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A Spectral Calculus for Lorentz Invariant Measures on Minkowski Space

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Abstract: This paper presents a spectral calculus for computing the spectra of causal Lorentz invariant Borel complex measures on Minkowski space, thereby enabling one to compute their densities with respect to Lebesque measure. The spectra of certain elementary convolutions involving Feynman propagators of scalar particles are computed. It is proved that the convolution of arbitrary causal Lorentz invariant Borel complex measures exists and the product of such measures exists in a wide class of cases. Techniques for their computation in terms of their spectral representation are presented.

Keywords: Lorentz invariant complex measures; Minkowski space; spectral decomposition; measure convolution; measure product; Feynman propagator

MSC: 28XX; 81XX; 22E15; 22E70

1. Introduction

Let $\mathcal{B}(\mathbf{R}^4)$ denote the Borel algebra of $\mathbf{R}^4$ (with respect to the Euclidean topology) [1] and let

$$\mathcal{B}_0(\mathbf{R}^4) = \{\Gamma \in \mathcal{B}(\mathbf{R}^4) : \Gamma \text{ is relatively compact}\}. \tag{1}$$

Let

$$H_m^+ = \{p \in \mathbf{R}^4 : p^2 = m^2, p^0 > 0\}, \tag{2}$$

and

$$H_m^- = \{p \in \mathbf{R}^4 : p^2 = m^2, p^0 < 0\}, \tag{3}$$

be the mass shells (cones) corresponding to mass $m > 0$ ($m = 0$) and let

$$H_{im}^+ = \{p \in \mathbf{R}^4 : p^2 = -m^2, p^0 > 0\}, \tag{4}$$

and

$$H_{im}^- = \{p \in \mathbf{R}^4 : p^2 = -m^2, p^0 < 0\}, \tag{5}$$

be the positive time (negative time) imaginary mass hyperboloids corresponding to mass $m > 0$.

Define measures on these hyperboloids (cones) by

$$\Omega_m^\pm(\Gamma) = \int_{\pi(H_m^\pm \cap \Gamma)} \frac{d\vec{p}}{\omega_m(\vec{p})}, m \geq 0, \tag{6}$$

$$\Omega_{im}^\pm(\Gamma) = \int_{\pi(H_{im}^\pm \cap \Gamma)} \frac{d\vec{p}}{\omega_{im}(\vec{p})}, m > 0, \tag{7}$$

and

$$\Omega_{im} = \Omega^{+}_{im} + \Omega^{-}_{im}, \tag{8}$$

where $\pi : \mathbf{R}^4 \to \mathbf{R}^3$ is defined by

$$\pi(p) = \pi(p^0, p^1, p^2, p^3) = \vec{p} = (p^1, p^2, p^3), \tag{9}$$

and $\omega_m : \mathbf{R}^3 \to [0, \infty)$ is defined by

$$\omega_m(\vec{p}) = (m^2 + |\vec{p}|^2)^{\frac{1}{2}}. \tag{10}$$

An equivalent set of definitions to these definitions of the measures $\Omega^{\pm}_m$ and $\Omega^{\pm}_{im}$ is to specify the effect of applying the measures to measurable functions $\psi : \mathbf{R}^4 \to \mathbf{C}$. In fact

$$< \Omega^{\pm}_m, \psi >= \int \psi(\pm\omega_m(\vec{p}), \vec{p}) \frac{d\vec{p}}{\omega_m(\vec{p})}, m \geq 0, \tag{11}$$

$$< \Omega^{\pm}_{im}, \psi >= \int \psi(\pm\omega_{im}(\vec{p}), \vec{p}) \frac{d\vec{p}}{\omega_{im}(\vec{p})}, m > 0. \tag{12}$$

Consider the following general form of a complex measure $\mu : \mathcal{B}_0(\mathbf{R}^4) \to \mathbf{C}$ on Minkowski space.

$$\mu(\Gamma) = c\delta(\Gamma) + \int_{m=0}^{\infty} \Omega^{+}_m(\Gamma)\, \sigma_1(dm) + \int_{m=0}^{\infty} \Omega^{-}_m(\Gamma)\, \sigma_2(dm) + \int_{m=0}^{\infty} \Omega_{im}(\Gamma)\, \sigma_3(dm), \tag{13}$$

where $c \in \mathbf{C}$ (the complex numbers), δ is the Dirac delta function (measure), $\sigma_1, \sigma_2, \sigma_3 : \mathcal{B}_0([0, \infty)) \to \mathbf{C}$ are Borel complex measures. Then μ is a Lorentz invariant Borel complex measure. Conversely [2] leads to the following.

Theorem 1. *The Spectral Theorem. Let $\mu : \mathcal{B}_0(\mathbf{R}^4) \to \mathbf{C}$ be a Lorentz invariant Borel complex measure. Then μ has the form of Equation (13) for some $c \in \mathbf{C}$ and Borel spectral measures σ_1, σ_2 and σ_3.*

(More generally, from [2] Lorentz invariant distributions in $\mathcal{D}^*(\mathbf{R}^4)$ are of the form of Equation (13) where $c \in \mathbf{C}$, $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{D}^*(\mathbf{R})$ with the possible addition of a distribution supported at the origin of the form $P(\Box)\delta$ where $\Box$ is the wave operator and P is some polynomial.)

In Section 3, we show how the Feynman scalar propagator in momentum space can be identified with the causal Lorentz invariant measure Ω_m. In Section 4, we will present a spectral calculus whereby the spectrum of a causal Lorentz invariant Borel complex measure on Minkowski space can be calculated, whereby causal is meant that the support of the measure is contained in the closed future null cone of the origin.

In Section 5 of the paper, we use the spectral calculus and other methods to compute the spectrum of the measure $\Omega_m * \Omega_m$ which is the convolution of the standard Lorentz invariant measure on the mass m mass shell (i.e., the Feynman scalar propagator corresponding to mass m on the space of positive energy functions) with itself, where $m > 0$. In Section 7, we use general arguments to compute the spectrum of $\Omega_{im} * \Omega_{im}, m > 0$. In Section 8, we will show how the density with respect to Lebesque measure associated with a causal Lorentz invariant Borel complex measure can be determined from its spectrum and in Section 9 we will show how the convolution and product of such measures can be computed.

Some of the work of this paper may be compared to the work of Scharf and others, dating back to the paper of Epstein and Glaser [3] on forming products of causal distributions.

The concept of spectral representation in quantum field theory (QFT) dates back to the work of Källén [4] and Lehmann [5] who, independently, proposed the representation

$$< 0|[\phi(x), \phi^{\dagger}(y)]|0 >= i \int_0^{\infty} dm'^2 \sigma(m'^2) \Delta_{m'}(x-y), \tag{14}$$

for the commutator of interacting fields where $\Delta_{m'}$ is the Feynman propagator corresponding to mass m'. Itzykson and Zuber [6] state, with respect to σ, "In general this is a positive measure with δ-function singularities." While Källén, Lehmann and others propose and use this decomposition they do not present a way to compute the spectral measure σ. As mentioned above one of the main results of the present paper is a presentation of a spectral calculus that enables one to compute the spectral function of a causal Lorentz invariant Borel complex measure on Minkowski space. This spectral calculus is quite easy to use in practice but it is somewhat tedious to prove rigorously its validity. This use in practice involves a general form of argument which is exemplified by the argument used in the case of the computation of the spectrum of the convolution $\Omega_m * \Omega_m$ which we call Argument 1. The validity of Argument 1 is proved in Section 6.

2. Related Work

Our work has some connection with the spectral theory of hyperbolic surfaces [7,8] and its multivarious ramifications in quantum physics, number theory, and discrete groups as the hyperboloid H_m is a higher dimensional hyperbolic space and the standard measure Ω_m on H_m is a fundamental solution of the Klein-Gordon equation on Minkowski space (whose solutions are eigenfunctions of the wave operator) whereas the spectral theory of hyperbolic surfaces is concerned with eigenfunctions of the Laplace operator.

Bollini et al. [9] describe how the convolution of two ultradistributions of exponential type (UET) can exist. They then define the product of two UETs in terms of the convolution of their Fourier transforms. They obtain expressions for the Fourier transform of Lorentz invariant UETs (generalizing Bochner's theorem). Kamiński and Mincheva-Kaminska [10] present results concerning the convolution of distributions such as the existence of the convolution of tempered distributions whose supports are polynomially compatible sets. Ortner and Wagner [11] consider the Fourier transform of $O(p,q)$ invariant distributions. They present a condition under which two Lorentz invariant tempered distributions are convolvable and a formula for their convolution.

Zinoviev [12] considers Lorentz invariant tempered distributions on $(\mathbf{R}^4)^k$ supported on the product of closed future light cones. Soloviev [13] discusses the theory of Lorentz covariant distributions, ultradistributions and hyperfunctions.

Harish-Chandra [14] realized the fruitfulness of regarding the space of invariant distributions as a module for the algebra of polynomial differential operators. In this context Kolk and Varadarajan [15] consider Lorentz invariant distributions supported on the boundary of the cone representing the causal future of the origin.

Our work does not consider the complexities and partial results of the general theory of Lorentz invariant distributions, ultradistributions and other such spaces but restricts attention to Lorentz invariant Borel complex measures. There are two reasons for this. Firstly one can obtain complete, unencumbered and "elegant" results. Secondly, many the distributional objects of interest in QFT (such as correlation functions) can, through Wick's theorem, or else the operator product expansion, be represented in terms of Feynman propagators and the propagators are Lorentz invariant measures (or else K invariant matrix-valued measures whose trace is Lorentz invariant) [16].

3. The Feynman Scalar Field Propagator as a Tempered Measure

In this section, we give a well-defined definition of the Feynman scalar propagator of QFT in terms of tempered measures and distributions. The propagator is viewed as being a complex tempered distribution. It is constructed from the Fourier transform of the tempered measure Ω_m.

Consider the Feynman scalar field propagator. It is written as ([6], p. 35)

$$\triangle_F(x) = -(2\pi)^{-4} \int \frac{e^{-ip.x}}{p^2 - m^2 + i\epsilon}\, dp. \tag{15}$$

This is to be understood with respect to the $i-$epsilon procedure described in Mandl and Shaw ([17], p. 57), and the dot product $p.x$ is given by

$$p.x = \eta_{\alpha\beta} p^\alpha x^\beta,$$

where $\eta = \mathrm{diag}(1, -1, -1, -1)$. Therefore $\triangle_F(x)$ is written as

$$\triangle_F(x) = -(2\pi)^{-4} \int_{\mathbf{R}^3} \int_{C_F} \frac{e^{-ip.x}}{p^2 - m^2}\, dp^0 d\overrightarrow{p}, \tag{16}$$

where C_F is the standard Feynman propagator contour. Thus $\triangle_F(x)$ is written as

$$\triangle_F(x) = -(2\pi)^{-4} \int_{\mathbf{R}^3} I(\overrightarrow{p}, x)\, d\overrightarrow{p}, \tag{17}$$

where

$$I(\overrightarrow{p}, x) = \int_{C_F} \frac{e^{-ip.x}}{(p^0)^2 - \omega_m(\overrightarrow{p})^2}\, dp^0, \tag{18}$$

and

$$\omega_m(\overrightarrow{p}) = (|\overrightarrow{p}|^2 + m^2)^{\frac{1}{2}}, m \geq 0. \tag{19}$$

The contour integral Equation (18) exists for $\omega_m(\overrightarrow{p}) \neq 0$ and is given by

$$I(\overrightarrow{p}, x) = -\frac{\pi i}{\omega_m(\overrightarrow{p})} \begin{cases} e^{-i(\omega_m(\overrightarrow{p})x^0 - \overrightarrow{p}.\overrightarrow{x})} & \text{if } x^0 > 0, \\ e^{-i(-\omega_m(\overrightarrow{p})x^0 - \overrightarrow{p}.\overrightarrow{x})} & \text{if } x^0 < 0. \end{cases} \tag{20}$$

To prove this consider the contour $C_1(R)$ given by

$$C_1(R) = \{Re^{it} : 0 \leq t \leq \pi\}.$$

We will show that

$$I_1(R) = \int_{C_1(R)} \frac{e^{-ip.x}}{(p^0)^2 - \omega_m(\overrightarrow{p})^2}\, dp^0 \to 0 \text{ as } R \to \infty,$$

as long as $x^0 < 0$. To this effect we note that

$$\begin{aligned} |I_1(R)| &= \left| \int_{t=0}^{\pi} \frac{e^{-iRe^{it}x^0 + i\overrightarrow{p}.\overrightarrow{x}}}{(Re^{it})^2 - \omega_m(\overrightarrow{p})^2} iRe^{it}\, dt \right| \\ &\leq \int_{t=0}^{\pi} \left| \frac{e^{R\sin t x^0}}{(Re^{it})^2 - \omega_m(\overrightarrow{p})^2} \right| R\, dt \\ &\leq \int_{t=0}^{\pi} \frac{1}{|R^2 - \omega_m(\overrightarrow{p})^2|} R\, dt \\ &= \frac{\pi R}{|R^2 - \omega_m(\overrightarrow{p})^2|}, \\ &\to 0 \text{ as } R \to \infty, \text{ if } x^0 < 0. \end{aligned} \tag{21}$$

Therefore, for $x^0 < 0$,

$$I(\overrightarrow{p}, x) = 2\pi i \mathrm{res}(p^0 \mapsto \frac{e^{-ip.x}}{(p^0)^2 - \omega_m(\overrightarrow{p})^2}, -\omega_m(p)).$$

Now

$$\frac{e^{-ip.x}}{(p^0)^2 - \omega_m(\overrightarrow{p})^2} = \frac{e^{-ip.x}}{(p^0 - \omega_m(\overrightarrow{p}))(p^0 + \omega_m(\overrightarrow{p}))}.$$

Thus

$$I(\overrightarrow{p}, x) = -\pi i \frac{e^{-i(-\omega_m(\overrightarrow{p})x^0 - \overrightarrow{p}.\overrightarrow{x})}}{\omega_m(\overrightarrow{p})}.$$

Similarly, if $x^0 > 0$, then

$$I(\overrightarrow{p}, x) = -\pi i \frac{e^{-i(\omega_m(\overrightarrow{p})x^0 - \overrightarrow{p}.\overrightarrow{x})}}{\omega_m(\overrightarrow{p})}.$$

Hence

$$\int_{\mathbf{R}^3} |I(\overrightarrow{p}, x)|\, d\overrightarrow{p} = \pi \int_{\mathbf{R}^3} \frac{1}{\omega_m(\overrightarrow{p})}\, d\overrightarrow{p} = \infty, \tag{22}$$

and so the integral Equation (17) defining $\triangle_F(x)$ does not exist as a Lebesgue integral.

We would like to give a well-defined interpretation of the propagator $\triangle_F$. $H_m^{\pm}$ for $m \geq 0$ are orbits of the action of the Lorentz group on Minkowski space (these orbits correspond to real mass orbits, there are also "imaginary mass" hyperboloid orbits H_{im}). $\Omega_m^{\pm}$ are Lorentz invariant measures for Minkowski space supported on $H_m^{\pm}$ ([18], p. 157). $\Omega_m^{\pm}$ is locally finite for $m \geq 0$. Now, for any non-negative measurable function $\psi : \mathbf{R}^4 \to [0, \infty]$,

$$\int_{\mathbf{R}^4} \psi(p)\, \Omega_m^{\pm}(dp) = \int_{\mathbf{R}^3} \psi(\pm\omega_m(\overrightarrow{p}), \overrightarrow{p}) \frac{d\overrightarrow{p}}{\omega_m(\overrightarrow{p})}. \tag{23}$$

Here, and for the rest of the section, the symbol ψ stands for a test function in Minkowski space. It follows from Equations (17), (20) and (23) that one may write

$$\triangle_F(x) = \begin{cases} (2\pi)^{-4}\pi i \int e^{-ip.x}\, \Omega_m^{+}(dp), & \text{if } x^0 > 0, \\ (2\pi)^{-4}\pi i \int e^{-ip.x}\, \Omega_m^{-}(dp), & \text{if } x^0 < 0. \end{cases} \tag{24}$$

Equations (16), (17) and (24) are all integral expressions equivalent to Equation (15) and none of them exist as Lebesgue integrals. However, formally, Equation (24) can be written as

$$\triangle_F(x) = \begin{cases} \pi i \overset{\vee}{\Omega}{}_m^{+}(-x) & \text{if } x^0 > 0, \\ \pi i \overset{\vee}{\Omega}{}_m^{-}(-x) & \text{if } x^0 < 0, \end{cases} \tag{25}$$

where $^{\vee}$ denotes the inverse Fourier transform operator (and we use the "physics"convention for the definition of the Fourier transform). Since Ω_m^{+} and Ω_m^{-} are tempered distributions their inverse Fourier transforms exist and are tempered distributions. Let $\mathcal{S}^{\pm}(\mathbf{R}^4) \subset \mathcal{S}(\mathbf{R}^4)$ be the space of test functions supported in $S^{\pm}$, where

$$S^{+} = \{x \in \mathbf{R}^4 : x^0 > 0\}, S^{-} = \{x \in \mathbf{R}^4 : x^0 < 0\}. \tag{26}$$

Then

$$< \triangle_F, \psi >= \pi i < \overset{\vee}{\Omega}{}_m^{\mp}, \psi >= \pi i < \Omega_m^{\mp}, \overset{\vee}{\psi} >, \tag{27}$$

for $\psi \in \mathcal{S}^{\pm}(\mathbf{R}^4)$, where $<\omega, \psi>$ denotes the evaluation of a distribution ω on its test function argument ψ. Therefore the momentum space scalar field propagator on $(\mathcal{S}^{\pm}(\mathbf{R}^4))^{\wedge}$ is

$$\overset{\wedge}{\triangle}_F = \pi i \Omega_m^{\mp}. \tag{28}$$

$(\mathcal{S}^{+}(\mathbf{R}^4))^{\wedge}$ is the space of wave functions with only positive frequency components while $(\mathcal{S}^{-}(\mathbf{R}^4))^{\wedge}$ is the space of wave functions with only negative frequency components.

This measure is a tempered measure, i.e., it is a tempered distribution as well as being a measure. Equations (15) and (25) lead to the *ansatz*

$$\frac{1}{p^2 - m^2 + i\epsilon} \to -\pi i \Omega_m^{\pm}(p). \tag{29}$$

4. A Spectral Calculus for Lorentz Invariant Measures

Suppose that μ is a Lorentz invariant Borel complex measure on Minkowski space. Then by the spectral theorem, it must have the form of Equation (13). If $\sigma_2 = \sigma_3 = 0$ then μ will be said to be *causal* or a type I measure. If $\sigma_1 = \sigma_3 = 0$ then μ will be said to be a type II measure and if $c = 0$ and $\sigma_1 = \sigma_2 = 0$ then μ will be said to be a type III measure. Thus any Lorentz invariant measure is a sum of a type I measure, a type II measure and a type III measure. In particular, any measure of the form

$$\mu(\Gamma) = \int_{m=0}^{\infty} \sigma(m) \Omega_m^{+}(\Gamma)\, dm, \tag{30}$$

where σ is locally integrable function and the integration is carried out with respect to the Lebesgue measure, is a causal Lorentz invariant Borel complex measure. If σ is polynomially bounded then μ is a tempered measure.

The spectral calculus that we will now explain is a way to compute the spectrum σ of a Lorentz invariant measure μ if we know that μ can be written in the form of Equation (30) and σ is continuous.

For $m > 0$ and $\epsilon > 0$ let $S(m, \epsilon)$ be the hyperbolic (hyper-)disc defined by

$$S(m, \epsilon) = \{p \in \mathbf{R}^4 : p^2 = m^2, |\vec{p}| < \epsilon, p^0 > 0\}, \tag{31}$$

where $\vec{p} = \pi(p) = \pi(p^0, p^1, p^2, p^3) = (p^1, p^2, p^3)$. For $a, b \in \mathbf{R}$ with $0 < a < b$ let $\Gamma(a, b, \epsilon)$ be the hyperbolic cylinder defined by

$$\Gamma(a, b, \epsilon) = \bigcup_{m \in (a,b)} S(m, \epsilon). \tag{32}$$

Now suppose that we have a measure in the form of Equation (30) where σ is continuous. Then we can write

$$\begin{aligned} \mu(\Gamma(a, b, \epsilon)) &= \int_{m=0}^{\infty} \sigma(m) \Omega_m(\Gamma(a, b, \epsilon))\, dm \\ &= \int_{m=0}^{\infty} \sigma(m) \int_{\pi(\Gamma(a,b,\epsilon) \cap H_m^{+})} \frac{d\vec{p}}{\omega_m(\vec{p})}\, dm \\ &= \int_a^b \sigma(m) \int_{B_\epsilon(\vec{0})} \frac{d\vec{p}}{\omega_m(\vec{p})}\, dm \\ &\approx \frac{4}{3}\pi\epsilon^3 \int_a^b \frac{\sigma(m)}{m}\, dm. \end{aligned} \tag{33}$$

where

$$\omega_m(\vec{p}) = (\vec{p}^{\,2} + m^2)^{\frac{1}{2}}, \tag{34}$$

and $B_\epsilon(\vec{0}) = \{\vec{p} \in \mathbf{R}^3 : |\vec{p}| < \epsilon\}$.

The approximation $\approx$ in the last line comes about because ω_m is not constant over $B_\epsilon(\vec{0})$. Thus if we define

$$g_a(b) = g(a,b) = \lim_{\epsilon \to 0} \epsilon^{-3} \mu(\Gamma(a,b,\epsilon)), \tag{35}$$

then we can retreive σ using the formula

$$\sigma(b) = \frac{3}{4\pi} b g_a'(b). \tag{36}$$

Thus we have proved the following fundamental theorem of the spectral calculus of causal Lorentz invariant measures.

Theorem 2. *Suppose that μ is a causal Lorentz invariant measure with continuous spectrum σ. Then σ can be calculated from the formula*

$$\sigma(b) = \frac{3}{4\pi} b g_a'(b), \tag{37}$$

where, for $a, b \in \mathbf{R}, 0 < a < b, g_a : (a, \infty) \to \mathbf{R}$ is given by Equation (35).

To make the proof of this theorem rigorous we prove the following.

Lemma 1. *Let $a, b \in \mathbf{R}, 0 < a < b$. Then*

$$\lim_{\epsilon \to 0} \epsilon^{-3} \int_{B_\epsilon(0)} \frac{d\vec{p}}{\omega_m(\vec{p})} = \frac{4\pi}{3} \frac{1}{m}, \tag{38}$$

uniformly for $m \in [a, b]$.

Proof. Define

$$I = I(m, \epsilon) = \int_{B_\epsilon(0)} \frac{d\vec{p}}{\omega_m(\vec{p})}. \tag{39}$$

Then

$$I = \int_{r=0}^{\epsilon} \frac{4\pi r^2\, dr}{(r^2 + m^2)^{\frac{1}{2}}}. \tag{40}$$

Now

$$I_1 < I < I_2,$$

where

$$I_1 = \int_{r=0}^{\epsilon} \frac{4\pi r^2\, dr}{(\epsilon^2 + m^2)^{\frac{1}{2}}} = \frac{4\pi}{(\epsilon^2 + m^2)^{\frac{1}{2}}} \frac{1}{3} \epsilon^3,$$

$$I_2 = \int_{r=0}^{\epsilon} \frac{4\pi r^2\, dr}{m} = \frac{4\pi}{m} \frac{1}{3} \epsilon^3.$$

Therefore

$$\frac{4\pi}{3(\epsilon^2 + m^2)^{\frac{1}{2}}} < \epsilon^{-3} I < \frac{4\pi}{3m}.$$

Thus

$$\frac{4\pi}{3m} - \frac{4\pi}{3(\epsilon^2 + m^2)^{\frac{1}{2}}} > \frac{4\pi}{3m} - \epsilon^{-3} I > 0.$$

Hence

$$\left| \epsilon^{-3} I - \frac{4\pi}{3m} \right| < \frac{4\pi}{3m} - \frac{4\pi}{3(\epsilon^2 + m^2)^{\frac{1}{2}}}. \tag{41}$$

We have

$$\begin{aligned}\frac{4\pi}{3m}-\frac{4\pi}{3(\epsilon^2+m^2)^{\frac{1}{2}}} &= \frac{4\pi}{3}\frac{(\epsilon^2+m^2)^{\frac{1}{2}}-m}{m(\epsilon^2+m^2)^{\frac{1}{2}}}\\ &= \frac{4\pi}{3}\frac{\epsilon^2}{m(\epsilon^2+m^2)^{\frac{1}{2}}((\epsilon^2+m^2)^{\frac{1}{2}}+m)}\\ &< \frac{4\pi}{3}\frac{\epsilon^2}{2m^3}\\ &\leq \frac{4\pi}{3}\frac{\epsilon^2}{2a^3},\ \text{for all } m\in[a,b].\end{aligned}$$

Therefore

$$\left|\epsilon^{-3}I-\frac{4\pi}{3m}\right|<\frac{2\pi\epsilon^2}{3a^3}, \tag{42}$$

for all $m\in[a,b]$ □

This lemma justifies the step of taking the limit under the integral sign (indicated by the symbol $\approx$) in the proof of Theorem 2.

More generally, suppose that $\mu:\mathcal{B}_0(\mathbf{R}^4)\to\mathbf{C}$ is a causal Lorentz invariant Borel measure on Minkowski space with spectrum σ. Then, by the Lebesgue decomposition theorem there exist unique measures $\sigma_c,\sigma_s:\mathcal{B}_0([0,\infty))\to\mathbf{C}$ such that $\sigma=\sigma_c+\sigma_s$ where σ_c, the continuous part of the spectrum of μ, is absolutely continuous with respect to Lebesque measure and σ_s, the singular part of the spectrum of μ, is singular with respect to σ_c.

It is straightforward to prove the following.

Theorem 3. *Suppose that $a',b'\in\mathbf{R}$ are such that $0<a'<b'$, $\sigma_c|_{(a',b')}$ is continuous. Then for all $a,b\in\mathbf{R}$ with $a'<a<b<b'$, $g_a(b)$ defined by Equation (35) exists and is continuously differentiable. Furthermore $\sigma_c|_{(a',b')}$ can be computed using the formula*

$$\sigma_c(b)=\frac{3}{4\pi}bg_a'(b), \tag{43}$$

and

$$\sigma_s(E)=0,\forall\ \text{Borel } E\subset(a',b'). \tag{44}$$

Conversely suppose that $a',b'\in\mathbf{R}$ are such that $0<a'<b'$ and for all $a,b\in\mathbf{R}$ with $a'<a<b<b'$, $g_a(b)$ defined by Equation (35) exists and is continuously differentiable. Then $\sigma_c|_{(a',b')}$ is continuous and can be retrieved using the formula of Equation (43).

5. Investigation of the Measure Defined by the Convolution $\Omega_m*\Omega_m$

5.1. Determination of Some Properties of $\Omega_m*\Omega_m$

Consider the measure defined by

$$\mu(\Gamma)=(\Omega_m*\Omega_m)(\Gamma)=\int\chi_\Gamma(p+q)\,\Omega_m(dp)\,\Omega_m(dq), \tag{45}$$

where, for any set Γ, χ_Γ denotes the characteristic function of Γ defined by

$$\chi_\Gamma(p)=\begin{cases}1 \text{ if } p\in\Gamma\\ 0 \text{ otherwise.}\end{cases} \tag{46}$$

μ exists as a Borel measure because as $|p|, |q| \to \infty$ with $p, q \in H_m^+$, $(p+q)^0 \to \infty$ and so $p + q$ is eventually $\notin \Gamma$ for any compact set $\Gamma \subset \mathbf{R}^4$. Now

$$\begin{aligned} \mu(\Lambda(\Gamma)) &= \int \chi_{\Lambda(\Gamma)}(p+q)\, \Omega_m(dp)\, \Omega_m(dq) \\ &= \int \chi_\Gamma(\Lambda^{-1}p + \Lambda^{-1}q)\, \Omega_m(dp)\, \Omega_m(dq) \\ &= \int \chi_\Gamma(p+q)\, \Omega_m(dp)\, \Omega_m(dq) \\ &= \mu(\Gamma), \end{aligned} \tag{47}$$

for all $\Lambda \in O(1,3)^{+\uparrow}, \Gamma \in \mathcal{B}_0(\mathbf{R}^4)$. Thus μ is a Lorentz invariant measure.

We will now show that μ is concentrated in the set

$$C_{2m} = \{p \in \mathbf{R}^4 : p^2 \geq 4m^2, p^0 > 0\}, \tag{48}$$

and therefore, that μ is causal. Let $U \subset \mathbf{R}^4$ be open. Then

$$\mu(U) = \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \chi_U(\omega_m(\vec{p}) + \omega_m(\vec{q}), \vec{p} + \vec{q})\, \frac{d\vec{p}}{\omega_m(\vec{p})}\, \frac{d\vec{q}}{\omega_m(\vec{q})}. \tag{49}$$

Therefore, using continuity, it follows that

$$\mu(U) > 0 \quad \Leftrightarrow \quad (\exists \vec{q}_1, \vec{q}_2 \in \mathbf{R}^3)\ (\omega_m(\vec{q}_1) + \omega_m(\vec{q}_2), \vec{q}_1 + \vec{q}_2) \in U.$$

Suppose that $p \in \operatorname{supp}(\mu)$ (the support of the measure μ) i.e., p is such that $\mu(U) > 0$ for all open neighborhoods U of p. Let U be an open neighborhood of p. Then, as $\mu(U) > 0$, there exists $q \in U$, $\vec{q}_1, \vec{q}_2 \in \mathbf{R}^3$ such that $q = (\omega_m(\vec{q}_1) + \omega_m(\vec{q}_2), \vec{q}_1 + \vec{q}_2)$. Clearly $q^0 \geq 2m$. As this is true for all neighborhoods U of p it follows that $p^0 \geq 2m$. By Lorentz invariance we may assume without loss of generality that $\vec{p} = 0$. Therefore $p^2 \geq 4m^2$. Thus $\operatorname{supp}(\mu) \subset C_{2m}$.

For the converse, let $p = (\omega_m(\vec{p}), \vec{p}), q = (\omega_m(\vec{p}), -\vec{p}) \in H_m^+$ for $\vec{p} \in \mathbf{R}^3$. As $\vec{p}$ ranges over $\mathbf{R}^3$, $p + q = (2\omega_m(\vec{p}), \vec{0})$ ranges over $\{(m', \vec{0}) : m' \geq 2m\}$. It follows using Lorentz invariance that $\operatorname{supp}(\mu) \supset C_{2m}$.

Therefore the support $\operatorname{supp}(\mu)$ of μ is C_{2m}. Therefore by the spectral theorem μ has a spectral representation of the form

$$\mu(\Gamma) = \int_{m'=2m}^{\infty} \Omega_{m'}(\Gamma)\, \sigma(dm'), \tag{50}$$

for some Borel measure $\sigma : \mathcal{B}_0([2m, \infty)) \to \mathbf{C}$.

5.2. Computation of the Spectrum of $\Omega_m * \Omega_m$ Using the Spectral Calculus

Let $a, b \in \mathbf{R}$ with $0 < a < b$. Let

$$g_a(b, \epsilon) = \mu(\Gamma(a, b, \epsilon)) \text{ for } \epsilon > 0. \tag{51}$$

We would like to calculate

$$g_a(b) = \lim_{\epsilon \to 0} \epsilon^{-3} g_a(b, \epsilon), \tag{52}$$

and then retreive the spectral function as

$$\sigma(b) = \frac{3}{4\pi} b g'(b). \tag{53}$$

To this effect we calculate

$$\begin{aligned}
g(a,b,\epsilon) &= \mu(\Gamma(a,b,\epsilon)) \\
&= \int \chi_{\Gamma(a,b,\epsilon)}(p+q)\,\Omega_m(dp)\,\Omega_m(dq) \\
&\approx \int \chi_{(a,b)\times B_\epsilon(\vec{0})}(p+q)\,\Omega_m(dp)\,\Omega_m(dq) \\
&= \int \chi_{(a,b)}(\omega_m(\vec{p})+\omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})}(\vec{p}+\vec{q})\,\frac{d\vec{p}}{\omega_m(\vec{p})}\frac{d\vec{q}}{\omega_m(\vec{q})} \\
&= \int \chi_{(a,b)}(\omega_m(\vec{p})+\omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p})\,\frac{d\vec{p}}{\omega_m(\vec{p})}\frac{d\vec{q}}{\omega_m(\vec{q})} \\
&\approx \int \chi_{(a,b)}(2\omega_m(\vec{q}))\frac{\frac{4}{3}\pi\epsilon^3}{\omega_m(\vec{q})^2}\,d\vec{q}.
\end{aligned}$$

We will call this argument Argument 1. This argument is intuitively reasonable but it needs to be justified rigorously. It is proved in the proof of Theorem 4 of the next section.

Given that Argument 1 is valid we now proceed to compute the spectrum of μ. We have

$$\begin{aligned}
a < 2\omega_m(\vec{q}) < b \quad &\Leftrightarrow \quad \left(\frac{a}{2}\right)^2 - m^2 < \vec{q}^{\,2} < \left(\frac{b}{2}\right)^2 - m^2 \\
&\Leftrightarrow \quad mZ(a) < |\vec{q}| < mZ(b),
\end{aligned}$$

where

$$Z(m') = (\frac{m'^2}{4m^2} - 1)^{\frac{1}{2}}, \text{ for } m' \geq 2m. \tag{54}$$

Thus

$$g(a,b,\epsilon) \approx \frac{16\pi^2}{3}\epsilon^3 \int_{r=mZ(a)}^{mZ(b)} \frac{r^2}{m^2+r^2}\,dr. \tag{55}$$

Hence

$$g_a(b) = \frac{16\pi^2}{3} \int_{r=mZ(a)}^{mZ(b)} \frac{r^2}{m^2+r^2}\,dr. \tag{56}$$

Therefore g_a is continuously differentiable and so Theorem 3 applies. Using the Leibniz integral rule

$$g_a'(b) = \frac{16\pi^2}{3}\frac{m^2Z^2(b)}{m^2+m^2Z^2(b)}mZ'(b) = \frac{16\pi^2}{3}\frac{mZ(b)}{b}. \tag{57}$$

Therefore we compute the spectrum σ of μ as

$$\sigma(b) = \begin{cases} 4\pi mZ(b) \text{ for } b \geq 2m \\ 0 \text{ otherwise.} \end{cases} \tag{58}$$

6. Proof of the Validity of Argument 1

The following theorem establishes that Argument 1 is justified.

Theorem 4. *Let $g(a,b,\epsilon)$ be defined by $g(a,b,\epsilon) = \mu(\Gamma(a,b,\epsilon))$ for $a,b \in \mathbf{R}, a < b, \epsilon > 0$, where $\mu = \Omega_m * \Omega_m$. Then the following formal argument (Argument 1)*

$$\begin{aligned}
g(a,b,\epsilon) &= \mu(\Gamma(a,b,\epsilon)) \\
&= \int \chi_{\Gamma(a,b,\epsilon)}(p+q)\,\Omega_m(dp)\,\Omega_m(dq) \\
&\approx \int \chi_{(a,b)\times B_\epsilon(0)}(p+q)\,\Omega_m(dp)\,\Omega_m(dq) \\
&= \int \chi_{(a,b)}(\omega_m(\vec{p})+\omega_m(\vec{q}))\chi_{B_\epsilon(0)}(\vec{p}+\vec{q})\,\frac{d\vec{p}}{\omega_m(\vec{p})}\frac{d\vec{q}}{\omega_m(\vec{q})} \\
&= \int \chi_{(a,b)}(\omega_m(\vec{p})+\omega_m(\vec{q}))\chi_{B_\epsilon(0)-\vec{q}}(\vec{p})\,\frac{d\vec{p}}{\omega_m(\vec{p})}\frac{d\vec{q}}{\omega_m(\vec{q})} \\
&\approx \int \chi_{(a,b)}(2\omega_m(\vec{q}))\frac{\frac{4}{3}\pi\epsilon^3}{\omega_m(\vec{q})^2}\,d\vec{q},
\end{aligned}$$

is justified in the sense that

$$\lim_{\epsilon\to 0}\epsilon^{-3}g(a,b,\epsilon) = \frac{4}{3}\pi\int \chi_{(a,b)}(2\omega_m(\vec{q}))\frac{1}{\omega_m(\vec{q})^2}\,d\vec{q}. \tag{59}$$

Proof. There are 2 $\approx$ signs that we have to consider. The first is in line 3 and arises because we are approximating the hyperbolic cylinder of radius ϵ between a and b with an ordinary cylinder of radius ϵ. We will show that the error is of order greater than ϵ^3. Let $\Gamma = \Gamma(a,b,\epsilon)$ be the aforementioned hyperbolic cylinder. Then

$$\Gamma = \bigcup_{m'\in(a,b)} S(m',\epsilon). \tag{60}$$

Let

$$\begin{aligned}
\Gamma' &= \bigcup_{m'\in(a,b)} \{m'\}\times B_\epsilon(\vec{0}) = (a,b)\times B_\epsilon(\vec{0}) \\
\Gamma'^- &= \bigcup_{m'\in(a,a^+)} \{(m',\vec{p}) : \vec{p}^{\,2} > m'^2 - a^2\} \\
&\subset \bigcup_{m'\in(a,a^+)} (\{m'\}\times B_\epsilon(\vec{0})) = (a,a^+)\times B_\epsilon(\vec{0}) \\
\Gamma'^+ &= \bigcup_{m'\in(b,b^+)} \{(m',\vec{p}) : \vec{p}^{\,2} > m'^2 - b^2\} \\
&\subset \bigcup_{m'\in(b,b^+)} (\{m'\}\times B_\epsilon(\vec{0})) = (b,b^+)\times B_\epsilon(\vec{0}),
\end{aligned}$$

in which

$$a^+ = (a^2+\epsilon^2)^{\frac{1}{2}}, b^+ = (b^2+\epsilon^2)^{\frac{1}{2}}. \tag{61}$$

Then Γ differs from $(\Gamma' \sim \Gamma'^-)\cup\Gamma'^+$ on a set of measure zero,

It is straightforward to show that if $\Gamma_1, \Gamma_2 \in \mathcal{B}_0(\mathbf{R}^4), \Gamma_1\cap\Gamma_2 = \emptyset$ then

$$\begin{aligned}
\int \chi_{\Gamma_1\cup\Gamma_2}(p+q)\,\Omega_m(dp)\,\Omega_m(dq) &= \int \chi_{\Gamma_1}(p+q)\,\Omega_m(dp)\,\Omega_m(dq) + \\
&\quad \int \chi_{\Gamma_2}(p+q)\,\Omega_m(dp)\,\Omega_m(dq).
\end{aligned}$$

Therefore

$$|\int \chi_\Gamma(p+q)\,\Omega_m(dp)\,\Omega_m(dq) - \int \chi_{\Gamma'}(p+q)\,\Omega_m(dp)\,\Omega_m(dq)| \leq$$
$$\int \chi_{\Gamma'^-}(p+q)\,\Omega_m(dp)\,\Omega_m(dq) + \int \chi_{\Gamma'^+}(p+q)\,\Omega_m(dp)\,\Omega_m(dq).$$

We will show that

$$\lim_{\epsilon\to 0}(\epsilon^{-3}\int \chi_{\Gamma'^\pm}(p+q)\,\Omega_m(dp)\Omega_m(dq)) = 0. \tag{62}$$

It suffices to consider the $-$ case. We have

$$\begin{aligned}\int \chi_{\Gamma'^-}(p+q)\,\Omega_m(dp)\Omega_m(dq) &\leq \int \chi_{(a,a^+)\times B_\epsilon(0)}(p+q)\,\Omega_m(dp)\,\Omega_m(dq)\\ &= \int \chi_{(a,a^+)}(\omega_m(\vec{p})+\omega_m(\vec{q}))\chi_{B_\epsilon(0)-\vec{q}}(\vec{p})\,\frac{d\vec{p}}{\omega_m(\vec{p})}\\ &\quad \frac{d\vec{q}}{\omega_m(\vec{q})}.\end{aligned} \tag{63}$$

We will come back to this equation later but will now return to the general argument Argument 1 and consider the second and final $\approx$. This $\approx$ arises because we are approximating $\vec{p}$ by $-\vec{q}$ since $\vec{p}$ ranges over a ball of radius ϵ centred on $-\vec{q}$.

Suppose that $\vec{p}$ and $\vec{q}$ are such that $\chi_{B_\epsilon(0)-\vec{q}}(\vec{p}) = 1$. Then $|\vec{p}+\vec{q}| < \epsilon$. Thus $||\vec{p}| - |\vec{q}|| < \epsilon$. Hence

$$\begin{aligned}|\omega_m(\vec{p}) - \omega_m(\vec{q})| &= |(\vec{p}^{\,2}+m^2)^{\frac{1}{2}} - (\vec{q}^{\,2}+m^2)^{\frac{1}{2}}|\\ &= \left|\frac{\vec{p}^{\,2}-\vec{q}^{\,2}}{(\vec{p}^{\,2}+m^2)^{\frac{1}{2}} + (\vec{q}^{\,2}+m^2)^{\frac{1}{2}}}\right|\\ &\leq \frac{|\vec{p}^{\,2}-\vec{q}^{\,2}|}{2m}\\ &= \frac{||\vec{p}|-|\vec{q}||(|\vec{p}|+|\vec{q}|)}{2m}\\ &< \frac{\epsilon}{2m}(|\vec{p}|+|\vec{q}|).\end{aligned}$$

We have $|\vec{p}| \in (|\vec{q}|-\epsilon, |\vec{q}|+\epsilon)$. Therefore $|\vec{p}|+|\vec{q}| < 2|\vec{q}|+\epsilon$. Thus

$$|\omega_m(\vec{p}) - \omega_m(\vec{q})| < \frac{\epsilon}{2m}(2|\vec{q}|+\epsilon).$$

Therefore

$$\begin{aligned}\omega_m(\vec{p}) + \omega_m(\vec{q}) &= \omega_m(\vec{p}) - \omega_m(\vec{q}) + \omega_m(\vec{q}) + \omega_m(\vec{q})\\ &\leq |\omega_m(\vec{p}) - \omega_m(\vec{q})| + 2\omega_m(\vec{q})\\ &< 2\omega_m(\vec{q}) + \frac{\epsilon}{2m}(2|\vec{q}|+\epsilon).\end{aligned} \tag{64}$$

Now let

$$\begin{aligned}
I(\epsilon) &= \int \chi_{(a,b)}(\omega_m(\vec{p}) + \omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p})\frac{d\vec{p}}{\omega_m(\vec{p})}\frac{d\vec{q}}{\omega_m(\vec{q})} \\
J(\epsilon) &= \int \chi_{(a,b)}(2\omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p})\frac{d\vec{p}}{\omega_m(\vec{p})}\frac{d\vec{q}}{\omega_m(\vec{q})} \\
K(\epsilon) &= \int \chi_{(a,b)}(2\omega_m(\vec{q}))\frac{\frac{4}{3}\pi\epsilon^3}{\omega_m(\vec{q})^2}\, d\vec{q}.
\end{aligned}$$

We will show that

$$\lim_{\epsilon\to 0}\epsilon^{-3}(I(\epsilon) - J(\epsilon)) = 0, \text{ and } \lim_{\epsilon\to 0}\epsilon^{-3}(J(\epsilon) - K(\epsilon)) = 0. \tag{65}$$

Concerning the first limit we note that $\chi_{(a,b)}(\omega_m(\vec{p}) + \omega_m(\vec{q}))$ differs from $\chi_{(a,b)}(2\omega_m(\vec{q}))$ if and only if

1. $\omega_m(\vec{p}) + \omega_m(\vec{q}) \in (a,b)$ but $2\omega_m(\vec{q}) \le a$ or
2. $\omega_m(\vec{p}) + \omega_m(\vec{q}) \in (a,b)$ but $2\omega_m(\vec{q}) \ge b$ or
3. $2\omega_m(\vec{q}) \in (a,b)$ but $\omega_m(\vec{p}) + \omega_m(\vec{q}) \le a$ or
4. $2\omega_m(\vec{q}) \in (a,b)$ but $\omega_m(\vec{p}) + \omega_m(\vec{q}) \ge b$.

Thus

$$|I(\epsilon) - J(\epsilon)| = I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon) + I_4(\epsilon), \tag{66}$$

where

$$\begin{gathered}
I_1(\epsilon) = \int \chi_{(a,b)}(\omega_m(\vec{p}) + \omega_m(\vec{q}))\chi_{(-\infty,a]}(2\omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p}) \\
\frac{d\vec{p}}{\omega_m(\vec{p})}\frac{d\vec{q}}{\omega_m(\vec{q})},
\end{gathered} \tag{67}$$

and I_2, I_3, I_4 are defined similarly. We will show that

$$\lim_{\epsilon\to 0}\epsilon^{-3}I_1(\epsilon) = 0. \tag{68}$$

I_2, I_3 and I_4 can be dealt with similarly.

Using Equation (64)

$$\omega_m(\vec{p}) + \omega_m(\vec{q}) \in (a,b) \text{ and } 2\omega_m(\vec{q}) \le a \Rightarrow a - \frac{\epsilon}{2m}(2|\vec{q}| + \epsilon) < 2\omega_m(\vec{q}) \le a.$$

Therefore

$$\begin{aligned}
I_1(\epsilon) &\le \int \chi_{(a-(2|\vec{q}|+\epsilon)\epsilon/(2m),a]}(2\omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p})\frac{1}{m^2}\, d\vec{p}\, d\vec{q} \\
&= \frac{4}{3}\pi\epsilon^3 \int \chi_{(a-(2|\vec{q}|+\epsilon)\epsilon/(2m),a]}(2\omega_m(\vec{q}))\frac{1}{m^2}\, d\vec{q}.
\end{aligned}$$

Hence

$$\epsilon^{-3}I_1(\epsilon) \le \frac{4}{3}\pi \int \chi_{(a-(2|\vec{q}|+\epsilon)\epsilon/(2m),a]}(2\omega_m(\vec{q}))\frac{1}{m^2}\, d\vec{q}. \tag{69}$$

The integrand is integrable for all $\epsilon > 0$, vanishes outside the compact set

$$C = \{\overrightarrow{q} \in \mathbf{R}^3 : 2\omega_m(\overrightarrow{q}) \leq a\},$$

is dominated by the integrable function

$$g(\overrightarrow{q}) = \frac{1}{m^2}\chi_{[0,a]}(2\omega_m(\overrightarrow{q})),$$

and converges pointwise to 0 everywhere on $\mathbf{R}^3$ as $\epsilon \to 0$ except on the set $\partial C = \{\overrightarrow{q} \in \mathbf{R}^3 : 2\omega_m(\overrightarrow{q}) = a\}$ which is a set of measure 0. Therefore by the dominated convergence theeorem

$$\lim_{\epsilon \to 0} \epsilon^{-3} I_1(\epsilon) = 0, \tag{70}$$

as required.

Now regarding the second limit in Equation (65) consider the function $f : [0, \infty) \to (0, m^{-1}]$ defined by

$$f(p) = (m^2 + p^2)^{-\frac{1}{2}}. \tag{71}$$

f is analytic. Therefore by Taylor's theorem for all $q, p \geq 0$

$$f(p) = f(q) + f'(q)(p - q) + \frac{1}{2} f''(\xi)(p - q)^2, \tag{72}$$

for some ξ between q and p. Now

$$\begin{aligned} f'(p) &= -p(m^2 + p^2)^{-\frac{3}{2}} \\ f''(p) &= (m^2 + p^2)^{-\frac{5}{2}}(2p^2 - m^2). \end{aligned}$$

Therefore

$$\begin{aligned} |f''(\xi)| &= (m^2 + \xi^2)^{-\frac{5}{2}} |2\xi^2 - m^2| \\ &\leq m^{-5}(2(q + \epsilon)^2 + m^2), \end{aligned}$$

as long as $|p - q| < \epsilon$. Thus

$$\begin{aligned} |f(p) - f(q)| &= |f'(q)(p - q) + \frac{1}{2} f''(\xi)(p - q)^2| \\ &< q(m^2 + q^2)^{-\frac{3}{2}} \epsilon + \frac{1}{2} m^{-5}(2(q + \epsilon)^2 + m^2)\epsilon^2 \\ &< m^{-1}\epsilon + \frac{1}{2} m^{-5}(2(q + \epsilon)^2 + m^2)\epsilon^2, \end{aligned}$$

as long as $|p - q| < \epsilon$. Hence

$$
\begin{aligned}
|J(\epsilon)-K(\epsilon)| &= |\int \chi_{(a,b)}(2\omega_m(\vec{q}))(\int \chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p})(\frac{1}{\omega_m(\vec{p})}-\frac{1}{\omega_m(\vec{q})})\,d\vec{p})\frac{d\vec{q}}{\omega_m(\vec{q})}| \\
&\leq \int \chi_{(a,b)}(2\omega_m(\vec{q}))(\int \chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p})(|f(|\vec{p}|)-f(|\vec{q}|)|)\,d\vec{p})\frac{d\vec{q}}{\omega_m(\vec{q})} \\
&\leq \int \chi_{(a,b)}(2\omega_m(\vec{q}))\int \chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p})(m^{-1}\epsilon+\frac{1}{2}m^{-5}(2(|\vec{q}|+\epsilon)^2+m^2)\epsilon^2) \\
&\quad d\vec{p}\,\frac{d\vec{q}}{\omega_m(\vec{q})} \\
&= \frac{4}{3}\pi\epsilon^3\int \chi_{(a,b)}(2\omega_m(\vec{q}))(m^{-1}\epsilon+\frac{1}{2}m^{-5}(2(|\vec{q}|+\epsilon)^2+m^2)\epsilon^2)\frac{d\vec{q}}{\omega_m(\vec{q})}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{\epsilon\to 0}\epsilon^{-3}|J(\epsilon)-K(\epsilon)| &= \lim_{\epsilon\to 0}\frac{4}{3}\pi\int \chi_(a,b)(2\omega_m(\vec{q}))(m^{-1}\epsilon+\frac{1}{2}m^{-5}(2(|\vec{q}|+\epsilon)^2+m^2)\epsilon^2) \\
&\quad \frac{d\vec{q}}{\omega_m(\vec{q})} \\
&= 0,
\end{aligned}
$$

as required. We have therefore dealt with the second $\approx$ in Argument 1.

To finish dealing with the first $\approx$ suppose that $\epsilon_1 > 0$ is given. Choose $c \in (a,b)$ such that

$$\frac{16}{3}\pi^2(Z(c)-Z(a)) < \frac{\epsilon_1}{2}. \tag{73}$$

Now choose $\delta_1 > 0$ such that if $0 < \epsilon < \delta_1$ then

$$
\begin{aligned}
&|\epsilon^{-3}\int \chi_{(a,c)}(\omega_m(\vec{p})+\omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p})\frac{d\vec{p}}{\omega_m(\vec{p})}\frac{d\vec{q}}{\omega_m(\vec{q})} - \\
&\frac{4}{3}\pi\int \chi_{(a,c)}(2\omega_m(\vec{q}))\frac{d\vec{q}}{\omega_m(\vec{q})^2}| < \frac{\epsilon_1}{2}.
\end{aligned}
$$

(We can do this because of the validity of the second $\approx$.) Choose $\delta_2 > 0$ such that if $0 < \epsilon < \delta_2$ then $a^+ = a^+(\epsilon) < c$. Let $\delta = \min(\delta_1, \delta_2)$.

Then if $\epsilon < \delta$ then

$$
\begin{aligned}
& |\epsilon^{-3} \int \chi_{(a,a^+)}(\omega_m(\vec{p}) + \omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p}) \frac{d\vec{p}}{\omega_m(\vec{p})} \frac{d\vec{q}}{\omega_m(\vec{q})}| \\
\leq\ & |\epsilon^{-3} \int \chi_{(a,c)}(\omega_m(\vec{p}) + \omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p}) \frac{d\vec{p}}{\omega_m(\vec{p})} \frac{d\vec{q}}{\omega_m(\vec{q})}| \\
\leq\ & |\epsilon^{-3} \int \chi_{(a,c)}(\omega_m(\vec{p}) + \omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p}) \frac{d\vec{p}}{\omega_m(\vec{p})} \frac{d\vec{q}}{\omega_m(\vec{q})} - \\
& \frac{4}{3}\pi \int \chi_{(a,c)}(2\omega_m(\vec{q})) \frac{d\vec{q}}{\omega_m(\vec{q})^2}| + |\frac{4}{3}\pi \int \chi_{(a,c)}(2\omega_m(\vec{q})) \frac{d\vec{q}}{\omega_m(\vec{q})^2}| \\
<\ & \frac{\epsilon_1}{2} + \frac{4}{3}\pi \int \chi_{(a,c)}(2\omega_m(\vec{q})) \frac{d\vec{q}}{\omega_m(\vec{q})^2} \\
=\ & \frac{\epsilon_1}{2} + \frac{4}{3}\pi \int_{Z(a)}^{Z(c)} \frac{4\pi r^2}{m^2 + r^2}\, dr \\
\leq\ & \frac{\epsilon_1}{2} + \frac{16}{3}\pi^2 (Z(c) - Z(a)) \\
<\ & \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} \\
=\ & \epsilon_1.
\end{aligned}
$$

Thus

$$
\lim_{\epsilon \to 0} \epsilon^{-3} \int \chi_{(a,a^+)}(\omega_m(\vec{p}) + \omega_m(\vec{q}))\chi_{B_\epsilon(\vec{0})-\vec{q}}(\vec{p}) \frac{d\vec{p}}{\omega_m(\vec{p})} \frac{d\vec{q}}{\omega_m(\vec{q})} = 0, \tag{74}
$$

thereby completing the proof of the validity of the first $\approx$ and therefore the validity of Argument 1. □

7. Investigation of the Measure Defined by the Convolution $\Omega_{im} * \Omega_{im}$

The measure Ω_{im}^+ is defined by

$$
\Omega_{im}^+(\Gamma) = \int_{\pi(\Gamma \cap H_{im}^+)} \frac{d\vec{p}}{\omega_{im}(\vec{p})} \text{ for } \Gamma \in \mathcal{B}_0(\mathbf{R}^4), \tag{75}
$$

where

$$
H_{im}^+ = \{p \in \mathbf{R}^4 : p^2 = -m^2, p^0 \geq 0\}. \tag{76}
$$

Ω_{im}^+ is a measure concentrated on the positive time imaginary mass hyperboloid H_{im}^+ corresponding to mass im. There is also a measure Ω_{im}^- on H_{im}^- and we may define $\Omega_{im} = \Omega_{im}^+ + \Omega_{im}^-$, for $m > 0$. Ω_{im} is a Lorentz invariant measure on $H_{im} = \{p \in \mathbf{R}^4 : p^2 = -m^2\}$.

Define, for $m \in \mathbf{C}$

$$
J_m^+ = \{p \in \mathbf{C}^4 : p^2 = m^2, \mathrm{Re}(p^0) \geq 0, \mathrm{Im}(p^0) \geq 0\}, \tag{77}
$$

where $p^2 = \eta_{\mu\nu} p^\mu p^\nu$ (in which $\eta_{\mu\nu}$ is the Minkowski space metric tensor). Then, for $m > 0$,

$$
J_m^+ \cap \mathbf{R}^4 = \{p \in \mathbf{R}^4 : p^2 = m^2, p^0 \geq 0\} = H_m^+, \tag{78}
$$

$$\begin{aligned} J_m^+ \cap (i\mathbf{R}^4) &= \{p \in i\mathbf{R}^4 : p^2 = m^2, \mathrm{Re}(p^0) \geq 0, \mathrm{Im}(p^0) \geq 0\} \\ &= \{iq : q \in \mathbf{R}^4, q^2 = -m^2, q^0 \geq 0\} \\ &= iH_{im}^+. \end{aligned} \tag{79}$$

Now if $\vec{p} \in \mathbf{R}^3, m > 0$, we may write

$$\omega_{im}(\vec{p}) = ((im)^2 + \vec{p}^2)^{\frac{1}{2}} = (-m^2 + \vec{p}^2)^{\frac{1}{2}} = (-(m^2 + (i\vec{p})^2))^{\frac{1}{2}} = i(m^2 + (i\vec{p})^2)^{\frac{1}{2}} = i\omega_m(i\vec{p}).$$

One may consider the measure Ω_m^+ to be defined on $i\mathbf{R}^4$ as well as $\mathbf{R}^4$ and for all $m \in \mathbf{R}$ or $m \in i\mathbf{R}$ according to

$$\Omega_m^+(\Gamma) = \int_{\pi(\Gamma \cap J_m^+)} \frac{d\vec{p}}{\omega_m(\vec{p})}. \tag{80}$$

Then from Equation (79)

$$\Omega_m^+(i\Gamma) = \int_{i\pi(\Gamma \cap H_{im}^+)} \frac{d\vec{p}}{\omega_m(\vec{p})}. \tag{81}$$

Now make the substitution $\vec{p} = i\vec{q}$. Then $d\vec{p} = -id\vec{q}$. Thus

$$\Omega_m^+(i\Gamma) = \int_{\pi(\Gamma \cap H_{im}^+)} \frac{-id\vec{q}}{-i\omega_{im}(\vec{q})} = \Omega_{im}^+(\Gamma). \tag{82}$$

Now suppose that

$$\psi = \sum_k c_k \chi_{E_k}, \tag{83}$$

where $c_i \in \mathbf{C}$ and $E_k \in \mathcal{B}_0(\mathbf{R}^4)$, is a simple function. Then

$$\begin{aligned} \int_{\mathbf{R}^4} \psi(p)\, \Omega_{im}^+(dp) &= \sum_k c_k \Omega_{im}^+(E_k) \\ &= \sum_k c_k \Omega_m^+(iE_k) \\ &= \sum_k c_k \int_{i\mathbf{R}^4} \chi_{iE_k}(p)\, \Omega_m^+(dp) \\ &= \sum_k c_k \int_{i\mathbf{R}^4} \chi_{E_k}(\frac{p}{i})\, \Omega_m^+(dp) \\ &= \int_{i\mathbf{R}^4} \psi(\frac{p}{i})\, \Omega_m^+(dp). \end{aligned} \tag{84}$$

As this is true for every such simple function ψ it follows that

$$\int_{\mathbf{R}^4} \psi(p)\, \Omega_{im}^+(dp) = \int_{i\mathbf{R}^4} \psi(\frac{p}{i})\, \Omega_m^+(dp), \tag{85}$$

for every function ψ which is integrable with respect to Ω_{im}^+. Therefore

$$\begin{aligned} (\Omega_{im}^+ * \Omega_{im}^+)(\Gamma) &= \int_{(\mathbf{R}^4)^2} \chi_\Gamma(p+q)\, \Omega_{im}^+(dp)\, \Omega_{im}^+(dq) \\ &= \int_{(i\mathbf{R}^4)^2} \chi_\Gamma\left(\frac{p+q}{i}\right) \Omega_m^+(dp)\, \Omega_m^+(dq) \\ &= \int_{(i\mathbf{R}^4)^2} \chi_{i\Gamma}(p+q) \Omega_m^+(dp) \Omega_m^+(dq) \\ &= (\Omega_m^+ * \Omega_m^+)(i\Gamma), \end{aligned} \tag{86}$$

for all $\Gamma \in \mathcal{B}_0(\mathbf{R}^4)$.

Now in general, suppose that a measure μ has a causal spectral representation of the form

$$\mu(\Gamma) = \int_{m'=0}^{\infty} \Omega_{m'}^{+}(\Gamma)\, \sigma(m'), \tag{87}$$

for some Borel spectral measure $\sigma : \mathcal{B}_0([0,\infty)) \to \mathbf{C}$. Then μ extends to a measure defined on $i\mathbf{R}^4$ by

$$\mu(i\Gamma) = \int_{m'=0}^{\infty} \Omega_{m'}^{+}(i\Gamma)\, \sigma(dm') = \int_{m'=0}^{\infty} \Omega_{im'}^{+}(\Gamma)\, \sigma(dm'), \tag{88}$$

for $\Gamma \in \mathcal{B}_0(\mathbf{R}^4)$. Therefore since, as we have determined above, $\Omega_m^+ * \Omega_m^+$ is a causal spectral measure with spectrum

$$\sigma(m') = \begin{cases} 4\pi m Z(m') \text{ for } m' \geq 2m \\ 0 \text{ otherwise,} \end{cases} \tag{89}$$

it follows that

$$(\Omega_m^+ * \Omega_m^+)(i\Gamma) = \int_{m'=0}^{\infty} \Omega_{im'}^{+}(\Gamma)\, \sigma(dm'). \tag{90}$$

Therefore using Equation (86) $\Omega_{im}^+ * \Omega_{im}^+$ is a measure with spectral representation

$$(\Omega_{im}^+ * \Omega_{im}^+)(\Gamma) = \int_{m'=0}^{\infty} \Omega_{im'}^{+}(\Gamma)\, \sigma(m')\, dm', \tag{91}$$

where σ is the spectral function given by Equation (89). Note that $\Omega_{im}^+ * \Omega_{im}^+$ is not causal, it is a type III measure, and

$$\operatorname{supp}(\Omega_{im}^+ * \Omega_{im}^+) = \{p \in \mathbf{R}^4 : p^2 \leq -4m^2, p^0 \geq 0\}. \tag{92}$$

8. Determination of the Density Defining a Causal Lorentz Invariant Borel Measure from Its Spectrum

Suppose that μ is of the form of Equation (30) where σ is a well behaved (e.g., locally integrable) function. We would like to see if μ can be defined by a density with respect to the Lebesgue measure, i.e., if there exists a function $g : \mathbf{R}^4 \to \mathbf{C}$ such that

$$\mu(\Gamma) = \int_{\Gamma} g(p)\, dp. \tag{93}$$

Well we have that

$$\mu(\Gamma) = \int_{m=0}^{\infty} \sigma(m) \Omega_m^+(\Gamma)\, dm = \int_{m=0}^{\infty} \sigma(m) \int_{\pi(\Gamma \cap H_m^+)} \frac{d\vec{p}}{\omega_m(\vec{p})}\, dm. \tag{94}$$

Now

$$\begin{aligned} \vec{p} \in \pi(\Gamma \cap H_m^+) \quad &\Leftrightarrow \quad (\exists p \in \mathbf{R}^4)\, \vec{p} = \pi(p), p \in H_m^+, p \in \Gamma \\ &\Leftrightarrow \quad (\omega_m(\vec{p}), \vec{p}) \in \Gamma \\ &\Leftrightarrow \quad \chi_\Gamma(\omega_m(\vec{p}), \vec{p}) = 1. \end{aligned}$$

Therefore

$$\mu(\Gamma) = \int_{m=0}^{\infty} \sigma(m) \int_{\mathbf{R}^3} \chi_\Gamma(\omega_m(\vec{p}), \vec{p}) \frac{1}{\omega_m(\vec{p})}\, d\vec{p}\, dm. \tag{95}$$

Now consider the transformation defined by the function $h : (0, \infty) \times \mathbf{R}^3 \to \mathbf{R}^4$ given by

$$h(m, \overrightarrow{p}) = (\omega_m(\overrightarrow{p}), \overrightarrow{p}). \tag{96}$$

Let

$$q = h(m, \overrightarrow{p}) = (\omega_m(\overrightarrow{p}), \overrightarrow{p}) = ((m^2 + \overrightarrow{p}^2)^{\frac{1}{2}}, \overrightarrow{p}). \tag{97}$$

Then

$$\frac{\partial q^0}{\partial m} = m\omega_m(\overrightarrow{p})^{-1}, \frac{\partial q^0}{\partial p^j} = p^j \omega_m(\overrightarrow{p})^{-1}, \frac{\partial q^i}{\partial m} = 0, \frac{\partial q^i}{\partial p^j} = \delta_{ij}, \tag{98}$$

for $i, j = 1, 2, 3$. Thus the Jacobian of the transformation is

$$J(m, \overrightarrow{p}) = m\omega_m(\overrightarrow{p})^{-1}. \tag{99}$$

Now $q = (\omega_m(\overrightarrow{p}), \overrightarrow{p})$. Therefore $q^2 = \omega_m(\overrightarrow{p})^2 - \overrightarrow{p}^2 = m^2$. So $m = (q^2)^{\frac{1}{2}}, q^2 > 0$. Thus

$$\begin{aligned} \mu(\Gamma) &= \int_{q \in \mathbf{R}^4, q^2 > 0, q^0 > 0} \chi_\Gamma(q) \frac{\sigma(m)}{\omega_m(\overrightarrow{p})} \frac{dq}{J(m, \overrightarrow{p})} \\ &= \int_{q^2 > 0, q^0 > 0} \chi_\Gamma(q) \frac{\sigma(m)}{m} \, dq. \end{aligned} \tag{100}$$

Hence

$$\begin{aligned} \mu(\Gamma) &= \int_{q^2 > 0, q^0 > 0} \chi_\Gamma(q) \frac{\sigma((q^2)^{\frac{1}{2}})}{(q^2)^{\frac{1}{2}}} \, dq \\ &= \int_\Gamma g(q) \, dq, \end{aligned}$$

where $g : \mathbf{R}^4 \to \mathbf{C}$ is defined by

$$g(q) = \begin{cases} (q^2)^{-\frac{1}{2}} \sigma((q^2)^{\frac{1}{2}}) \text{ if } q^2 > 0, q^0 > 0 \\ 0 \text{ otherwise.} \end{cases} \tag{101}$$

We have therefore shown how, given a spectral representation of a causal Lorentz invariant Borel complex measure in which the spectrum is a complex function, one can obtain and equivalent representation of the measure in terms of a density with respect to Lebesgue measure.

9. Convolutions and Products of Causal Lorentz Invariant Borel Measures

9.1. Convolution of Measures

Let μ and ν be causal Lorentz invariant Borel complex measures. Then (up to possible atoms at the origin which can be dealt with in a straightforward way) there exist Borel spectral measures $\sigma, \rho : \mathcal{B}_0([0, \infty)) \to \mathbf{C}$ such that

$$\begin{aligned} \mu &= \int_{m=0}^{\infty} \Omega_m \, \sigma(dm), \\ \nu &= \int_{m=0}^{\infty} \Omega_m \, \rho(dm). \end{aligned} \tag{102}$$

We will assume, without loss of generality, that σ and ρ are complex measures, i.e. $\sigma, \rho : \mathcal{B}([0, \infty)) \to \mathbf{C}$ and are countably additive. The convolution of μ and ν, if it exists, is given by

$$(\mu * \nu)(\Gamma) = \int \chi_\Gamma(p + q) \, \mu(dp) \, \nu(dq). \tag{103}$$

Now let $\psi = \sum_i c_i \chi_{E_i}$ with $c_i \in \mathbf{C}, E_i \in \mathcal{B}_0(\mathbf{R}^4)$ be a simple function. Then

$$\begin{aligned} \int \psi(p)\, \mu(dp) &= \int \sum_i c_i \chi_{E_i}\, \mu(dp) \\ &= \sum_i c_i \mu(E_i) \\ &= \sum_i c_i \int_{m=0}^{\infty} \Omega_m(E_i)\, \sigma(dm) \\ &= \sum_i c_i \int_{m=0}^{\infty} \int_{\mathbf{R}^4} \chi_{E_i}(p)\, \Omega_m(dp)\, \sigma(dm) \\ &= \int_{m=0}^{\infty} \int_{\mathbf{R}^4} \psi(p)\, \Omega_m(dp)\, \sigma(dm). \end{aligned}$$

Therefore for any sufficiently well behaved measurable function $\psi : \mathbf{R}^4 \to \mathbf{C}$ (e.g. bounded measurable functions of compact support)

$$\int \psi(p)\mu(dp) = \int \psi(p)\, \Omega_m(dp)\, \sigma(dm). \tag{104}$$

(Note that the integral exists because σ is a Borel measure.) Hence for all $\Gamma \in \mathcal{B}_0(\mathbf{R}^4)$

$$\begin{aligned} (\mu * \nu)(\Gamma) &= \int \chi_\Gamma(p+q)\, \mu(dp)\, \nu(dq) \\ &= \int \chi_\Gamma(p+q)\, \Omega_m(dp)\, \sigma(dm)\, \Omega_{m'}(dq)\, \rho(dm') \\ &= \int \chi_\Gamma(p+q)\, \Omega_m(dp)\, \Omega_{m'}(dq) \sigma(dm)\, \rho(dm'), \end{aligned} \tag{105}$$

by Fubini's theorem, as long as

$$\int \chi_\Gamma(p+q)\, \Omega_m(dp)\, \Omega_{m'}(dq) |\sigma|(dm) < \infty, \forall m' \in [0, \infty), \tag{106}$$

where $|\sigma|$ is the total variations of the measure σ.

Suppose that $\Gamma \in \mathcal{B}_0(\mathbf{R}^4)$. Then there exists $a, R \in (0, \infty)$ such that $\Gamma \subset (-a, a) \times B_R(\vec{0})$, where $B_R(\vec{0}) = \{\vec{p} \in \mathbf{R}^3 : |\vec{p}| < R\}$. Now

$$\int \chi_\Gamma(p+q)\, \Omega_m(dp) = \int_{\Gamma - q} \Omega_m(dp) = \Omega_m(\Gamma - q) < \infty, \tag{107}$$

for all $q \in \mathbf{R}^4$ because Ω_m is Borel and Γ is compact.

Now suppose that $m, m' > a$. Then

$$p \in H_m^+, q \in H_{m'}^+ \Rightarrow (p+q)^0 = p^0 + q^0 \geq m + m' > 2a \Rightarrow (p+q) \notin \Gamma. \tag{108}$$

Thus

$$\int \chi_\Gamma(p+q)\, \Omega_m(dp)\, \Omega_{m'}(dq) = 0. \tag{109}$$

Therefore since σ and ρ are Borel, $(\mu * \nu)(\Gamma)$ exists, is finite and is given by Equation (105).

Now let $\Lambda \in O(1,3)^{+\uparrow}$, $\psi : \mathbf{R}^4 \to \mathbf{C}$ be a measurable function of compact support. Then

$$\begin{aligned} < \mu * \nu, \Lambda\psi > &= \int \psi(\Lambda^{-1}(p+q))\, \Omega_m(dp)\, \Omega_{m'}(dq)\, \sigma(dm)\, \rho(dm') \\ &= \int \psi(p+q)\, \Omega_m(dp)\, \Omega_{m'}(dq)\, \sigma(dm)\, \rho(dm'). \\ &= < \mu * \nu, \psi > \end{aligned}$$

Therefore $\mu * \nu$ is Lorentz invariant. It can be shown, by an argument similar to that used for the case $\Omega_m * \Omega_m$ that $\mu * \nu$ is causal.

We have therefore shown that the convolution of two causal Lorentz invariant Borel complex measures exists and is a causal Lorentz invariant Borel complex measure.

9.2. Product of measures

We now turn to the problem of computing the product of two causal Lorentz invariant Borel complex measures. The problem of computing the product of measures or distributions is difficult in general and has attracted a large amount of research [10,19,20]. In such work one generally seeks a definition of the product of measures or distributions which agrees with the ordinary product when the measures or distributions are functions (i.e., densities with respect to Lebesgue measure). The most common approach is to use the fact that, for Schwartz functions $f, g \in \mathcal{S}(\mathbf{R}^4)$ multiplication in the spatial domain corresponds to convolution in the frequency domain, i.e., $(fg)^\wedge = f^\wedge * g^\wedge$ (where $\wedge$ denotes the Fourier transform operator). Thus one defines the product of measures or distributions μ, ν as

$$\mu\nu = (\mu^\wedge * \nu^\wedge)^\vee. \tag{110}$$

However, this definition is only successful when the convolution that it involves exists which may not be the case in general. If μ, ν are tempered measures then $\mu^\wedge$ and $\nu^\wedge$ exist as tempered distributions, however, they are generally not causal, even if μ, ν are causal.

We will therefore not use the “frequency space" approach to define the product of measures but will use a different approach. Our approach is just as valid as the frequency space approach because our product will coincide with the usual function product when the measures are defined by densities. Furthermore, our approach is useful for the requirements of QFT because measures and distributions in QFT are frequently Lorentz invariant and causal.

Let $\mathrm{int}(C) = \{p \in \mathbf{R}^4 : p^2 > 0, p^0 > 0\}$. Suppose that $f : \mathrm{int}(C) \to \mathbf{C}$ is a Lorentz invariant locally integrable function. Then it defines a causal Lorentz invariant Borel measure μ_f which, by the spectral theorem, must have a representation of the form

$$\mu_f(\Gamma) = \int_\Gamma f(p)\, dp = \int_{m=0}^{\infty} \Omega_m(\Gamma)\, \sigma(dm), \tag{111}$$

for some spectral measure $\sigma : \mathcal{B}_0([0,\infty)) \to \mathbf{C}$. As μ_f is absolutely continuous with respect to Lebesgue measure it follows that σ must be non-singular, i.e., a function. By the result of the previous section a density defining μ_f is $\widetilde{f} : \mathrm{int}(C) \to \mathbf{C}$ defined by

$$\widetilde{f}(p) = (p^2)^{-\frac{1}{2}}\sigma((p^2)^{\frac{1}{2}}), p \in \ \mathrm{int}(C). \tag{112}$$

We must have that $\widetilde{f} = f$ (almost everywhere). Therefore (almost everywhere on $\mathrm{int}(C)$)

$$f(p) = (p^2)^{-\frac{1}{2}}\sigma((p^2)^{\frac{1}{2}}). \tag{113}$$

Without loss of generality, it can be assumed that equality holds everywhere in Equation (113). $f(p)$ depends only on p^2. Therefore for all $m > 0$, $\sigma(m) = mf(p)$ for all $p \in \mathrm{int}(C)$ such that $p^2 = m^2$. In particular

$$\sigma(m) = mf((m, \vec{0})^T), \forall m > 0. \tag{114}$$

Now we are seeking a definition of product which has useful properties. Two such properties would be that it is distributive with respect to generalized sums such as integrals and also that it agrees with the ordinary product when the measures are defined by functions. Suppose that we had such a product.

Let $f, g : \mathrm{int}(C) \to \mathbf{C}$ be Lorentz invariant locallly integrable functions. Let $\mu, \nu : \mathcal{B}_0(\mathrm{int}(C)) \to \mathbf{C}$ be the associated measures with spectra σ, ρ. Then

$$\begin{aligned}
\mu\nu &= \int_{m=0}^{\infty} \Omega_m \, \sigma(dm) \int_{m'=0}^{\infty} \Omega_{m'} \, \rho(dm') \\
&= \int_{m=0}^{\infty} \Omega_m \, m f((m, \vec{0})^T) \, dm \int_{m'=0}^{\infty} \Omega_{m'} \, m' g((m', \vec{0})^T) \, dm' \\
&= \int_{m=0}^{\infty} \int_{m'=0}^{\infty} \Omega_m \Omega_{m'} m f((m, \vec{0})^T) m' g((m', \vec{0})^T) \, dm \, dm'.
\end{aligned}$$

Now we want this to be equal to

$$\int_{m=0}^{\infty} \Omega_m m (fg)((m, \vec{0})^T) \, dm \tag{115}$$

This will be the case (formally) if we have

$$\Omega_m \Omega_{m'} = \frac{1}{m} \delta(m - m') \Omega_m, \forall m, m' > 0. \tag{116}$$

Physicists will be familiar with such a formula (e.g., the equal time commutation relations). Rather than attempting to define its meaning in a rigorous way, we will simply carry out the following formal computation for general Lorentz invariant Borel measures μ, ν with spectra σ, ρ

$$\begin{aligned}
\mu\nu &= \int_{m=0}^{\infty} \Omega_m \, \sigma(dm) \int_{m'=0}^{\infty} \Omega_{m'} \, \rho(dm') \\
&= \int_{m=0}^{\infty} \int_{m'=0}^{\infty} \Omega_m \Omega_{m'} \sigma(m) \rho(m') \, dm \, dm' \\
&= \int_{m=0}^{\infty} \int_{m'=0}^{\infty} \frac{1}{m} \Omega_m \delta(m - m') \sigma(m) \rho(m') \, dm' \, dm \\
&= \int_{m=0}^{\infty} \frac{1}{m} \Omega_m \sigma(m) \rho(m) \, dm.
\end{aligned}$$

Therefore we can simply define the product $\mu\nu$ in general by

$$\mu\nu = \int_{m=0}^{\infty} \frac{1}{m} \Omega_m \, (\sigma\rho)(dm), \tag{117}$$

i.e.,

$$(\mu\nu)(\Gamma) = \int_{m=0}^{\infty} \frac{1}{m} \Omega_m(\Gamma) \, (\sigma\rho)(dm), \tag{118}$$

for $\Gamma \in \mathcal{B}_0(\mathbf{R}^4)$.

We have therefore reduced the problem of computing the product of measures on $\mathrm{int}(C)$ to the problem of computing the product of their 1D spectral measures. The problem of multiplying 1D measures is somewhat less problematic than the problem of multiplying 4D measures. A large class of 1D measures is made up of measures which are of the form of a function plus a finite number of "atoms" (singularities of the form $c\delta_a$ where $c \in \mathbf{C} \backslash \{0\}, a \in [0, \infty)$, where δ_a is the Dirac delta function (measure) concentrated at a). There are other pathological types of the 1D measure but these may not be of interest for physical applications.

In the general non-pathological case, if μ, ν are causal Lorentz invariant Borel measures with spectra $\sigma(m) = \xi(m) + \sum_{i=1}^{k} c_i \delta(m - a_i), \rho(m) = \zeta(m) + \sum_{j=1}^{l} d_j \delta(m - b_j)$ where $\xi, \zeta : [0, \infty) \to \mathbf{C}$ are locally integrable functions, $c_i, d_j \in \mathbf{C} \backslash \{0\}, k, l \geq 0, a_i, b_j \in [0, \infty)$ are such that $a_i \neq b_j, \forall i, j$ then we may define the product of μ and ν to be the causal Lorentz invariant measure $\mu\nu$ given by

$$\mu\nu = \int_{m=0}^{\infty} \Omega_m \tau(dm), \tag{119}$$

where

$$\begin{aligned}\tau(m) &= \frac{1}{m}(\xi(m)\zeta(m) + \zeta(m)\sum_{i=1}^{k} c_i\delta(m-a_i) + \xi(m)\sum_{j=1}^{l} d_j\delta(m-b_j)) \\ &= \frac{1}{m}(\xi(m)\zeta(m) + \sum_{i=1}^{k}\zeta(a_j)c_i\delta(m-a_i) + \sum_{j=1}^{l}\xi(b_j)d_j\delta(m-b_j)),\end{aligned}$$

for $m > 0$.

10. Conclusions

We have defined a spectral calculus that enables one to compute the spectrum of any causal Lorentz invariant Borel complex measure on Minkowski space whose spectrum is a continuous function. This calculus can be used in many applications in QFT and leads to a method called spectral regularization [21].

We have computed the spectra associated with certain elementary convolutions involving Feynman propagators of mass m scalar particles. It has been shown how one can compute the density associated with a causal Lorentz invariant Borel complex measure from its spectrum.

We have shown that the convolution of arbitrary measures of the prescribed type exists and how their product exists in a wide class of cases of physical interest. Methods for the computation of these objects from the spectra of their components have been presented.

The spectral calculus can be used to compute the spectrum, and hence density, associated with the contraction of the vacuum polarization tensor [21]. A generalization of the spectral calculus to Lorentz invariant tensor valued measures on Minkowski space can be used to compute the form of the vacuum polarization tensor and therefore to compute the vacuum polarization function. This function is shown to have a close agreement, up to finite renormalization, with the vacuum polarization function obtained using dimensional regularization /renormalization. This can be used to compute the Uehling potential function without using renormalization from which the Uehling contribution to the Lamb shift for the H atom can be computed exactly.

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