf{I}}$*, given by*

$$\tilde{\psi}_x = 2\left(u_x - \frac{\mathrm{e}^{-\mathrm{i}u} f - g}{\sin u}\psi\right)\tilde{\psi}, \qquad \tilde{\psi}_y = -2\frac{\mathrm{e}^{-\mathrm{i}u} g - h}{\sin u}\psi\tilde{\psi},$$

defines the nonlocal conservation law

$$\omega = \left(u_x - \frac{e^{-iu} f - g}{\sin u} \psi \right) dx - \frac{e^{-iu} g - h}{\sin u} \psi \, dy$$

of the Gauss-Mainardi-Codazzi equations.

Example 6. *The last example shows that the above described techniques fail for infinite-dimensional coverings (such coverings are typical for equations of dimension greater than two).*

Consider the equation

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$$

that arises in the theory of integrable hydrodynamical chains, see [21]. This equation admits the covering τ *with the nonlocal variables* w^i, $i = 0, 1, \ldots$, *that enjoy the defining relations*

$$\begin{aligned} &w_t^0 + u_y w_x^1 = 0, \quad w_y^0 + u_x w_x^1 = 0, \\ &w_x^i = w^{i+1}, \qquad i \geq 0, \\ &w_t^i + D_x^i(u_y w_x^1) = 0, \quad w_y^i + D_x^i(u_x w_x^1) = 0, \qquad i \geq 1. \end{aligned}$$

see [22]. This is a linear covering, but its projectivization does not lead to construction of conservation laws.

5. Discussion

We described a procedure that allows one to associate, in an algorithmic way, with any nontrivial finite-dimensional covering over a differentially connected equation a nonlocal conservation law. Nevertheless, this method fails in the case of infinite-dimensional coverings. It is unclear, at the moment at least, whether this is an immanent property of such coverings or a disadvantage of the method. I hope to clarify this in future research.

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