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# An Integral Operational Matrix of Fractional-Order Chelyshkov Functions and Its Applications

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Received: 26 July 2020; Accepted: 16 October 2020; Published: 23 October 2020



**Abstract:** Fractional differential equations have been applied to model physical and engineering processes in many fields of science and engineering. This paper adopts the fractional-order Chelyshkov functions (FCHF) for solving the fractional differential equations. The operational matrices of fractional integral and product for FCHF are derived. These matrices, together with the spectral collocation method, are used to reduce the fractional differential equation into a system of algebraic equations. The error estimation of the presented method is also studied. Furthermore, numerical examples and comparison with existing results are given to demonstrate the accuracy and applicability of the presented method.

**Keywords:** fractional-order chelyshkov functions; fractional differential equations; operational matrix; collocation method; error estimation

**MSC:** 65L05; 65L60

## 1. Introduction

Fractional differential and integral operators are generalizations of differential and integral operators of integer order. In physical sciences, differential equations are used to model phenomena. However, differential equations of integer order cannot give acceptable results for complex systems. So, fractional differential equations are used to improve these models [1–4]. Several papers have been devoted for studying existence and uniqueness of solutions to the fractional differential equations [5,6]. Solving fractional differential equations is a desirable objective. There are some issues in solving these equations analytically except for a limited number of these equations. So, several studies have been reported for the development of new methods for finding numerical or approximate solutions, such as fractional differential transformation method [7], Adomian decomposition method [8], homotopy perturbation method [9], homotopy analysis method [10], variational iteration method [11], Galerkin method [12], collocation method [13], wavelet method [14], B-Spline operational matrix method [15] and Jacobi operational matrix method [16].

Spectral methods, finite differences and finite element methods are the main methods of discretization that provide a numerical solution of differential equations. The spectral methods have the advantage that they are accurate for a given number of unknowns [17,18]. These methods exhibit exponential rates of convergence/spectral accuracy for smooth problems, and this differs from finite difference and finite element methods, which offer only algebraic convergence rates [19]. In spectral methods, the solution of differential equations is approximated as an expansion in terms of orthogonal polynomials. The most commonly used spectral schemes are the collocation, tau and Galerkin approaches [20,21].

Orthogonal functions play an important role in finding numerical solution of differential equations. These functions simplify the treatment of differential equations via converting its solution into the solution of a system of algebraic equations. Some examples are shifted Legendre polynomials [22], Chebyshev wavelets [23], shifted Chebyshev polynomials [17], shifted Jacobi polynomials [24], block pulse operational matrix [25], etc. Orthogonal Chelyshkov polynomials were presented in [26]. Using these polynomials, a solution has been given of weakly singular integral equations in [27], Volterra–Fredholm integral equations in [28], linear functional integro-differential equations with variable coefficients in [29] and Volterra–Fredholm–Hammerstein integral equations in [30]. Furthermore, the operational matrix of fractional derivatives based on Chelyshkov polynomials for solving multi-order fractional differential equations has been presented in [31] and the operational matrix of fractional integration to solve a class of nonlinear fractional differential equations has been introduced in [32]. Recently, Talaei [33] has proposed a numerical algorithm based on FCHFs for solving linear weakly singular Volterra integral equation.

Solving fractional differential equations by using series expansion of the form  $\sum_{i=0}^n q_i x^{i\alpha}(x)$ ,  $\alpha > 0$ , such as Adomian’s decomposition method, homotopy perturbation method and He’s variational iteration method, is a common and efficient method [34]. So, building orthogonal functions of fractional-order may be useful in solving fractional differential equations more successfully.

The motivation of this paper is to construct the FCHFs based on Chelyshkov polynomials. Additionally, the operational matrices of the fractional integration and product for FCHFs are derived. In addition to that, an application of FCHFs for solving linear and nonlinear fractional differential equations is presented. The paper is organized as follows. Following this introduction, some definitions and preliminaries of fractional calculus, as well as the FCHFs and their properties are presented in Section 2. In Section 3, we investigate the error of the proposed method. The FCHFs operational matrices of fractional integration and product are given in Section 4. Section 5 presents a new technique to use the FCHFs operational matrices for solving multi-order fractional differential equation. The performance of the proposed algorithm is reported in Section 6 with satisfactory numerical results. Finally, the paper ends with conclusions in Section 7.

## 2. Preliminaries

### 2.1. Fractional Calculus

Now, we will present some definitions and properties of fractional integration and differentiation which will be used throughout this article [35,36]. In the literature, one can find many definitions of fractional integration of order  $\alpha \geq 0$ , but these definitions may not equivalent to each other [37]. The definition of Riemann–Liouville is the more widely used, which can be stated as follows

**Definition 1.** The Riemann–Liouville fractional integral operator of order  $\alpha \geq 0$  of a function  $y(x)$  is given by

$$I^\alpha y(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t) dt, & \alpha > 0, x > 0, \\ y(x), & \alpha = 0. \end{cases} \quad (1)$$

For Riemann–Liouville fractional integral operator  $I^\alpha$  we have

$$I^\alpha x^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} x^{\eta+\alpha}, \quad \eta \geq -1, \quad (2)$$

$$I^\alpha I^\gamma y(x) = I^{\alpha+\gamma} y(x) = I^\gamma I^\alpha y(x), \quad \alpha, \gamma \geq 0, \quad (3)$$

$$I^\alpha (\lambda_1 y_1(x) + \lambda_2 y_2(x)) = \lambda_1 I^\alpha y_1(x) + \lambda_2 I^\alpha y_2(x), \quad (4)$$

where  $\lambda_1$  and  $\lambda_2$  are constants.

**Definition 2.** The fractional derivative of order  $\alpha > 0$  in the Caputo sense is defined in the form

$$D^\alpha y(x) = I^{[\alpha]-\alpha} D^{[\alpha]} y(x) = \frac{1}{\Gamma([\alpha]-\alpha)} \int_0^x (x-t)^{[\alpha]-\alpha-1} y^{([\alpha])}(t) dt, \quad [\alpha]-1 < \alpha \leq [\alpha], x > 0 \quad (5)$$

where  $[\alpha]$  is the smallest integer greater than or equal to  $\alpha$ .

Some characteristics of the operator  $D^\alpha$  are given as follows

$$D^\alpha x^\eta = \begin{cases} 0, & \eta \in N_0 \text{ \& } \eta < [\alpha], \\ \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)} x^{\eta-\alpha}, & \eta \in N_0 \text{ \& } \eta \geq [\alpha] \text{ or } \eta \notin N \text{ \& } \eta > [\alpha], \end{cases} \quad (6)$$

$$D^\alpha C = 0, \quad C \text{ is constant}, \quad (7)$$

$$D^\alpha I^\alpha y(x) = y(x), \quad (8)$$

$$I^\alpha D^\alpha y(x) = y(x) - \sum_{i=0}^{[\alpha]-1} \frac{y^{(i)}(0^+)}{i!} x^i, \quad x > 0, \quad (9)$$

$$D^\alpha (\mu_1 y_1(x) + \mu_2 y_2(x)) = \mu_1 D^\alpha y_1(x) + \mu_2 D^\alpha y_2(x), \quad (10)$$

where  $[\alpha]$  denotes the largest integer less than or equal to  $\alpha$ ,  $N = \{1, 2, \dots\}$ ,  $N_0 = \{0, 1, 2, \dots\}$  and  $\mu_1, \mu_2$  are constants.

**Definition 3.** (Generalized Taylor's formula). Assume that  $D^{j\eta} y(x) \in C(0, 1]$  for  $j = 0, 1, \dots, n+1$  and  $0 < \eta \leq 1$ , then

$$y(x) = \sum_{l=0}^n \frac{x^{l\eta}}{\Gamma(l\eta+1)} D^{l\eta} y(0^+) + \frac{x^{(n+1)\eta}}{\Gamma((n+1)\eta+1)} D^{(n+1)\eta} y(z), \quad 0 < z \leq x, \quad \forall x \in (0, 1], \quad (11)$$

where  $D^{l\eta} = \underbrace{D^\eta D^\eta \dots D^\eta}_{l \text{ times}}$ .

In the case of  $\eta = 1$ , the generalized Taylor's formula reduces to the classical Taylor's formula.

## 2.2. Fractional Chelyshkov Functions

The set  $\mathbb{C}_n = \{C_{ni}\}_{i=0}^n$  of Chelyshkov polynomials is a set of orthogonal polynomials of degree  $n$  over the interval  $[0, 1]$  with the weight function  $\omega(z) = 1$  and has the form

$$C_{ni}(z) = \sum_{j=i}^n (-1)^{j-i} \binom{n-i}{j-i} \binom{n+j+1}{n-i} z^j, \quad i = 0, 1, \dots, n. \quad (12)$$

It can be seen that, all members of the set  $\mathbb{C}_n$  have the same degree  $n$ , which represents the main difference with other sets of orthogonal polynomials. The orthogonality condition of these polynomials is

$$\int_0^1 C_{nl}(z) C_{nr}(z) dz = \begin{cases} 0, & l \neq r, \\ \frac{1}{l+r+1}, & l = r, \end{cases} \quad l, r = 0, 1, \dots, n. \quad (13)$$

Chelyshkov polynomials  $C_{ni}(z)$  are related to Jacobi polynomials  $P_m^{(\alpha, \beta)}(z)$ , where  $\alpha, \beta > -1$  and  $m \geq 0$  by the following relation

$$C_{ni}(z) = (-1)^{n-i} z^i P_{n-i}^{(0, 2i+1)}(2z-1), \quad i = 0, 1, \dots, n. \quad (14)$$

The fractional-order Chelyshkov functions (FCHFs) can be built by substituting  $z = x^\eta$  and  $\eta > 0$  in the Chelyshkov polynomials. Suppose that  $C_{ni}^\eta(x)$  denotes the FCHFs  $C_{ni}(x^\eta)$ . Then, from Equation (12),  $C_{ni}^\eta(x)$  can be defined as

$$C_{ni}^\eta(x) = \sum_{j=i}^n (-1)^{j-i} \binom{n-i}{j-i} \binom{n+j+1}{n-i} x^{j\eta}, \quad i = 0, 1, \dots, n. \quad (15)$$

These functions are orthogonal with respect to the weight function  $\omega_\eta(x) = x^{\eta-1}$  over the interval  $[0, 1]$  and satisfy the orthogonality property

$$\int_0^1 C_{nl}^\eta(x) C_{nr}^\eta(x) \omega_\eta(x) dx = \begin{cases} 0, & l \neq r, \\ \frac{1}{\eta(l+r+1)}, & l = r, \end{cases} \quad l, r = 0, 1, \dots, n. \quad (16)$$

Let the FCHFs vector be

$$C_\eta(x) = \left( C_{n0}^\eta(x), C_{n1}^\eta(x), \dots, C_{nn}^\eta(x) \right)^T, \quad (17)$$

which can take the form

$$C_\eta(x) = EX_\eta(x), \quad (18)$$

where the vector  $X_\eta(x)$  and the upper triangular matrix  $E$  are given by

$$X_\eta(x) = \left( 1, x^\eta, x^{2\eta}, \dots, x^{n\eta} \right)^T, \quad (19)$$

$$E = [e_{ij}]_{i,j=0}^n, \quad e_{ij} = \begin{cases} 0, & 0 \leq j < i, \\ (-1)^{j-i} \binom{n-i}{j-i} \binom{n+j+1}{n-i}, & i \leq j \leq n. \end{cases} \quad (20)$$

### 3. Function Approximation and Error Estimation

Assume that  $\Lambda = [0, 1]$  and define the weighted space  $L_{\omega_\eta}^2(\Lambda)$  by

$$L_{\omega_\eta}^2(\Lambda) = \left\{ \rho : \Lambda \longrightarrow \mathbb{R}; \rho \text{ is measurable on } \Lambda \text{ \& } \int_0^1 |\rho(x)|^2 \omega_\eta(x) dx < \infty \right\}, \quad (21)$$

with inner product and norm

$$\langle \rho, \sigma \rangle_{\omega_\eta} = \int_0^1 \rho(x) \sigma(x) \omega_\eta(x) dx, \quad (22)$$

$$\|\rho\|_{\omega_\eta} = \langle \rho, \rho \rangle_{\omega_\eta}^{\frac{1}{2}}. \quad (23)$$

Suppose that  $\mathbb{S}_n = \text{span} \{ C_{n0}^\eta(x), C_{n1}^\eta(x), \dots, C_{nn}^\eta(x) \}$ , which is finite dimensional and closed subspace of  $L_{\omega_\eta}^2(\Lambda)$ , then for each  $y(x) \in L_{\omega_\eta}^2(\Lambda)$  there is a unique best approximation  $y_n(x) \in \mathbb{S}_n$  that satisfies

$$\|y - y_n\|_{\omega_\eta} \leq \|y - \xi\|_{\omega_\eta}, \quad \forall \xi \in \mathbb{S}_n. \quad (24)$$

Additionally,  $y_n(x) \in \mathbb{S}_n$ , thus it can be expanded in terms of FCHFs as

$$y_n(x) = \sum_{i=0}^n q_i C_{ni}^\eta(x) = Q^T C_\eta(x), \quad (25)$$

where the coefficient vector  $Q$  is given by

$$Q = (q_0, q_1, \dots, q_n)^T, \quad q_i = \frac{\langle y, C_{ni}^\eta \rangle_{\omega_\eta}}{\langle C_{ni}^\eta, C_{ni}^\eta \rangle_{\omega_\eta}}, \quad i = 0, 1, \dots, n. \quad (26)$$

The following theorem gives an upper bound for estimating the error.

**Theorem 1.** Assume that  $D^{j\eta}y(x) \in C(0, 1]$  for  $j = 0, 1, \dots, n+1$  and  $y_n(x)$  is the best approximation to  $y$  out of  $\mathbb{S}_n$  then the error bound is given by

$$\|y(x) - y_n(x)\|_{\omega_\eta} \leq \frac{M_\eta}{\Gamma((n+1)\eta+1)} \sqrt{\frac{1}{(2n+3)\eta}}, \quad (27)$$

where

$$M_\eta = \sup \left\{ |D^{(n+1)\eta}y(x)| \right\}, \quad x \in (0, 1]. \quad (28)$$

**Proof.** Let  $p(x)$  be the generalized Taylor's formula of  $y(x)$ , then from Definition 3 we get

$$p(x) = \sum_{l=0}^n \frac{x^{l\eta}}{\Gamma(l\eta+1)} D^{l\eta}y(0^+), \quad x \in (0, 1], \quad (29)$$

with error bound as

$$\begin{aligned} |y(x) - p(x)| &= \left| \frac{x^{(n+1)\eta}}{\Gamma((n+1)\eta+1)} D^{(n+1)\eta}y(z) \right|, \quad 0 < z \leq x \\ &\leq \frac{M_\eta x^{(n+1)\eta}}{\Gamma((n+1)\eta+1)}. \end{aligned} \quad (30)$$

However,  $p(x) \in \mathbb{S}_n$  and  $y_n$  is the best approximation to  $y$  from  $\mathbb{S}_n$  then

$$\begin{aligned} \|y(x) - y_n(x)\|_{\omega_\eta}^2 &\leq \|y(x) - p(x)\|_{\omega_\eta}^2 \\ &= \int_0^1 (y(x) - p(x))^2 \omega_\eta(x) dx \\ &\leq \int_0^1 \left( \frac{M_\eta x^{(n+1)\eta}}{\Gamma((n+1)\eta+1)} \right)^2 \omega_\eta(x) dx \\ &= \frac{M_\eta^2}{\Gamma((n+1)\eta+1)^2} \int_0^1 x^{(2n+3)\eta-1} dx \\ &= \frac{M_\eta^2}{\Gamma((n+1)\eta+1)^2 (2n+3)\eta}, \end{aligned} \quad (31)$$

and by taking the square roots, we have the desired result.  $\square$

#### 4. The Fractional Integration Operational Matrix of FCHFs

In this section, the FCHFs operational matrices of fractional integration and product are obtained, which can be built easily and their numerical results are accurate.

**Theorem 2.** Suppose  $C_\eta(x)$  is the FCHFs vector, then

$$I^\alpha C_\eta(x) \simeq P^{(\alpha)} C_\eta(x), \quad (32)$$

where  $P^{(\alpha)} = [p_{ik}]_{i,k=0}^n$  is the  $(n+1) \times (n+1)$  operational matrix of fractional integration of order  $\alpha > 0$  in the Riemann–Liouville sense and its elements are given by

$$p_{ik} = \sum_{l=i}^n \sum_{s=k}^n (-1)^{l+s-i-k} \frac{\eta(2k+1)\Gamma(l\eta+1)}{(\eta(l+s+1)+\alpha)\Gamma(l\eta+\alpha+1)} \binom{n-i}{l-i} \binom{n+l+1}{n-i} \binom{n-k}{s-k} \binom{n+s+1}{n-k}. \quad (33)$$

**Proof.** By integrating Equation (18), we obtain

$$\begin{aligned} I^\alpha C_\eta(x) &= I^\alpha EX_\eta(x) \\ &= EI^\alpha X_\eta(x), \end{aligned} \quad (34)$$

and by using Equation (2), we get

$$\begin{aligned} I^\alpha X_\eta(x) &= \left( \frac{1}{\Gamma(\alpha+1)} x^\alpha, \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} x^{\eta+\alpha}, \frac{\Gamma(2\eta+1)}{\Gamma(2\eta+\alpha+1)} x^{2\eta+\alpha}, \dots, \frac{\Gamma(n\eta+1)}{\Gamma(n\eta+\alpha+1)} x^{n\eta+\alpha} \right)^T \\ &= B\bar{X}_\eta(x), \end{aligned} \quad (35)$$

where

$$B = \text{diag} \left( \frac{1}{\Gamma(\alpha+1)}, \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)}, \frac{\Gamma(2\eta+1)}{\Gamma(2\eta+\alpha+1)}, \dots, \frac{\Gamma(n\eta+1)}{\Gamma(n\eta+\alpha+1)} \right), \quad (36)$$

$$\bar{X}_\eta(x) = \left( x^\alpha, x^{\eta+\alpha}, x^{2\eta+\alpha}, \dots, x^{n\eta+\alpha} \right)^T. \quad (37)$$

The vector  $\bar{X}_\eta(x)$  can be approximated in terms of FCHFs as follows

$$x^{r\eta+\alpha} \simeq \sum_{k=0}^n a_{rk} C_{nk}^\eta(x), \quad r = 0, 1, \dots, n, \quad (38)$$

where  $a_{rk}$  can be obtained from Equations (15) and (26) as follows

$$\begin{aligned} a_{rk} &= \eta(2k+1) \int_0^1 x^{r\eta+\alpha} C_{nk}^\eta(x) \omega_\eta(x) dx \\ &= \eta(2k+1) \sum_{s=k}^n (-1)^{s-k} \binom{n-k}{s-k} \binom{n+s+1}{n-k} \int_0^1 x^{(r+s+1)\eta+\alpha-1} dx \\ &= \eta(2k+1) \sum_{s=k}^n \frac{(-1)^{s-k}}{\eta(r+s+1)+\alpha} \binom{n-k}{s-k} \binom{n+s+1}{n-k}. \end{aligned} \quad (39)$$

Then,

$$\bar{X}_\eta(x) \simeq AC_\eta(x), \quad (40)$$

where  $A = [a_{rk}]_{r,k=0}^n$  is a  $(n+1) \times (n+1)$  matrix and its elements are given by Equation (39). From Equations (34), (35) and (40), we can write

$$I^\alpha C_\eta(x) \simeq EBAC_\eta(x), \quad (41)$$

and by using Equations (20), (36) and (39) for multiplying the matrices  $E$ ,  $B$  and  $A$ , we get

$$I^\alpha C_\eta(x) \simeq P^{(\alpha)} C_\eta(x). \quad (42)$$

□

**Remark 1.** It can be noted that the operational matrix of fractional integration in [32] is a special case of the operational matrix of fractional integration of FCHFs with  $\eta = 1$ .

In the following, we present the operational matrix of product of two FCHFs vectors, which will be useful to reduce the fractional differential equation into a set of algebraic equations.

**Theorem 3.** Suppose  $U = [u_0, u_1, \dots, u_n]^T$  is an arbitrary vector, then

$$C_\eta(x)C_\eta^T(x)U \simeq \bar{U}C_\eta(x), \quad (43)$$

where

$$\bar{U} = [\bar{u}_{rl}]_{r,l=0}^n, \quad \bar{u}_{rl} = \sum_{k=0}^n \sum_{j=0}^n \sum_{i=0}^n u_j e_{ji} e_{rk} h_{ikl}, \quad h_{ikl} = (2l+1) \sum_{s=l}^n \frac{(-1)^{s-l}}{i+k+s+1} \binom{n-l}{s-l} \binom{n+s+1}{n-l}. \quad (44)$$

**Proof.** From Equations (18)–(20), we have

$$\begin{aligned} C_\eta(x)C_\eta^T(x)U &= EX_\eta(x)X_\eta^T(x)E^T U \\ &= \begin{pmatrix} \sum_{k=0}^n \sum_{j=0}^n \sum_{i=0}^n u_j e_{ji} e_{0k} x^{(i+k)\eta} \\ \sum_{k=0}^n \sum_{j=0}^n \sum_{i=0}^n u_j e_{ji} e_{1k} x^{(i+k)\eta} \\ \vdots \\ \sum_{k=0}^n \sum_{j=0}^n \sum_{i=0}^n u_j e_{ji} e_{nk} x^{(i+k)\eta} \end{pmatrix}. \end{aligned} \quad (45)$$

Now, we approximate  $x^{(i+k)\eta}$  for  $i, k = 0, 1, 2, \dots, n$ , in terms of FCHFs, as follows

$$x^{(i+k)\eta} \simeq \sum_{l=0}^n h_{ikl} C_{nl}^\eta(x), \quad (46)$$

where

$$\begin{aligned} h_{ikl} &= \eta(2l+1) \int_0^1 x^{(i+k)\eta} C_{nl}^\eta(x) \omega_\eta(x) dx \\ &= \eta(2l+1) \sum_{s=l}^n (-1)^{s-l} \binom{n-l}{s-l} \binom{n+s+1}{n-l} \int_0^1 x^{(i+k+s+1)\eta-1} dx \\ &= (2l+1) \sum_{s=l}^n \frac{(-1)^{s-l}}{i+k+s+1} \binom{n-l}{s-l} \binom{n+s+1}{n-l}. \end{aligned} \quad (47)$$

Then, for every  $r = 0, 1, \dots, n$ , we get

$$\begin{aligned} \sum_{k=0}^n \sum_{j=0}^n \sum_{i=0}^n u_j e_{ji} e_{rk} x^{(i+k)\eta} &\simeq \sum_{k=0}^n \sum_{j=0}^n \sum_{i=0}^n u_j e_{ji} e_{rk} \sum_{l=0}^n h_{ikl} C_{nl}^\eta(x) \\ &= \sum_{l=0}^n \left( \sum_{k=0}^n \sum_{j=0}^n \sum_{i=0}^n u_j e_{ji} e_{rk} h_{ikl} \right) C_{nl}^\eta(x) \\ &= \sum_{l=0}^n \bar{u}_{rl} C_{nl}^\eta(x). \end{aligned} \quad (48)$$

By using Equation (48) into Equation (45) we obtain

$$\begin{aligned} C_\eta(x)C_\eta^T(x)U &\simeq \begin{pmatrix} \sum_{l=0}^n \bar{u}_{0l}C_{nl}^\eta(x) \\ \sum_{l=0}^n \bar{u}_{1l}C_{nl}^\eta(x) \\ \vdots \\ \sum_{l=0}^n \bar{u}_{nl}C_{nl}^\eta(x) \end{pmatrix} \\ &= \bar{U}C_\eta(x), \end{aligned} \quad (49)$$

which completes the proof.  $\square$

## 5. Solution Method

In this section, the operational matrices of fractional integral and product for FCHFs together with the spectral collocation method are applied for solving the fractional differential equations.

Consider the multi-order fractional differential equation

$$F(x, y(x), D^{\alpha_1}y(x), D^{\alpha_2}y(x), \dots, D^{\alpha_m}y(x), D^\alpha y(x)) = 0, \quad (50)$$

with initial conditions

$$y^{(i)}(0) = \lambda_i, \quad i = 0, 1, \dots, [\alpha] - 1, \quad (51)$$

where,  $y(x)$  is the unknown function,  $0 < \alpha_1 < \alpha_2 < \dots < \alpha_m < \alpha$ ,  $x \in [0, 1]$  and  $D^\alpha$  denotes the Caputo fractional derivative of order  $\alpha$ . To find a solution of problem (50), we approximate  $D^\alpha y(x)$  as

$$D^\alpha y(x) \simeq \sum_{k=0}^n q_k C_{nk}^\eta(x) = Q^T C_\eta(x). \quad (52)$$

By using the Riemann–Liouville integral operator  $I^\alpha$  and Equations (9) and (32), we get

$$\begin{aligned} y(x) - \sum_{i=0}^{[\alpha]-1} \frac{y^{(i)}(0)}{i!} x^i &\simeq Q^T I^\alpha C_\eta(x) \\ &\simeq Q^T P^{(\alpha)} C_\eta(x). \end{aligned} \quad (53)$$

Employing Equation (51) in Equation (53) gives

$$y(x) \simeq Q^T P^{(\alpha)} C_\eta(x) + \sum_{i=0}^{[\alpha]-1} \frac{\lambda_i}{i!} x^i. \quad (54)$$

Therefore, for  $j = 1, 2, \dots, m$

$$\begin{aligned} D^{\alpha_j} y(x) &\simeq Q^T P^{(\alpha-\alpha_j)} C_\eta(x) + \sum_{i=0}^{[\alpha]-1} \frac{\lambda_i}{i!} D^{\alpha_j} x^i, \\ &= Q^T P^{(\alpha-\alpha_j)} C_\eta(x) + \sum_{i=[\alpha_j]}^{[\alpha]-1} \frac{\lambda_i}{\Gamma(i-\alpha_j+1)} x^{i-\alpha_j}, \quad [\alpha_j] \leq [\alpha] - 1, \end{aligned} \quad (55)$$



clearly if  $\lceil \alpha_j \rceil \geq \lceil \alpha \rceil$  the second term of Equation (55) will vanish. Combining Equations (54) and (55) yields

$$D^{\alpha_j} y(x) \simeq Q^T P^{(\alpha-\alpha_j)} C_\eta(x) + \sum_{i=\lceil \alpha_j \rceil}^{\lceil \alpha \rceil-1} \frac{\lambda_i}{\Gamma(i-\alpha_j+1)} x^{i-\alpha_j}, \quad j = 0, 1, \dots, m, \quad (56)$$

where  $\alpha_0 = 0$ . Now, approximate  $\sum_{i=\lceil \alpha_j \rceil}^{\lceil \alpha \rceil-1} \frac{\lambda_i}{\Gamma(i-\alpha_j+1)} x^{i-\alpha_j}$  in terms of FCHFs as

$$\sum_{i=\lceil \alpha_j \rceil}^{\lceil \alpha \rceil-1} \frac{\lambda_i}{\Gamma(i-\alpha_j+1)} x^{i-\alpha_j} \simeq \sum_{r=0}^n g_{jr} C_{nr}^\eta(x) = G_j^T C_\eta(x), \quad j = 0, 1, \dots, m, \quad (57)$$

where the vectors  $G_j = [g_{j0}, g_{j1}, \dots, g_{jn}]^T$  are given by

$$\begin{aligned} g_{jr} &= \eta(2r+1) \sum_{i=\lceil \alpha_j \rceil}^{\lceil \alpha \rceil-1} \frac{\lambda_i}{\Gamma(i-\alpha_j+1)} \int_0^1 x^{i-\alpha_j} C_{nr}^\eta(x) \omega_\eta(x) dx, \\ &= \eta(2r+1) \sum_{i=\lceil \alpha_j \rceil}^{\lceil \alpha \rceil-1} \sum_{l=r}^n \frac{(-1)^{l-r} \lambda_i}{\Gamma(i-\alpha_j+1)} \binom{n-r}{l-r} \binom{n+l+1}{n-r} \int_0^1 x^{(l+1)\eta+i-\alpha_j-1} dx, \\ &= \eta(2r+1) \sum_{i=\lceil \alpha_j \rceil}^{\lceil \alpha \rceil-1} \sum_{l=r}^n \frac{(-1)^{l-r} \lambda_i}{(\eta(l+1)+i-\alpha_j) \Gamma(i-\alpha_j+1)} \binom{n-r}{l-r} \binom{n+l+1}{n-r}, \quad j = 0, 1, \dots, m, \quad r = 0, 1, \dots, n. \end{aligned} \quad (58)$$

Then, Equation (56) becomes

$$D^{\alpha_j} y(x) = \left( Q^T P^{(\alpha-\alpha_j)} + G_j^T \right) C_\eta(x), \quad j = 0, 1, \dots, m. \quad (59)$$

Additionally, from Theorem 3, we can write

$$\begin{aligned} (D^{\alpha_j} y(x))^2 &= \left( Q^T P^{(\alpha-\alpha_j)} + G_j^T \right) C_\eta(x) C_\eta^T(x) \left( \left( P^{(\alpha-\alpha_j)} \right)^T Q + G_j \right), \\ &= \left( Q^T P^{(\alpha-\alpha_j)} + G_j^T \right) \bar{Q}_1 C_\eta(x), \end{aligned} \quad (60)$$

where  $\bar{Q}_1$  is the operational matrix of the product, also

$$\begin{aligned} (D^{\alpha_j} y(x))^3 &= \left( Q^T P^{(\alpha-\alpha_j)} + G_j^T \right) \bar{Q}_1 C_\eta(x) C_\eta^T(x) \left( \left( P^{(\alpha-\alpha_j)} \right)^T Q + G_j \right), \\ &= \left( Q^T P^{(\alpha-\alpha_j)} + G_j^T \right) \bar{Q}_2 C_\eta(x), \end{aligned} \quad (61)$$

⋮

$$\begin{aligned} (D^{\alpha_j} y(x))^s &= \left( Q^T P^{(\alpha-\alpha_j)} + G_j^T \right) \bar{Q}_{s-2} C_\eta(x) C_\eta^T(x) \left( \left( P^{(\alpha-\alpha_j)} \right)^T Q + G_j \right), \\ &= \left( Q^T P^{(\alpha-\alpha_j)} + G_j^T \right) \bar{Q}_{s-1} C_\eta(x), \quad s \in N. \end{aligned} \quad (62)$$

By substituting as needed from Equations (52), (59) and (62) in Equation (50), the multi-order fractional differential Equation (50) is converted into the following algebraic equation

$$F \left( x, \left( Q^T P^{(\alpha)} + G_0^T \right) C_\eta(x), \left( Q^T P^{(\alpha-\alpha_1)} + G_1^T \right) C_\eta(x), \dots, \left( Q^T P^{(\alpha-\alpha_m)} + G_m^T \right) C_\eta(x), Q^T C_\eta(x) \right) = 0. \quad (63)$$

To find the unknown vector  $Q$ , let  $\Psi_n = \{x_i : x_i = \frac{i}{n}, i = 0, 1, \dots, n\}$  be a set of equidistant nodes and collocate Equation (63) at the nodes  $x_i, i = 0, 1, \dots, n$ , which gives

$$F\left(x_i, \left(Q^T P^{(\alpha)} + G_0^T\right) C_\eta(x_i), \left(Q^T P^{(\alpha-\alpha_1)} + G_1^T\right) C_\eta(x_i), \dots, \left(Q^T P^{(\alpha-\alpha_m)} + G_m^T\right) C_\eta(x_i), Q^T C_\eta(x_i)\right) = 0. \quad (64)$$

Equation (64) represents a system of  $n + 1$  nonlinear algebraic equations and can be solved to find the vector  $Q$ . So, an approximate solution of Equation (50) can be determined.

## 6. Illustrative Examples

In order to demonstrate the efficiency of the proposed method, we apply it to solve some linear and nonlinear fractional differential equations. A comparison with other methods reveals that the presented method is effective and accurate.

**Example 1.** Consider the inhomogeneous Bagley–Torvik equation [16,22]

$$\begin{aligned} D^2 y(x) + D^{\frac{3}{2}} y(x) + y(x) &= x + 1, \\ y(0) = y'(0) &= 1. \end{aligned} \quad (65)$$

The exact solution of this problem is  $y(x) = x + 1$ . By using the technique presented in Section 5, we have

$$\begin{aligned} D^2 y(x) &= Q^T C_\eta(x), \\ D^{\frac{3}{2}} y(x) &= \left(Q^T P^{(\frac{1}{2})} + G_1^T\right) C_\eta(x), \\ y(x) &= \left(Q^T P^{(2)} + G_0^T\right) C_\eta(x). \end{aligned} \quad (66)$$

Substituting Equations (66) in (65), gives

$$Q^T \left(I + P^{(\frac{1}{2})} + P^{(2)}\right) C_\eta(x) + \left(G_1^T + G_0^T\right) C_\eta(x) - x - 1 = 0. \quad (67)$$

By taking  $n = 2, \eta = 1$ , we obtain

$$G_0 = \left(\frac{1}{3}, \frac{5}{4}, \frac{35}{12}\right)^T, \quad (68)$$

$$G_1 = (0, 0, 0)^T, \quad (69)$$

$$P^{(\frac{1}{2})} = \frac{1}{11\sqrt{\pi}} \begin{pmatrix} \frac{124}{21} & \frac{3172}{315} & \frac{340}{63} \\ \frac{832}{64} & \frac{7}{32} & \frac{320}{27} \\ -\frac{945}{32} & \frac{7}{32} & \frac{27}{3} \end{pmatrix}, \quad (70)$$

$$P^{(2)} = \frac{1}{7} \begin{pmatrix} \frac{1}{15} & \frac{17}{30} & 3 \\ -\frac{180}{1} & -\frac{20}{1} & \frac{36}{5} \\ \frac{60}{1} & -\frac{12}{12} & \frac{12}{12} \end{pmatrix}. \quad (71)$$

Using Equations (68)–(71) and collocate Equation (67) at the nodes  $x_i = \frac{i}{2}$ ,  $i = 0, 1, 2$ , we obtain

$$\begin{aligned} &\left(\frac{104}{105} + \frac{124}{231\sqrt{\pi}}\right)q_0 - \left(\frac{1}{1260} + \frac{832}{10395\sqrt{\pi}}\right)q_1 + \left(\frac{1}{420} + \frac{32}{3465\sqrt{\pi}}\right)q_2 = 0, \\ &\left(-\frac{55}{168} + \frac{172}{315\sqrt{\pi}}\right)q_0 + \left(\frac{17}{21} + \frac{3232}{3465\sqrt{\pi}}\right)q_1 + \left(\frac{107}{420} - \frac{1024}{3465\sqrt{\pi}}\right)q_2 + \frac{5}{6} = 0, \\ &\left(\frac{281}{210} + \frac{388}{3465\sqrt{\pi}}\right)q_0 - \left(\frac{103}{140} - \frac{64}{385\sqrt{\pi}}\right)q_1 + \left(\frac{451}{420} - \frac{3104}{3465\sqrt{\pi}}\right)q_2 = 0. \end{aligned} \quad (72)$$

Solving Equation (72) gives  $Q = (0, 0, 0)^T$ , and therefore

$$y(x) = G_0^T C_1(x) = \begin{pmatrix} \frac{1}{3} & \frac{5}{4} & \frac{35}{12} \end{pmatrix} \begin{pmatrix} 3 - 12x + 10x^2 \\ 4x - 5x^2 \\ x^2 \end{pmatrix} = x + 1, \quad (73)$$

which is the exact solution. Additionally, for the same  $n = 2$  with  $\eta = \frac{1}{2}$  we can obtain the exact solution, where

$$G_0 = \left(\frac{1}{3}, 1, \frac{8}{3}\right)^T, \quad (74)$$

$$G_1 = (0, 0, 0)^T, \quad (75)$$

$$P^{(\frac{1}{2})} = \frac{1}{3\sqrt{\pi}} \begin{pmatrix} \frac{2}{3} & \frac{1}{2} & \frac{335 - 108\pi}{6} \\ -\frac{1}{3} & 2 & \frac{18\pi - 50}{3} \\ \frac{1}{15} & -\frac{2}{5} & \frac{10}{3} \end{pmatrix}, \quad (76)$$

$$P^{(2)} = \frac{1}{9} \begin{pmatrix} -\frac{1}{21} & \frac{1}{5} & -\frac{1}{42} \\ \frac{19}{420} & -\frac{41}{140} & \frac{11}{6} \\ \frac{5}{84} & -\frac{1}{4} & \frac{5}{6} \end{pmatrix}. \quad (77)$$

Similarly, by collocating Equation (67) at the nodes  $x_i = \frac{i}{2}$ ,  $i = 0, 1, 2$  and solving the resulting system, we get the solution  $Q = (0, 0, 0)^T$ , which yields

$$y(x) = G_0^T C_{\frac{1}{2}}(x) = \begin{pmatrix} \frac{1}{3} & 1, \frac{8}{3} \end{pmatrix} \begin{pmatrix} 3 - 12\sqrt{x} + 10x \\ 4\sqrt{x} - 5x \\ x \end{pmatrix} = x + 1. \quad (78)$$

**Example 2.** Consider the following linear fractional differential equation [38,39]

$$\begin{aligned} D^{\frac{1}{2}}y(x) + y(x) &= \sqrt{x} + \frac{\sqrt{\pi}}{2}, \\ y(0) &= 0. \end{aligned} \quad (79)$$

The exact solution of this problem is  $y(x) = \sqrt{x}$ . Using the proposed method yields

$$\begin{aligned} D^{\frac{1}{2}}y(x) &= Q^T C_{\eta}(x), \\ y(x) &= \left( Q^T P^{(\frac{1}{2})} + G_0^T \right) C_{\eta}(x). \end{aligned} \quad (80)$$

Equation (79) can be written after using Equation (80) in the form

$$Q^T \left( I + P^{(\frac{1}{2})} \right) C_{\eta}(x) + G_0^T C_{\eta}(x) - \frac{2\sqrt{x} + \sqrt{\pi}}{2} = 0. \quad (81)$$

With  $n = 1$  and  $\eta = \frac{1}{2}$ , we get

$$G_0 = (0, 0)^T, \quad (82)$$

$$P^{(\frac{1}{2})} = \begin{pmatrix} \frac{\sqrt{\pi}}{8} & \frac{32 - 9\pi}{8\sqrt{\pi}} \\ -\frac{\sqrt{\pi}}{24} & \frac{3\sqrt{\pi}}{8} \end{pmatrix}, \quad (83)$$

and the generated set of linear algebraic equations is

$$\begin{aligned} \left( 1 + \frac{\sqrt{\pi}}{8} \right) q_0 - \frac{\sqrt{\pi}}{24} q_1 - \frac{\sqrt{\pi}}{4} &= 0, \\ \left( 1 - \frac{16 - 5\pi}{4\sqrt{\pi}} \right) q_0 - \left( 1 + \frac{5\sqrt{\pi}}{12} \right) q_1 + 1 + \frac{\sqrt{\pi}}{2} &= 0. \end{aligned} \quad (84)$$

The solution of Equation (84) is  $Q = \left( \frac{\sqrt{\pi}}{4}, \frac{3\sqrt{\pi}}{4} \right)^T$ . Then

$$y(x) = Q^T P^{(\frac{1}{2})} C_{\frac{1}{2}}(x) = \left( \frac{\sqrt{\pi}}{4}, \frac{3\sqrt{\pi}}{4} \right) \begin{pmatrix} \frac{\sqrt{\pi}}{8} & \frac{32 - 9\pi}{8\sqrt{\pi}} \\ -\frac{\sqrt{\pi}}{24} & \frac{3\sqrt{\pi}}{8} \end{pmatrix} \begin{pmatrix} 2 - 3\sqrt{x} \\ \sqrt{x} \end{pmatrix} = \sqrt{x}, \quad (85)$$

which is the exact solution.

**Example 3.** Consider the nonlinear initial value problem [16,22,31]

$$\begin{aligned} D^3y(x) + D^{\frac{5}{2}}y(x) + y^2(x) &= x^4, \\ y(0) = y'(0) &= 0, \quad y''(0) = 2. \end{aligned} \quad (86)$$

The exact solution of the problem is  $y(x) = x^2$ . By applying the method described in Section 5, we obtain

$$\begin{aligned} D^3y(x) &= Q^T C_{\eta}(x), \\ D^{\frac{5}{2}}y(x) &= \left( Q^T P^{(\frac{1}{2})} + G_1^T \right) C_{\eta}(x), \\ y(x) &= \left( Q^T P^{(3)} + G_0^T \right) C_{\eta}(x), \\ y^2(x) &= \left( Q^T P^{(3)} + G_0^T \right) \bar{Q}_1 C_{\eta}(x). \end{aligned} \quad (87)$$

where  $\bar{Q}_1$  can be calculated from Equation (43). Equation (87) transforms Equation (86) to

$$Q^T \left( I + P^{(\frac{1}{2})} + P^{(3)} \bar{Q}_1 \right) C_{\eta}(x) + \left( G_1^T + G_0^T \bar{Q}_1 \right) C_{\eta}(x) - x^4 = 0. \quad (88)$$

With  $n = 3$  and  $\eta = 1$ , the resulting system of nonlinear algebraic equations can be written in the form

$$\begin{aligned}
 & \frac{1}{3} \left( 1 + \frac{359}{3024} q_0 + \frac{13}{432} q_1 + \frac{11}{5040} q_2 + \frac{1}{3024} q_3 \right)^2 + \left( 17 - \frac{34816}{3003\sqrt{\pi}} \right) q_0 - \left( 21 + \frac{1102592}{45045\sqrt{\pi}} \right) q_1 + \left( 11 + \frac{566864}{225225\sqrt{\pi}} \right) q_2 \\
 & + \left( 1 + \frac{23552}{45045\sqrt{\pi}} \right) q_3 - \frac{1}{3} = 0, \\
 & \frac{1}{12} \left( 4 + \frac{55}{108} q_0 + \frac{269}{945} q_1 + \frac{991}{12600} q_2 + \frac{47}{7560} q_3 \right)^2 + \left( 2 + \frac{12192}{9009\sqrt{\pi}} \right) q_0 + \left( -3 + \frac{90544}{45045\sqrt{\pi}} \right) q_1 + \left( 4 + \frac{234884}{9009\sqrt{\pi}} \right) q_2 \\
 & + \left( 2 + \frac{67264}{45045\sqrt{\pi}} \right) q_3 - \frac{4}{3} = 0, \\
 & \left( 1 + \frac{125}{1008} q_0 + \frac{67}{720} q_1 + \frac{211}{5040} q_2 + \frac{13}{1680} q_3 \right)^2 + \left( -1 + \frac{512}{1001\sqrt{\pi}} \right) q_0 + \left( 1 + \frac{1024}{3003\sqrt{\pi}} \right) q_1 + \left( -1 + \frac{2084}{45045\sqrt{\pi}} \right) q_2 \\
 & + \left( 1 + \frac{8192}{9009\sqrt{\pi}} \right) q_3 - 1 = 0, \\
 & 4 \left( \frac{1}{2520} q_1 - \frac{1}{10080} q_2 - \frac{1}{10080} q_3 \right)^2 + \left( 1 + \frac{32}{77\sqrt{\pi}} \right) q_0 - \frac{304}{4095\sqrt{\pi}} q_1 + \frac{128}{9009\sqrt{\pi}} q_2 - \frac{64}{45045\sqrt{\pi}} q_3 = 0.
 \end{aligned} \tag{89}$$

The solution of system (89) is  $Q = (0, 0, 0, 0)^T$ , which leads to the exact solution.

**Example 4.** Consider the nonlinear multi-order fractional differential equation [40]

$$\begin{aligned}
 D^\alpha y(x) + y^2(x) &= x + \left( \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} \right)^2, \quad 0 < \alpha \leq 2, \\
 y(0) &= y'(0) = 0.
 \end{aligned} \tag{90}$$

The problem has the exact solution  $y(x) = \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}$ . Applying the proposed technique in Section 5 for this problem, we get

$$\begin{aligned}
 D^\alpha y(x) &= Q^T C_\eta(x), \\
 y(x) &= (Q^T P^{(\alpha)} + G_0^T) C_\eta(x), \\
 y^2(x) &= (Q^T P^{(\alpha)} + G_0^T) \overline{Q}_1 C_\eta(x).
 \end{aligned} \tag{91}$$

Then (90) takes the form

$$Q^T \left( I + P^{(\alpha)} \overline{Q}_1 \right) C_\eta(x) + G_0^T \overline{Q}_1 C_\eta(x) - x - \left( \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} \right)^2 = 0. \tag{92}$$

Consider  $\alpha = 1$  with the exact solution  $y(x) = \frac{x^2}{2}$ . For solving this case we take  $n = 2$  and  $\eta = 1$ , which leads to the following system of nonlinear algebraic equations

$$\begin{aligned} \frac{1}{36}q_0^2 + \frac{1}{144}q_1^2 + \frac{1}{3600}q_2^2 - \frac{1}{36}q_0q_1 + \frac{1}{180}q_0q_2 + \frac{1}{360}q_1q_2 + 3q_0 &= 0, \\ \frac{25}{144}q_0^2 + \frac{49}{576}q_1^2 + \frac{121}{14400}q_2^2 + \frac{35}{144}q_0q_1 + \frac{55}{720}q_0q_2 + \frac{77}{1440}q_1q_2 - \frac{1}{2}q_0 + \frac{3}{4}q_1 + \frac{1}{4}q_2 - \frac{33}{64} &= 0, \\ \frac{1}{36}q_0^2 + \frac{25}{144}q_1^2 + \frac{361}{3600}q_2^2 + \frac{5}{36}q_0q_1 + \frac{19}{180}q_0q_2 + \frac{95}{360}q_1q_2 + q_0 - q_1 + q_2 - \frac{5}{4} &= 0. \end{aligned} \quad (93)$$

Solving these equations yields  $Q = \left(0, \frac{1}{4}, \frac{5}{4}\right)^T$ , which leads to the exact solution.

**Example 5.** Consider the following fractional differential equation [31]

$$\begin{aligned} Dy(x) + D^{\frac{1}{4}}y(x) + y(x) &= \frac{5}{2}x^{\frac{3}{2}} + x^{\frac{5}{2}} + \frac{15}{8} \frac{\sqrt{\pi}}{\Gamma(\frac{13}{4})} x^{\frac{9}{4}}, \\ y(0) &= 0. \end{aligned} \quad (94)$$

The exact solution of this problem is  $y(x) = x^2\sqrt{x}$ . Table 1 shows a comparison of the results obtained by the present method and those in [31] in terms of  $L_{\omega_\eta}^\infty$  and  $L_{\omega_\eta}^2$  errors for different values of  $n$  and  $\eta$ . Note that the results for the method that we compare have been executed by the method in [31] and we use these results to make a direct comparison with the presented method. Symbol “–” means that the result for  $n$  is unavailable for that method. From Table 1, it can be seen that the errors achieved by the presented method are less than those in [31] for all values of  $n$ . In addition to that, when  $n$  increases, the errors are reduced until they become zero at  $n = 10$ ,  $\eta = 0.25$  and  $n = 13$ ,  $\eta = 0.25$ . This means that the presented method is more accurate than that in [31] for this problem.

**Table 1.** Comparison of the results for Example 5.

$n$	Our Method			Taleai's [31]	
	$\eta$	$L_{\omega_\eta}^\infty$	$L_{\omega_\eta}^2$	$L^\infty$	$L^2$
4	1.0	$3.82 \times 10^{-4}$	$3.81 \times 10^{-4}$	$1.21 \times 10^{-3}$	$5.92 \times 10^{-4}$
8	0.5	$1.18 \times 10^{-7}$	$4.06 \times 10^{-7}$	$5.80 \times 10^{-5}$	$2.50 \times 10^{-5}$
10	0.25	0.0	0.0	–	–
13	0.25	0.0	0.0	–	–
16	0.25	$8.13 \times 10^{-17}$	$5.36 \times 10^{-17}$	$2.45 \times 10^{-6}$	$9.89 \times 10^{-7}$
20	0.25	$1.78 \times 10^{-15}$	0.0	$8.59 \times 10^{-7}$	$3.42 \times 10^{-7}$

**Example 6.** Consider the fractional differential equation [31,41,42]

$$\begin{aligned} D^\alpha y(x) + y(x) &= x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{24}{\Gamma(5-\alpha)}x^{4-\alpha}, \quad 0 < \alpha < 1, \\ y(0) &= 0. \end{aligned} \quad (95)$$

The exact solution of this problem is  $y(x) = x^4 - \frac{1}{2}x^3$ . Table 2 describes the results obtained by the present method and those in [31,41] in terms of  $L_{\omega_\eta}^2$  errors for different values of  $n$  with  $\alpha = \frac{1}{4}$ . It can be seen that the new method performs better than the method in [31] and much better than the technique in [41]. The error achieved by the presented method becomes smaller and smaller with the

increment of  $n$  and converge to zero at  $n = 20$ . Which means that the introduced method is more coincidental with exact solutions than those in [31,41].

**Table 2.** Comparison of the results for Example 6.

$n$	Our Method		Talaei's [31]	Chen's [41]
	$\eta$	$L^2_{\omega_\eta}$	$L^2$	$L^2$
8	0.5	$5.68 \times 10^{-6}$	$3.07 \times 10^{-7}$	$4.50 \times 10^{-3}$
16	0.25	$6.70 \times 10^{-17}$	$2.87 \times 10^{-9}$	$1.80 \times 10^{-3}$
20	0.25	0.0	—	—

## 7. Conclusions

In this paper, the FCHFs have been adopted for solving fractional differential equations. The operational matrices of fractional integral and product for FCHFs have been derived. These matrices are used to approximate solutions of fractional differential equations where the fractional derivatives are considered in Caputo sense. The introduced method is a spectral collocation method which reduces the fractional differential equation to a system of algebraic equations. The efficiency and applicability of the presented method are tested on some problems. Comparison with some other methods has been performed. The numerical results show that the new method is more efficient, its performance is quite satisfactory, and only a small number of FCHFs is needed to obtain good results. Finally, we can say that employing fractional-order basis functions achieves more accurate results than the corresponding integer-order basis functions, which shows the applicability of this method for solving the fractional problems.

**Author Contributions:** Conceptualization, M.S.A.-S. and A.I.A.; methodology, M.S.A.-S. and A.I.A.; software, M.S.A.-S. and A.I.A.; validation, M.S.A.-S., A.I.A. and M.S.S.; formal analysis, M.S.A.-S., A.I.A. and M.S.S.; data curation, M.S.A.-S. and A.I.A.; writing—original draft preparation, A.I.A.; writing—review and editing, M.S.A.-S., A.I.A. and M.S.S.; supervision, M.S.S.; project administration, M.S.S. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Conflicts of Interest:** The authors declare no conflict of interest.

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