On Iteration S_n for Operators with Condition (D)

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Abstract: A recently introduced nonexpansive-type condition is subjected to an in-depth analysis. New examples are provided to highlight the relationship with Suzuki-type mappings. Furthermore, a convergence survey is conducted based on the iteration procedure S_n . Issues related to data dependence and the stability of this iterative process are also being studied. Our study is performed in the framework of Banach spaces, in which the symmetry of the associated metric is a fundamental axiom and plays a key role while proving many results of this paper.

Keywords: nonexpansive mapping; Suzuki mapping; condition (D); Opial property; iteration procedure; convergence; stability; data dependence

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1. Introduction

It is a well-known fact that various problems of applied mathematics could be generally expressed by means of systems of equations. Obviously, reaching the exact solutions is the main goal. However, quite often, the mathematical tools are insufficiently developed to provide the proper answer for this issue. For this reason, sometimes, we are content with less; we aim to confirm at least the existence and, possibly, the uniqueness of the solution. A powerful instrument in this regard is Banach's Contraction Principle. It requires, first of all, expression of the problem as a fixed point equation,

$$Tx = x,$$

in a properly defined setting (usually a complete metric space) and, secondly, evaluating the iteration function T as being a contraction. However, the contractive property is sometimes too restrictive. This is the reason why, in the past 50 years, the study of fixed points for more general classes of mappings has become an important research direction.

We will consider below a nonempty subset C of a Banach space X . A self-mapping $T: C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. The mapping is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$, where $F(T)$ is the set of fixed points of T , i.e., $F(T) = \{x \in C \mid Tx = x\}$. It is easy to see that every nonexpansive mapping that has a fixed point is quasi-nonexpansive. It is also known that if X is uniformly convex and C is a closed, bounded, and convex subset of X , then $F(T)$ is nonempty for a nonexpansive operator.

In 2008, Suzuki [1] introduced a class of generalized nonexpansive mappings; this definition relies on the so-called condition (C). Let C be a nonempty subset of a Banach space X . A mapping $T: C \rightarrow C$ is said to satisfy condition (C) if

$$\frac{1}{2} \|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

Suzuki showed that the class of mappings satisfying condition (C) properly includes the nonexpansive mappings; still, it is properly included in the class of quasi-nonexpansive operators (provided $F(T) \neq \emptyset$). He also obtained fixed point theorems and convergence theorems for the newly defined contractive condition.

In 2018, Donghan et al. [2] provided a new generalized nonexpansiveness condition, which they called condition (D_a) . If C is a nonempty subset of a Banach space X and $a \in \left(\frac{1}{2}, 1\right)$, the mapping $T: C \rightarrow C$ is said to satisfy condition (D_a) if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $\alpha \in [a, 1]$, $x \in C$ and $y \in C(T, x, \alpha)$, where

$$C(T, x, \alpha) = \{(1 - \alpha)p + \alpha Tq : p, q \in C, \|Tp - p\|, \|Tq - q\| \leq \|Tx - x\|\}.$$

Inspired by this, in 2020, Bejenaru and Postolache [3] introduced a more general class of mappings by fixing $\alpha = 1$, resulting in the so-called condition (D). Let C be a nonempty subset of a Banach space X . A mapping $T: C \rightarrow C$ is said to be endowed with property (D) if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x \in C$ and $y \in C(T, x)$, where

$$C(T, x) = \{Tp : p \in C, \|Tp - p\| \leq \|Tx - x\|\}.$$

The authors also noted that the new class of mappings relates to nonexpansiveness and quasi-nonexpansiveness, as does the Suzuki class of mappings: each nonexpansive mapping satisfies condition (D); also, if T satisfies condition (D), then T is quasi-nonexpansive. However, in the setting of Banach spaces, the authors of [3] did not perform an in-depth analysis of the inclusion or exclusion relationship between properties (C) and (D). This will be one of the objectives of this paper.

In addition to the existence and uniqueness statement, Banach's Contraction Principle also points out that the fixed points of a contraction can be obtained using Picard iterations. Unlike contractions, successive iterations for nonexpansive mapping do not necessarily converge at a fixed point. One of the most important results for the approximation of fixed points for nonexpansive mapping was established by Krasnosel'skii [4]. The results show that if X is a uniformly convex Banach space and $T: X \rightarrow X$ is a nonexpansive mapping, then the successive iterations of the function $\frac{1}{2}(I + T)$ are convergent to a fixed point of T .

Since then, more and more complex iterative processes have been developed to serve to approximate the fixed points of nonexpansive mappings. An example is the Mann iteration scheme [5]. Another iterative process of this kind that is widely used is owed to Ishikawa [6]. In 2000, Noor [7] introduced a three-step iterative scheme in connection with variational inequalities. An overview of this paper reveals that this procedure had a more general effect: It provided a more rapid iterative procedure for reckoning the fixed points of nonexpansive mappings. Another three-step iterative process involving nonexpansive mappings was provided by Abbas and Nazir [8] in 2014.

In 2016, Thakur et al. [9] introduced another three-step iteration process by means of two parametric sequences.

This time, the procedure was connected to Suzuki-type mappings. Similar approaches involving other iteration procedures were recently performed in [10,11].

In 2016, Sintunavarat and Pitea [12] introduced an iterative scheme in connection with Berinde-type operators (for details, please see [13]). For an arbitrary $x_1 \in C$, the sequence $\{x_n\}$ results from the three-step procedure

$$\begin{cases} x_{n+1} = (1 - \alpha_n) Tz_n + \alpha_n Ty_n \\ z_n = (1 - \gamma_n) x_n + \gamma_n y_n \\ y_n = (1 - \beta_n) x_n + \beta_n Tx_n, \end{cases} \quad (1)$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences $(0, 1)$. We will denote this iterative process further with S_n .

In the following, we will reconsider the iterative process (1) in connection with mappings satisfying condition (D). Overall, we adopt the same type of reasoning as in [9], performing convergence, stability, and data dependence analyses. Not least, we shall conduct a comparative analysis between the classes of mappings satisfying properties (C) and (D), respectively. This analysis is meant to emphasize the usefulness of considering mappings with property (D); an important step is proving that they are not included within the Suzuki-type class, although both these properties are properly extending the nonexpansiveness condition. We think that the present study opens new research perspectives. For instance, connections with split common fixed point problems (see [14–16]), proximal split feasibility problems (see [17,18]), or split equality fixed point problems (see [19]) could be further researched by means of this study. Other problems, such as finding the intersection points of the fixed point set and zero set of a sum for nonlinear operators (see [20]) or achieving common solution points of convex programming problems (see [21]), can also be considered as natural extensions.

2. Preliminaries

In the section, we shall provide a short description of the general setting, as well as several required tools to make the approach self-contained.

Definition 1 ([22,23]). A Banach space X is called uniformly convex if, for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, the following inequality holds true:

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

Definition 2 ([24]). A Banach space X is said to satisfy the Opial property provided that each weakly convergent sequence $\{x_n\}$ in X with weak limit x satisfies the following inequality:

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|, \text{ for all } y \in X \text{ with } y \neq x.$$

Proposition 1 ([3], Lemma 3.8). Let C be a nonempty subset of a Banach space X and $T: C \rightarrow C$. If T satisfies condition (D), then:

- (i) $\|T^2x - Tx\| \leq \|Tx - x\|$, for all $x \in C$;
- (ii) For all $x, y \in C$, at least one of the inequalities $\|T^2x - Ty\| \leq \|Tx - y\|$ or $\|T^2y - Tx\| \leq \|Ty - x\|$ is satisfied;
- (iii) $\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$ whenever $\|Tx - x\| \leq \|Ty - y\|$;
- (iv) $\|Tx - Ty\| \leq 2 \min \{\|Tx - x\|, \|Ty - y\|\} + \|x - y\|$, for all $x, y \in C$.

First of all, we will reinforce the conclusion in point (iii) for all $x, y \in C$.

Lemma 1. If T satisfies condition (D), then:

$$\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|, \forall x, y \in C.$$

Proof. Indeed, by applying property (iv), one finds

$$\begin{aligned}\|x - Ty\| &\leq \|x - Tx\| + \|Tx - Ty\| \\ &\leq \|x - Tx\| + 2 \min\{\|Tx - x\|, \|Ty - y\|\} + \|x - y\| \\ &\leq \|x - Tx\| + 2\|x - Tx\| + \|x - y\| \\ &= 3\|Tx - x\| + \|x - y\|,\end{aligned}$$

hence the proof. $\square$

Lemma 2. Let T be a mapping on a subset C of a Banach space X with the Opial property. Assume that T satisfies condition (D). If the sequence $\{x_n\}$ converges weakly to z and $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$, then $Tz = z$.

Proof. By Lemma 1 we have, for each $n \in \mathbb{N}$,

$$\|x_n - Tz\| \leq 3\|Tx_n - x_n\| + \|x_n - z\|.$$

Furthermore, by taking $\liminf$, we find

$$\liminf_{n \rightarrow \infty} \|x_n - Tz\| \leq \liminf_{n \rightarrow \infty} \|x_n - z\|,$$

and using the Opial property, we conclude that $Tz = z$. $\square$

Lemma 3 ([25], Lemma 1.3). Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$, for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that the inequalities $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\limsup_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then, $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Remark 1. Let C be a nonempty closed convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in X . For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} \|x - x_n\|.$$

(i) The asymptotic radius of $\{x_n\}$ relative to C is defined as

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) \mid x \in C\}.$$

(ii) The asymptotic center of $\{x_n\}$ relative to C is given by

$$A(C, \{x_n\}) = \{x \in C \mid r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

(iii) Edelstein [26] showed that, for a nonempty closed convex subset C of a uniformly convex Banach space and for each bounded sequence $\{x_n\}$, the set $A(C, \{x_n\})$ is a singleton.

3. Examples

In this section, we analyze the relationship between mappings satisfying conditions (C) and (D), respectively. First, let us prove that the mapping class that meets the (D) condition is not included in the Suzuki mapping class. In doing so, we emphasize the relevance of conducting a study of operators with property (D).

Example 1. Consider the mapping

$$T: [0, 1] \rightarrow [0, 1], \quad Tx = \begin{cases} 1, & x = 0 \\ \frac{1}{2}, & x \in \left(0, \frac{1}{2}\right) \\ x, & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We would like to prove that T is not a Suzuki mapping, but satisfies condition (D).

Proof. Let us point out the following two properties, resulting directly from condition (D):

- (1) $C(T, x) = \{Tp : d(p, Tp) \leq d(x, Tx)\} \subseteq \text{Im}T$;
- (2) $F(T) \subseteq C(T, x)$.

For the example above, we identify

$$\text{Im}T = \left[\frac{1}{2}, 1\right]$$

and

$$F(T) = \left[\frac{1}{2}, 1\right].$$

Therefore, $C(T, x) = \left[\frac{1}{2}, 1\right]$, for all $x \in [0, 1]$.

Let us prove next that property (D) is satisfied, i.e. for all $x \in [0, 1]$ and $y \in C(T, x)$, we have $d(Tx, Ty) \leq d(x, y)$.

Case I: Let $x = 0$ and $y \in C(T, x)$. Through direct computations, we find

$$d(Tx, Ty) = |1 - y| = 1 - y$$

and

$$d(x, y) = |0 - y| = y.$$

Since $y \geq \frac{1}{2}$, we have $1 - y \leq y$, so $d(Tx, Ty) \leq d(x, y)$.

Case II: Let $x \in \left(0, \frac{1}{2}\right)$ and $y \in C(T, x)$. Again, we compute

$$d(Tx, Ty) = \left|\frac{1}{2} - y\right| = y - \frac{1}{2}$$

and

$$d(x, y) = |x - y| = y - x.$$

Since $x < \frac{1}{2}$, it follows that $y - \frac{1}{2} < y - x$, so $d(Tx, Ty) \leq d(x, y)$.

Case III: For the conditions $x \in \left[\frac{1}{2}, 1\right]$ and $y \in C(T, x) = \left[\frac{1}{2}, 1\right]$, the conclusion comes out immediately.

Given these three cases, it follows that T is a mapping that satisfies condition (D). We prove next that T does not satisfy condition (C).

Take $x = \frac{1}{4}$ and $y = 0$. Then, $Tx = \frac{1}{2}$ and $Ty = 1$. We have

$$\frac{1}{2}d(x, Tx) = \frac{1}{2} \cdot \left|\frac{1}{4} - \frac{1}{2}\right| = \frac{1}{8}$$

and

$$d(x, y) = \left| \frac{1}{4} - 0 \right| = \frac{1}{4}.$$

It is obvious that

$$\frac{1}{2}d(x, Tx) \leq d(x, y).$$

Still, $d(Tx, Ty) = \left| \frac{1}{2} - 1 \right| = \frac{1}{2} > d(x, y)$. Therefore, we conclude that T is not a Suzuki mapping. $\square$

Obviously, the two examined classes are not completely disjoint, as both contain the nonexpansive operators. The following example is meant to prove that there are also common elements that are not necessarily nonexpansive.

Example 2. Let us consider the Banach space $X = L^\infty(\mathbb{R})$ with the essential supremum norm

$$\|f\|_\infty = \text{ess sup}_{\mathbb{R}} |f| = \inf \{M : |f(x)| \leq M \text{ almost everywhere on } \mathbb{R}\}.$$

Let C be set of all the functions $f: \mathbb{R} \rightarrow [0, 11]$, satisfying $f(x) = f(0)$, for all $x \leq 0$, almost everywhere on $\mathbb{R}$, and define the mapping

$$T: C \rightarrow C, \quad Tf(x) = \begin{cases} f(x), & x > 0 \\ \frac{4}{11}f(0), & x \leq 0, \quad f(0) \neq 11 \\ 5, & x \leq 0, \quad f(0) = 11. \end{cases}$$

We prove that T is not a nonexpansive mapping, but satisfies both conditions (C) and (D).

Proof. By taking the constant functions $f \equiv 10$ and $g \equiv 11$, we find

$$\|f - g\|_\infty = 1.$$

On the other hand,

$$Tf(x) = \begin{cases} 10, & x > 0 \\ \frac{40}{11}, & x \leq 0 \end{cases} \quad \text{and} \quad Tg(x) = \begin{cases} 11, & x > 0 \\ 5, & x \leq 0, \end{cases}$$

leading to the conclusion

$$\|Tf - Tg\|_\infty = \frac{15}{11}.$$

Hence, T is not a nonexpansive mapping.

Let us check the condition (C) next. Suppose the inequality

$$\frac{1}{2} \|f - Tf\|_\infty \leq \|g - f\|_\infty$$

is satisfied. This is further equivalent with

$$\frac{1}{2} |Tf(0) - f(0)| \leq \max \left\{ |f(0) - g(0)|, \text{ess sup}_{(0, \infty)} |f(x) - g(x)| \right\}.$$

Thus, two cases are worth analyzing.

Case I. Let us assume that

$$\|g - f\|_\infty = \max \left\{ |f(0) - g(0)|, \text{ess sup}_{(0, \infty)} |f(x) - g(x)| \right\} = |f(0) - g(0)|.$$

For T to satisfy condition (C), this must imply

$$\max \left\{ |Tf(0) - Tg(0)|, \operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)| \right\} \leq |f(0) - g(0)|,$$

so, as before, there are two sub-cases that need to be considered. Only one of these is nontrivial, i.e.,

$$\frac{1}{2} |Tf(0) - f(0)| \leq |f(0) - g(0)| \Rightarrow |Tf(0) - Tg(0)| \leq |f(0) - g(0)|.$$

If $f(0) \neq 11$ and $g(0) \neq 11$, or $f(0) = 11$ and $g(0) = 11$, it can be easily noticed that T is nonexpansive, and, therefore, condition (C) is automatically fulfilled.

For $f(0) \neq 11$ and $g(0) = 11$, T is nonexpansive just for $0 \leq f(0) \leq \frac{66}{7}$ and condition (C) is satisfied. For $\frac{66}{7} < f(0) < 11$ and $g(0) = 11$, we will have

$$\frac{1}{2} |Tf(0) - f(0)| \leq |f(0) - g(0)| \Rightarrow \frac{7}{22} f(0) \leq 11 - f(0).$$

However, this inequality is never satisfied, since, for $\frac{66}{7} < f(0) < 11$, one has $3 < \frac{7}{22} f(0) < \frac{7}{2}$ and $0 < 11 - f(0) < \frac{11}{7}$. Hence, this particular range is not relevant for the nonexpansiveness analysis.

The same result is obtained if we take $f(0) = 11$ and $g(0) \neq 11$. Considering all the situations analyzed, we conclude that T is a Suzuki mapping for the current case.

Case II. Suppose

$$\|g - f\|_{\infty} = \max \left\{ |f(0) - g(0)|, \operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)| \right\} = \operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)|.$$

For T to satisfy condition (C), the assumption

$$\frac{1}{2} |Tf(0) - f(0)| \leq \operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)|$$

should imply

$$\max \left\{ |Tf(0) - Tg(0)|, \operatorname{ess\,sup}_{(0,\infty)} |Tf(x) - Tg(x)| \right\} \leq \operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)|,$$

which ultimately comes to

$$|Tf(0) - Tg(0)| \leq \operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)|.$$

By assuming the opposite, namely $|Tf(0) - Tg(0)| > \operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)|$, and keeping in mind that

$$\operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)| \geq |f(0) - g(0)|$$

and

$$\operatorname{ess\,sup}_{(0,\infty)} |f(x) - g(x)| \geq \frac{1}{2} |Tf(0) - f(0)|,$$

we reach contradictions. Indeed, if $f(0) \neq 11$ and $g(0) \neq 11$, then $|Tf(0) - Tg(0)| = \frac{4}{11} |f(0) - g(0)|$.

If $f(0) = 11$ and $g(0) \neq 11$, then $|Tf(0) - Tg(0)| = 5 - \frac{4}{11}g(0)$, $|f(0) - g(0)| = 11 - g(0)$, $\frac{1}{2} |Tf(0) - f(0)| = 3$; the assumption above will lead to $\frac{66}{7} < g(0) < \frac{11}{2}$, which is absurd. Similarly,

when $f(0) \neq 11$ and $g(0) = 11$, it follows that $|Tf(0) - Tg(0)| = 5 - \frac{4}{11}f(0)$, $|f(0) - g(0)| = 11 - f(0)$, and $\frac{1}{2}|Tf(0) - f(0)| = \frac{7}{22}f(0)$, which leads to $\frac{66}{7} < f(0) < \frac{22}{3}$, which is, again, impossible.

So, T is a Suzuki mapping in this case too. Given these two cases and the above considerations, we have that T is a Suzuki operator.

Finally, let us prove that T also satisfies condition (D). Let us recall that $C(T, f) \subseteq \text{Im}T$ for each $f \in C$, and let us notice that

$$\text{Im}T = \{f \in C : f(0) \in [0, 4] \cup \{5\}\} \subset \{f \in C : f(0) \in [0, 5]\}.$$

Condition (D) asks the nonexpansiveness condition to be satisfied for each $f \in C$ and each $g \in C(T, f)$. We will prove a little more than this; namely, we shall test the nonexpansiveness for all $f \in C$ and for all $g \in \{f \in C : f(0) \in [0, 5]\}$. The nontrivial case to be analyzed is when $f(0) = 11$. If so, then the nonexpansiveness is ensured whenever $g(0) \leq \frac{66}{7}$, which is obviously true for the selected range. $\square$

4. Fixed Point and Convergence Results

In this section, inspired by the results obtained in [12] via the iteration procedure (1) involving Berinde mappings, we phrase and prove some convergence outcomes regarding mappings satisfying condition (D). It is worth mentioning that these results are valid in the broader context of quasi-nonexpansive operators. However, since our aim in this paper, besides studying the combination of the iterative procedure (1) with condition (D), is also to compare the conditions (C) and (D), we have adopted this phrasing.

Lemma 4. *Let C be a nonempty closed convex subset of a Banach space X and let $T : C \rightarrow C$ be a mapping satisfying condition (D) with $F(T) \neq \emptyset$. For an arbitrary $x_1 \in C$, let the sequence $\{x_n\}$ be generated by (1). Then, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$. Since T satisfies condition (D), it is also quasi-nonexpansive, hence

$$\|p - Tz\| \leq \|p - z\|, \quad \forall z \in C.$$

Then,

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n Tx_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\| \end{aligned} \quad (2)$$

and

$$\begin{aligned} \|z_n - p\| &= \|(1 - \gamma_n)x_n + \gamma_n y_n - p\| \\ &\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \quad (3)$$

Using (2) and (3), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)Tz_n + \alpha_n Ty_n - p\| \\ &\leq (1 - \alpha_n)\|Tz_n - p\| + \alpha_n\|Ty_n - p\| \\ &\leq (1 - \alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

That implies that the sequence $\{\|x_n - p\|\}$ is bounded and nonincreasing for each $p \in F(T)$. Hence, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. $\square$

Theorem 1. Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T : C \rightarrow C$ be a mapping satisfying condition (D). For an arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by (1) for all $n \geq 1$, where $\{\beta_n\}$ is a sequence of real numbers in $[a, b]$ for some a and b with $0 < a \leq b < 1$. Then, $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$.

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma 4, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded.

Set

$$\lim_{n \rightarrow \infty} \|x_n - p\| = r. \quad (4)$$

From (2) and (4), we get

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = r. \quad (5)$$

By quasi-nonexpansiveness, we have

$$\limsup_{n \rightarrow \infty} \|Tx_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = r. \quad (6)$$

On the other hand,

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n) \|z_n - p\| + \alpha_n \|y_n - p\| \\ &= (1 - \alpha_n) \|(1 - \gamma_n)x_n + \gamma_n y_n - p\| + \alpha_n \|y_n - p\| \\ &\leq (1 - \gamma_n)(1 - \alpha_n) \|x_n - p\| + [(1 - \alpha_n)\gamma_n + \alpha_n] \|y_n - p\| \\ &= \|x_n - p\| + [1 - (1 - \gamma_n)(1 - \alpha_n)] [\|y_n - p\| - \|x_n - p\|]. \end{aligned}$$

Since $\{\alpha_n\}$ and $\{\gamma_n\}$ are included in $(0, 1)$, it follows that $1 - (1 - \gamma_n)(1 - \alpha_n) \neq 0$, $\forall n \in \mathbb{N}$; thus,

$$\frac{\|x_{n+1} - p\| - \|x_n - p\|}{1 - (1 - \gamma_n)(1 - \alpha_n)} \leq \|y_n - p\| - \|x_n - p\|.$$

Furthermore,

$$\|x_{n+1} - p\| - \|x_n - p\| \leq \frac{\|x_{n+1} - p\| - \|x_n - p\|}{1 - (1 - \gamma_n)(1 - \alpha_n)} \leq \|y_n - p\| - \|x_n - p\|,$$

hence $\|x_{n+1} - p\| \leq \|y_n - p\|$.

Therefore,

$$r \leq \liminf_{n \rightarrow \infty} \|y_n - p\|. \quad (7)$$

By (5) and (7), we get

$$\lim_{n \rightarrow \infty} \|\beta_n(Tx_n - p) + (1 - \beta_n)(x_n - p)\| = \lim_{n \rightarrow \infty} \|y_n - p\| = r. \quad (8)$$

From (4), (6), (8), and Lemma 3, we get that $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. Let $p \in A(C, \{x_n\})$. By Lemma 1, we have:

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n \rightarrow \infty} \|x_n - Tp\| \\ &\leq \limsup_{n \rightarrow \infty} (3\|Tx_n - x_n\| + \|x_n - p\|) \\ &= \limsup_{n \rightarrow \infty} \|x_n - p\| \\ &= r(p, \{x_n\}). \end{aligned}$$

That implies that $Tp \in A(C, \{x_n\})$. Since X is uniformly convex, $A(C, \{x_n\})$ is a singleton; hence, we have $Tp = p$. This completes the proof. $\square$

Theorem 2. Let C be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property, and let T and $\{x_n\}$ be as in Theorem 1, with the additional assumption $F(T) \neq \emptyset$. Then, $\{x_n\}$ converges weakly to a fixed point of T .

Proof. The proof does not differ at all from the proof of Theorem 3.3 in [9]. $\square$

Theorem 3. Let C be a nonempty, compact, and convex subset of a uniformly convex Banach space X , and let T and $\{x_n\}$ be as in Theorem 1. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. $F(T) \neq \emptyset$, so by Theorem 1, we have

$$\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0.$$

Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges strongly to an element $p \in C$. Using Lemma 1, we find

$$\|x_{n_k} - Tp\| \leq 3\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\|, \text{ for all } n \geq 1.$$

Letting $k \rightarrow \infty$, we get that $\{x_{n_k}\}$ converges to Tp . This implies that $Tp = p$, so $p \in F(T)$. In addition, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists by Theorem 1; thus, p is the strong limit of the sequence $\{x_n\}$ itself. $\square$

Other interesting results concerning the strong convergence of this iterative procedure under additional assumptions (as property I [27], for instance) could be found in [9].

5. The Stability of the S_n Iteration Procedure

Broadly speaking, an iteration process that converges to a unique fixed point is stable if the convergence of the procedure is not influenced by the numerical errors that occur during each iteration step. Harder and Hicks [28], who have important contributions in the domain, were the ones who first introduced the notion of T -stability for both metric spaces and normed spaces. The second definition (the one for normed spaces) is used in this paper.

Definition 3 ([28]). Let C be a nonempty subset of E , where $(E, \|\cdot\|)$ is a normed space, and let T be a self-mapping on C with a fixed point. Let $\{t_n\}$ be an arbitrary sequence in C and

$$\varepsilon_n = \|t_{n+1} - f(T, t_n)\|,$$

for $n \geq 0$. Then, the iteration procedure $x_{n+1} = f(T, x_n)$ converging to a fixed point p of T is said to be T -stable provided that the following equivalence holds true:

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} t_n = p.$$

Berinde [29] and Olatinwo and Postolache [30] are some of the mathematicians that have recently developed important and innovative results regarding the T -stability of iterative processes in uniformly convex metric spaces. Their work is a big step in the field.

The next lemma will be used in the following.

Lemma 5 ([31]). *Let $\{\psi_n\}$ and $\{\varphi_n\}$ be nonnegative real number sequences satisfying the following inequality:*

$$\psi_{n+1} \leq (1 - \phi_n) \psi_n + \varphi_n,$$

where $\phi_n \in (0, 1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$, and $\lim_{n \rightarrow \infty} \frac{\varphi_n}{\phi_n} = 0$. Then,

$$\lim_{n \rightarrow \infty} \psi_n = 0.$$

We start by proving that, for contractive mappings, the iteration procedure (1) really converges to the unique fixed point of T . Further on, we shall analyze the stability of the iteration procedure S_n (1).

Theorem 4. *Let C be a nonempty closed convex subset of a Banach space X and let $T : C \rightarrow C$ be a contraction mapping. Let $\{x_n\}$ be an iterative sequence generated by (1), with $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$, satisfying $\sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T .*

Proof. The existence and uniqueness of the fixed point p are provided by Banach's Contraction Principle. We shall prove that $\{x_n\}$ strongly converges to p . Let $\theta \in (0, 1)$ be the contraction coefficient, i.e.,

$$\|Tx - Ty\| \leq \theta \|x - y\|,$$

for all $x, y \in C$.

We have

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n Tx_n - (1 - \beta_n)p - \beta_n p\| \\ &= \|(1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p)\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \theta\beta_n\|x_n - p\| \\ &= (1 - \beta_n(1 - \theta))\|x_n - p\| \end{aligned} \quad (9)$$

and

$$\begin{aligned} \|z_n - p\| &= \|(1 - \gamma_n)x_n + \gamma_n y_n - (1 - \gamma_n)p - \gamma_n p\| \\ &= \|(1 - \gamma_n)(x_n - p) + \gamma_n(y_n - p)\| \\ &\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n(1 - \beta_n(1 - \theta))\|x_n - p\| \\ &= (1 - \gamma_n\beta_n(1 - \theta))\|x_n - p\|. \end{aligned} \quad (10)$$

Using (9) and (10), we find

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)Tz_n + \alpha_n Ty_n - (1 - \alpha_n)p - \alpha_n p\| \\ &\leq (1 - \alpha_n)\|Tz_n - p\| + \alpha_n\|Ty_n - p\| \\ &\leq \theta(1 - \alpha_n)\|z_n - p\| + \theta\alpha_n\|y_n - p\| \\ &\leq \theta(1 - \alpha_n)(1 - \gamma_n\beta_n(1 - \theta))\|x_n - p\| + \theta\alpha_n(1 - \beta_n(1 - \theta))\|x_n - p\| \\ &\leq \theta(1 - \alpha_n\beta_n\gamma_n(1 - \theta))\|x_n - p\|. \end{aligned} \quad (11)$$

We can easily see that

$$\|x_n - p\| \leq \theta^{n+1} \|x_1 - p\| \prod_{k=1}^n [1 - (1 - \theta)\alpha_k\beta_k\gamma_k], \quad (12)$$

where $1 - (1 - \theta)\alpha_k\beta_k\gamma_k < 1$ since $\theta \in (0, 1)$ and $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\} \in (0, 1)$ for each k .

Using the inequality $1 - x < e^{-x}$ for all $x \in (0, 1)$, from (12), it follows that

$$\|x_{n+1} - p\| \leq \|x_1 - p\| \theta^n e^{-(1-\theta)\sum_{k=1}^n \alpha_k\beta_k\gamma_k}. \quad (13)$$

Taking the limit for $n \rightarrow \infty$ in relation (13) and knowing that $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$, we get that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = 0,$$

so $\{x_n\}$ converges strongly to p . $\square$

We are ready to state and prove our stability result.

Theorem 5. Let C be a nonempty closed convex subset of a Banach space X and let $T : C \rightarrow C$ be a contraction mapping. Let $\{x_n\}$ be an iterative sequence generated by (1), with $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ in $(0, 1)$ such that $\alpha_n\beta_n\gamma_n \geq a > 0$, $\forall n$. Then, the iterative procedure (1) is T -stable.

Proof. Let $\{t_n\}$ be an arbitrary sequence in C and $\varepsilon_n = \|t_{n+1} - f(T, t_n)\|$ for all $n \geq 0$. We have to prove that

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} t_n = p.$$

Suppose $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. By (11), we find

$$\begin{aligned} \|t_{n+1} - p\| &\leq \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - p\| \\ &= \varepsilon_n + \|f(T, t_n) - p\| \\ &\leq \varepsilon_n + \theta(1 - \alpha_n\beta_n\gamma_n(1 - \theta))\|t_n - p\| \\ &\leq \varepsilon_n + (1 - \alpha_n\beta_n\gamma_n(1 - \theta))\|t_n - p\|. \end{aligned}$$

Define $\psi_n = \|t_n - p\|$, $\phi_n = \alpha_n\beta_n\gamma_n(1 - \theta)$ and $\varphi_n = \varepsilon_n$, for all n . Since $\alpha_n\beta_n\gamma_n \in [a, 1)$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, it follows that $\sum_{n=1}^{\infty} \phi_n = \infty$ and $\lim_{n \rightarrow \infty} \frac{\varphi_n}{\phi_n} = 0$. Hence, by Lemma 4.3, we have $\lim_{n \rightarrow \infty} \psi_n = 0$, so $\lim_{n \rightarrow \infty} t_n = p$.

Conversely, let $\lim_{n \rightarrow \infty} t_n = p$. It follows that

$$\begin{aligned} \varepsilon_n &= \|t_{n+1} - f(T, t_n)\| \\ &\leq \|t_{n+1} - p\| + \|p - f(T, t_n)\| \\ &\leq \|t_{n+1} - p\| + \theta(1 - \alpha_n\beta_n\gamma_n(1 - \theta))\|t_n - p\|. \end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$; hence, the iteration procedure (1) is T -stable. $\square$

6. Data Dependence Analysis

In this section, some aspects related to data dependence will be considered. There are a lot of important results regarding this subject obtained by Rus and Mureşan [32], Berinde [33], Olatinwo [34], and Usurelu et al. [10,35], as the subject has been considered a big point of interest over the last years. The motivation of this problem is given by the fact that the practical implementation of algorithms works with approximations instead of theoretical, unperturbed operators. The data dependency analysis aims to answer the following question: To what extent is the achievement of the real fixed point affected by the use of a disturbed operator? In other words, by deviating from the actual

mapping T to a perturbed mapping $\tilde{T}$, the numerical simulation should keep the output close enough to the actual solution. Obviously, the errors would reach a minimum level in the situation where the procedure would depend only on the initial estimate and not on the operator itself. A formal statement regarding data dependence analysis is included in the definitions below.

Definition 4 ([29]). Let $T, \tilde{T} : X \rightarrow X$ be two mappings. We say that $\tilde{T}$ is an approximate mapping of T if for some $\varepsilon > 0$, provided that

$$\|Tx - \tilde{T}x\| \leq \varepsilon,$$

for all $x \in X$. We shall refer to ε as the maximum admissible error for T .

Definition 5 ([35]). Let X be a Banach space and T be a self-mapping with a fixed point p . Let $\tilde{T}$ be an approximate mapping of T with maximum admissible error $\varepsilon > 0$, admitting a fixed point $\tilde{p}$. Assume that f defines some iteration procedure such that, for $x_1 \in X$, the sequences $x_{n+1} = f(T, x_n)$ and $\tilde{x}_{n+1} = f(\tilde{T}, \tilde{x}_n)$ converge to p and $\tilde{p}$, respectively. We call the iteration procedure for f data independent if

$$\|p - \tilde{p}\| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

The lemma below will be an important further tool.

Lemma 6 ([36]). Let $\{\psi_n\}$ and $\{\phi_n\}$ be nonnegative real number sequences for which one assumes there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the following inequality holds:

$$\psi_{n+1} \leq (1 - \phi_n) \psi_n + \phi_n \varphi_n,$$

where $\phi_n \in (0, 1)$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} \phi_n = \infty$. Then,

$$0 \leq \limsup_{n \rightarrow \infty} \psi_n \leq \limsup_{n \rightarrow \infty} \varphi_n.$$

The following theorem provides an estimate of the deviation of the fixed point in terms of the maximum admissible error, also proving the data independency of the iteration procedure S_n .

Theorem 6. Let C be a nonempty closed convex subset of a Banach space X and let $T : C \rightarrow C$ be a contraction mapping with fixed point p . Let $\tilde{T}$ be an approximate mapping of the contraction mapping T with maximum admissible error ε , let $\{x_n\}$ be an iterative sequence generated by (1), and define an iterative sequence $(\tilde{x}_n)_n$ as follows

$$\begin{cases} \tilde{y}_n = (1 - \beta_n) \tilde{x}_n + \beta_n \tilde{T} \tilde{x}_n \\ \tilde{z}_n = (1 - \gamma_n) \tilde{x}_n + \gamma_n \tilde{y}_n \\ \tilde{x}_{n+1} = (1 - \alpha_n) \tilde{T} \tilde{z}_n + \alpha_n \tilde{T} \tilde{y}_n, \end{cases}$$

for an arbitrary $\tilde{x}_1 \in C$, with real number sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $(0, 1)$, satisfying $s_n = \beta_n(\alpha_n + \gamma_n - \alpha_n \gamma_n) \geq \frac{1}{\lambda - \theta}$ for some $\lambda > \theta$.

If $\lim_{n \rightarrow \infty} \tilde{x}_n = \tilde{p}$, then

$$\|p - \tilde{p}\| \leq \frac{\lambda \varepsilon}{1 - \theta}.$$

Proof. The following inequalities hold true:

$$\begin{aligned}
\|y_n - \tilde{y}_n\| &= \|(1 - \beta_n)x_n + \beta_n T x_n - (1 - \beta_n)\tilde{x}_n - \beta_n \tilde{T} \tilde{x}_n\| \\
&= \|(1 - \beta_n)(x_n - \tilde{x}_n) + \beta_n(T x_n - \tilde{T} \tilde{x}_n)\| \\
&\leq (1 - \beta_n)\|x_n - \tilde{x}_n\| + \beta_n\|T x_n - \tilde{T} \tilde{x}_n\| \\
&\leq (1 - \beta_n)\|x_n - \tilde{x}_n\| + \beta_n[\|T x_n - T \tilde{x}_n\| + \|T \tilde{x}_n - \tilde{T} \tilde{x}_n\|] \\
&\leq (1 - \beta_n)\|x_n - \tilde{x}_n\| + \theta \beta_n\|x_n - \tilde{x}_n\| + \beta_n\|T \tilde{x}_n - \tilde{T} \tilde{x}_n\| \\
&\leq (1 - \beta_n(1 - \theta))\|x_n - \tilde{x}_n\| + \beta_n \varepsilon.
\end{aligned} \tag{14}$$

Using (14), we have

$$\begin{aligned}
\|z_n - \tilde{z}_n\| &= \|(1 - \gamma_n)x_n + \gamma_n y_n - (1 - \gamma_n)\tilde{x}_n - \gamma_n \tilde{y}_n\| \\
&\leq (1 - \gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n\|y_n - \tilde{y}_n\| \\
&\leq (1 - \gamma_n)\|x_n - \tilde{x}_n\| + \gamma_n(1 - \beta_n(1 - \theta))\|x_n - \tilde{x}_n\| + \gamma_n \beta_n \varepsilon \\
&= (1 - \gamma_n \beta_n(1 - \theta))\|x_n - \tilde{x}_n\| + \gamma_n \beta_n \varepsilon.
\end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned}
\|x_{n+1} - \tilde{x}_{n+1}\| &= \|(1 - \alpha_n)T z_n + \alpha_n T y_n - (1 - \alpha_n)\tilde{T} \tilde{z}_n - \alpha_n \tilde{T} \tilde{y}_n\| \\
&\leq (1 - \alpha_n)\|T z_n - \tilde{T} \tilde{z}_n\| + \alpha_n\|T y_n - \tilde{T} \tilde{y}_n\| \\
&\leq (1 - \alpha_n)[\|T z_n - T \tilde{z}_n\| + \|T \tilde{z}_n - \tilde{T} \tilde{z}_n\|] + \alpha_n[\|T y_n - T \tilde{y}_n\| + \|T \tilde{y}_n - \tilde{T} \tilde{y}_n\|] \\
&\leq (1 - \alpha_n)[\theta\|z_n - \tilde{z}_n\| + \varepsilon] + \alpha_n[\theta\|y_n - \tilde{y}_n\| + \varepsilon] \\
&= \theta(1 - \alpha_n)\|z_n - \tilde{z}_n\| + \theta \alpha_n\|y_n - \tilde{y}_n\| + \varepsilon.
\end{aligned} \tag{16}$$

Using (14), (15), and (16), we find

$$\begin{aligned}
\|x_{n+1} - \tilde{x}_{n+1}\| &\leq \theta(1 - \alpha_n)[(1 - \gamma_n \beta_n(1 - \theta))\|x_n - \tilde{x}_n\| + \gamma_n \beta_n \varepsilon] \\
&\quad + \theta \alpha_n[(1 - \beta_n(1 - \theta))\|x_n - \tilde{x}_n\| + \beta_n \varepsilon] + \varepsilon \\
&= \theta(1 - \alpha_n)(1 - \gamma_n \beta_n(1 - \theta))\|x_n - \tilde{x}_n\| + \theta \alpha_n(1 - \beta_n(1 - \theta))\|x_n - \tilde{x}_n\| \\
&\quad + \theta(1 - \alpha_n)\gamma_n \beta_n \varepsilon + \theta \alpha_n \beta_n \varepsilon + \varepsilon \\
&= \theta[1 - \beta_n(\alpha_n + \gamma_n - \alpha_n \gamma_n)(1 - \theta)]\|x_n - \tilde{x}_n\| + \varepsilon[1 + \theta \beta_n(\alpha_n + \gamma_n - \alpha_n \gamma_n)] \\
&= \theta[1 - s_n(1 - \theta)]\|x_n - \tilde{x}_n\| + \varepsilon[1 + \theta s_n].
\end{aligned} \tag{17}$$

Under the hypothesis that $s_n \geq \frac{1}{\lambda - \theta}$, for some $\lambda > \theta$ and the fact that $\theta < 1$, the inequality (17) becomes

$$\|x_{n+1} - \tilde{x}_{n+1}\| < [1 - s_n(1 - \theta)]\|x_n - \tilde{x}_n\| + \varepsilon \lambda s_n. \tag{18}$$

Let us denote $\psi_n = \|x_n - \tilde{x}_n\|$, $\phi_n = s_n(1 - \theta)$, and $\varphi_n = \frac{\lambda \varepsilon}{1 - \theta}$ in (18). Since all the conditions of Lemma 4.4 are satisfied, it follows that

$$0 \leq \limsup_{n \rightarrow \infty} \|x_n - \tilde{x}_n\| \leq \limsup_{n \rightarrow \infty} \frac{\lambda \varepsilon}{1 - \theta} = \frac{\lambda \varepsilon}{1 - \theta}.$$

By assuming that $\lim_{n \rightarrow \infty} x_n = p$ and $\lim_{n \rightarrow \infty} \tilde{x}_n = \tilde{p}$, we have

$$\|p - \tilde{p}\| \leq \|p - x_n\| + \|x_n - \tilde{x}_n\| + \|\tilde{x}_n - \tilde{p}\|,$$

and, by taking $\limsup_{n \rightarrow \infty}$, we find $\|p - \tilde{p}\| \leq \frac{\lambda \varepsilon}{1 - \theta}$, hence the proof. $\square$

Remark 2. Let us note the fact that Theorem 6 provides two control parameters for the deviation from the solution: the maximum admissible error ε of the approximate operator $\tilde{T}$ and λ , which is a control element for the iteration coefficients. More precisely, we note that s_n and λ are inversely proportional, while the deviations

$\|p - \tilde{p}\|$ and λ are directly proportional. By rewriting $s_n = \beta_n[1 - (1 - \alpha_n)(1 - \gamma_n)]$, we notice that we can improve the performance of the algorithm (i.e., the distance $\|p - \tilde{p}\|$ should be as small as possible) by taking α_n , β_n , and γ_n close enough to 1.

7. Conclusions

This paper analyzes the S_n iteration procedure in the context of mappings that satisfy a very recently introduced generalized nonexpansiveness condition known as property (D). One direction of the approach refers to the stability and data dependence of the iteration procedure. An interesting outcome highlights the existence of two control parameters that could help us adjust the performance of the algorithms resulting from this procedure. On the other hand, a more detailed analysis is performed in connection with condition (D); through examples, it is proved that the new class of mappings is not included in the Suzuki class, although the set of common elements is consistent, strictly containing the nonexpansive operators. Finally, the iteration procedure combined with the new type of non-expansive operator is subject to a convergence analysis, resulting in several outcomes related to weak or strong convergence or fixed points' existence.

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