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Co-Compact Separation Axioms and Slight Co-Continuity

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Received: 10 August 2020; Accepted: 21 September 2020; Published: 29 September 2020

Abstract: Via co-compact open sets we introduce co-T_2 as a new topological property. We show that this class of topological spaces strictly contains the class of Hausdorff topological spaces. Using compact sets, we characterize co-T_2 which forms a symmetry. We show that co-T_2 propoerty is preserved by continuous closed injective functions. We show that a closed subspace of a co-T_2 topological space is co-T_2. We introduce co-regularity as a weaker form of regularity, s-regularity as a stronger form of regularity and co-normality as a weaker form of normality. We obtain several characterizations, implications, and examples regarding co-regularity, s-regularity and co-normality. Moreover, we give several preservation theorems under slightly coc-continuous functions.

Keywords: Coc-open; T_2 property; regularity; normality

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1. Introduction and Preliminaries

Defining a new type of generalized open sets and utilizing it to define new topological concepts is now a very hot research topic [1–11]. As a new type of generalized open sets, Al-Ghour and Samarah in [12] defined coc-open sets as follows: A subset A of a topological space (X, τ) is called coc-open set if A is a union of sets of the form $V - C$, where $V \in \tau$ and C is a compact subset of X. Authors in [12] proved that the family of all coc-open sets of a topological space (X, τ) forms a topology on X finer than τ, and via this class of sets they obtained a decomposition theorem of continuity. Research via coc-open sets was and still a hot area of research, indeed authors in [13] introduced and coc-closed, coc-open functions and coccompact spaces, authors in [14] defined new types of connectedness, in [15] authors have studied s-coc-separation axioms, coc-convergence as a new type of convergence for nets and filters was introduced in [16], new types of dimension theory were introduced in [17], in [18] new classes of functions were introduced, and in [19] the suthors generalized co-compact open sets. In this paper, we use coc-open sets to define and investigate new separation axioms and new class of functions.

The material of this paper lies in four chapters. In section two, we define and investigate co-T_2 as a new topological property and as a generalization of T_2 topological property. In section three, we introduce and investigate co-regularity as a weaker form of regularity, s-regularity as a stronger form of regularity and co-normality as a weaker form of normality. In section four, we introduce and investigate slightly coc-continuous functions.

In this paper, for a subset A of a topological space (X, τ), $cl_\tau(A)$ will denote the closure of A is (X, τ) and $\tau_{|A}$ will denote the relative topology of (X, τ) on A.

The following definition and theorem will be used in this sequel:

Definition 1. *[12] A subset A of a topological space (X, τ) is called co-compact open set (notation: coc-open) if for every $x \in A$, there exists an open set $U \subseteq X$ and a compact subset K of (X, τ) such that $x \in U - K \subseteq A$.*

The complement of a coc-open subset is called coc-closed. The family of all coc-open subsets of a topological space (X, τ) will be denoted by τ^k.

Theorem 1. *[12] Let (X, τ) be a topological space. Then*

(a) *The collection τ^k forms a topology on X with $\tau \subseteq \tau^k$.*
(b) *$\{U - K : U \in \tau$ and K is compact in $(X, \tau)\}$ forms a base for τ^k.*

2. Co-T_2 Topological Spaces

Let us start by the following main definition:

Definition 2. *A topological space (X, τ) is called co-T_2 if and only if for all $x, y \in X$ with $x \neq y$ there exist $U, V \in \tau^k$ such that $\{U, V\} \cap \tau \neq \emptyset$, $x \in U, y \in V$ and $U \cap V = \emptyset$.*

Theorem 2. *Every T_2 topological space is co-T_2.*

The following example shows that the converse of Theorem 2 is not true in general:

Example 1. *Let $X = \mathbb{R}$, $\mathcal{B} = \{\{x\} : x \neq 0\} \cup \{\mathbb{R}\}$, and τ is the topology on X that having $\mathcal{B}$ as a base. Then*

(a) *(X, τ) is co-T_2.*
(b) *(X, τ) is not T_2.*

Proof. (a) Let $x, y \in \mathbb{R}$ with $x \neq y$.

Case 1. $x = 0$. Let $U = \mathbb{R} - \{y\}$ and $V = \{y\}$. Then $U, V \in \tau^k$, $\{U, V\} \cap \tau = \{V\} \neq \emptyset, x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Case 2. $y = 0$. Let $U = \{x\}$ and $V = \mathbb{R} - \{x\}$. Then $U, V \in \tau^k$, $\{U, V\} \cap \tau = \{U\} \neq \emptyset, x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Case 3. $x \neq 0$ and $y \neq 0$. Let $U = \{x\}$ and $V = \{y\}$. Then $U, V \in \tau^k$, $\{U, V\} \cap \tau = \{U, V\} \neq \emptyset$, $x \in U, y \in V$ and $U \cap V = \emptyset$.

(b) Suppose to the contrary that (X, τ) is T_2. Then there are $U, V \in \tau$ such that $0 \in U, 1 \in V$ and $U \cap V = \emptyset$. Since $0 \in U \in \tau$, then $U = \mathbb{R}$ and $1 \in U$. Thus, $U \cap V \neq \emptyset$, a contradiction. □

Theorem 3. *[12] If (X, τ) is a hereditarily compact topological space, then τ^k is the discrete topology on X.*

Theorem 4. *If (X, τ) is a hereditarily compact topological space, then $\left(X, \tau^k\right)$ is T_2.*

Proof. Let (X, τ) be a hereditarily compact topological space. Then by Theorem 3, τ^k is the discrete topology on X. Thus, $\left(X, \tau^k\right)$ is T_2. □

Theorem 5. *If (X, τ) is hereditarily compact and T_1 topological space, then (X, τ) is co-T_2.*

Proof. Let $x, y \in X$ with $x \neq y$. Take $U = \{x\} \in \tau^k$ and $V = X - \{x\} \in \tau$. Since (X, τ) is hereditarily compact, then by Theorem 4, $U \in \tau^k$. Since (X, τ) is T_1, then $V \in \tau$. Thus, we have $U, V \in \tau^k$, $\{U, V\} \cap \tau = \{U\} \neq \emptyset, x \in U, y \in V$ and $U \cap V = \emptyset$. □

Theorem 6. *If X is any non-empty set and τ is the cofinite topology on X, then (X, τ) is hereditarily compact.*

Proof. If A is a non-empty subset of X, then $\tau_{|A}$ is the cofinite topology on A and hence A is a compact subset of (X, τ). Therefore, (X, τ) is hereditarily compact space. □

Corollary 1. *If X is any non-empty set and τ is the cofinite topology on X, then τ^k is the discrete topology on X.*

Proof. Theorems 3 and 6. $\square$

Corollary 2. *If X is any non-empty set and τ is the cofinite topology on X, then (X, τ) is co-T_2.*

Proof. Theorems 3 and 6 and the fact that (X, τ) is T_1. $\square$

Corollary 3. *If X is any infinite set and τ is the cofinite topology on X, then (X, τ) is an example on a co-T_2 topological space that is not T_2.*

Theorem 7. *If (X, τ) is a co-T_2 topological space, then (X, τ^k) is T_2.*

Proof. Suppose that (X, τ) is co-T_2 and let $x, y \in X$ with $x \neq y$. Since (X, τ) is co-T_2, there exist $U, V \in \tau^k$, such that $\{U, V\} \cap \tau = \{U\} \neq \emptyset, x \in U, y \in V$ and $U \cap V = \emptyset$. Therefore, (X, τ^k) is T_2. $\square$

The following example shows that Theorem 7 is not reversible:

Example 2. *Let $X = \mathbb{N}$ and τ be the topology on $\mathbb{N}$ having $\{\{2n-1, 2n\} : n \in \mathbb{N}\}$ as a base. It is proved in example 2.16 of [2] that τ^k is the discrete topology on $\mathbb{N}$ and so (X, τ^k) is T_2. If (X, τ) is co-T_2, then there are $U, V \in \tau^k$ such that $\{U, V\} \cap \tau \neq \emptyset, 1 \in U, 2 \in V$ and $U \cap V = \emptyset$. Without loss of generality, we can assume that $U \in \tau$. Choose a basic element B such that $1 \in B \subseteq U$. Then $B = \{1, 2\}$ and so $2 \in U \cap V$. Therefore, (X, τ) is not co-T_2.*

Theorem 8. *A topological space (X, τ) is co-T_2 if and only if for all $x, y \in X$ with $x \neq y$ there exist $U, V \in \tau$ and a compact set K of (X, τ) such that ($x \in U - K, y \in V$ and $(U - K) \cap V = \emptyset$) or ($x \in U, y \in V - K$ and $U \cap (V - K) = \emptyset$).*

Proof. $\Longrightarrow$) Assume that (X, τ) is co-T_2 and let $x, y \in X$ with $x \neq y$. Then there exist $U, V \in \tau^k$ such that $\{U, V\} \cap \tau \neq \emptyset, x \in U, y \in V$ and $U \cap V = \emptyset$. Without loss of generality we may assume that $U \in \tau$. Since $V \in \tau^k$, there are $W \in \tau$ and a compact set K of (X, τ) such that $y \in W - K \subseteq V$. Since $U \cap V = \emptyset$, then $U \cap (W - K) = \emptyset$.

$\Longleftarrow$) Let $x, y \in X$ with $x \neq y$. Then there exist $U, V \in \tau$ and a compact set K of (X, τ) such that ($x \in U - K, y \in V$ and $(U - K) \cap V = \emptyset$) or ($x \in U, y \in V - K$ and $U \cap (V - K) = \emptyset$). If ($x \in U - K, y \in V$ and $(U - K) \cap V = \emptyset$), then we have $U - K, V \in \tau^k$, $\{U - K, V\} \cap \tau = \{V\} \neq \emptyset, x \in U - K, y \in V$ and $(U - K) \cap V = \emptyset$. If $x \in U, y \in V - K$ and $U \cap (V - K) = \emptyset$, then $U, V - K \in \tau^k, \{U, V - K\} \cap \tau = \{U\} \neq \emptyset, x \in U, y \in V - K$ and $U \cap (V - K) = \emptyset$. Therefore, (X, τ) is co-T_2.

$\square$

Theorem 9. *A topological space (X, τ) is co-T_2 if and only if for all $x, y \in X$ with $x \neq y$ there exist $U \in \tau$ and a compact set K of (X, τ) such that ($x \in U - K$ and $y \notin cl_\tau(U - K)$) or ($x \notin cl_\tau(U - K)$ and $y \in U - K$).*

Proof. $\Longrightarrow$) Suppose that (X, τ) is co-T_2 and let $x, y \in X$ with $x \neq y$. Since (X, τ) is co-T_2, by Theorem 8 there are $U, V \in \tau$ and a compact set K of (X, τ) such that ($x \in U - K, y \in V$ and $(U - K) \cap V = \emptyset$) or ($x \in U, y \in V - K$ and $U \cap (V - K) = \emptyset$).

Case 1. $x \in U - K, y \in V$ and $(U - K) \cap V = \emptyset$. We have $U - K \subseteq X - V$ and so

$$cl_\tau(U - K) \subseteq cl_\tau(X - V) = X - V.$$

Since $y \notin X - V$, then $y \notin cl_\tau(U - K)$.

Case 2. $x \in U, y \in V - K$ and $U \cap (V - K) = \emptyset$. We have $V - K \subseteq X - U$ and so

$$cl_\tau(V - K) \subseteq cl_\tau(X - U) = X - U.$$

Since $x \notin X - U$, then $x \notin cl_\tau(V - K)$.

$\Longleftarrow$) Let $x, y \in X$ with $x \neq y$. By assumption there exist $U \in \tau$ and a compact set K of (X, τ) such that ($x \in U - K$ and $y \notin cl_\tau(U - K)$) or ($x \notin cl_\tau(U - K)$ and $y \in U - K$).

Case 1. $x \in U - K$ and $y \notin cl_\tau(U - K)$. Set $V = X - cl_\tau(U - K)$. Then $V \in \tau$ and $y \in V$.

Case 2. $x \notin cl_\tau(U - K)$ and $y \in U - K$. Set $V = X - cl_\tau(U - K)$. Then $V \in \tau$ and $x \in V$.

Therefore, by Theorem 8 we conclude that (X, τ) is co-T_2. □

Lemma 1. *[20] Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a closed function in which its fiber $f^{-1}(\{y\})$ is a compact subset of (X, τ) for all $y \in Y$. If B is a compact subset of (Y, σ), then $f^{-1}(B)$ is a compact subset of (X, τ).*

Theorem 10. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be continuous, closed and injective function where (Y, σ) is a co-T_2 topological space, then (X, τ) is co-T_2.*

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is injective, $f(x_1) \neq f(x_2)$. Since (Y, σ) is a co-T_2, then by Theorem 8 there exist $V_1, V_2 \in \sigma$ and compact set K of (Y, σ) such that ($f(x_1) \in V_1 - K, f(x_2) \in V_2$ and $(V_1 - K) \cap V_2 = \emptyset$) or ($f(x_1) \in V_1, f(x_2) \in V_2 - K$ and $V_1 \cap (V_2 - K) = \emptyset$). Without loss of generality we can assume that $f(x_1) \in V_1 - K, f(x_2) \in V_2$ and $(V_1 - K) \cap V_2 = \emptyset$. Since f is continuous, $f^{-1}(V_1), f^{-1}(V_2) \in \tau$. Since f is injective, then its fibers are compact. Hence, by Lemma 1, $f^{-1}(K)$ is compact in (X, τ). We have $x_1 \in f^{-1}(V_1 - K) = f^{-1}(V_1) - f^{-1}(K), x_2 \in f^{-1}(V_2)$ and

$$\left(f^{-1}(V_1) - f^{-1}(K)\right) \cap f^{-1}(V_2) = f^{-1}(V_1 - K) \cap f^{-1}(V_2) = f^{-1}((V_1 - K) \cap V_2) = f^{-1}(\emptyset) = \emptyset.$$

This ends the proof according to Theorem 8. □

Corollary 4. *Being 'co-T_2' is a topological property.*

Lemma 2. *[12] Let (X, τ) be a topological space and A be a closed subset of (X, τ). Then $(\tau |_A)^k = \tau^k |_A$.*

Theorem 11. *If (X, τ) is a co-T_2 topological space and A is a closed subset of (X, τ), then $(A, \tau |_A)$ is co-T_2.*

Proof. Let $x, y \in A$ with $x \neq y$. Then $x, y \in X$ with $x \neq y$. Since (X, τ) is *co-T_2*, then there exist $U, V \in \tau^k$ such that $\{U, V\} \cap \tau \neq \emptyset$, $x \in U, y \in V$ and $U \cap V = \emptyset$. Set $U_1 = U \cap A$ and $V_1 = V \cap A$. Then $U_1, V_1 \in \tau^k |_A$and by Lemma 2 $U_1, V_1 \in (\tau |_A)^k$. Since $\{U, V\} \cap \tau \neq \emptyset$, $\{U_1, V_1\} \cap (\tau |_A) \neq \emptyset$. Moreover, $x \in U_1, y \in V_1$ and

$$U_1 \cap V_1 = (U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \emptyset \cap A = \emptyset.$$

□

3. Co-Regulerity and Co-Normality

We start by the following main definition:

Definition 3. *A topological space (X, τ) is said to be co-regular if for each closed set F of (X, τ) and each $x \in X - F$ there exist $U, V \in \tau^k$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$.*

Theorem 12. *If (X, τ) is a topological space such that τ^k is the discrete topology on X, then (X, τ) is co-regular.*

Proof. Let F be a closed set in (X,τ) and let $x \in X - F$. Let $U = \{x\}$ and $V = F$. Then $U, V \in \tau^k$, $x \in U, F \subseteq V$ and $U \cap V = \emptyset$. Therefore, (X,τ) is co-regular. $\square$

Corollary 5. *If (X,τ) is a hereditarily compact topological space, then $\left(X,\tau^k\right)$ is co-regular.*

Proof. Theorems 3 and 12 $\square$

Corollary 6. *If X is any non-empty set and τ is the cofinite topology on X, then (X,τ) is co-regular.*

Proof. Corollary 1 and Theorem 12 $\square$

Theorem 13. *Every regular topological space is co-reguler.*

Proof. Let (X,τ) be a regular topological space. Let F be a closed set in (X,τ) and let $x \in X - F$. Since (X,τ) is regular, there exist $U, V \in \tau$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$. Since $\tau \subseteq \tau^k$, it follows that (X,τ) is co-reguler. $\square$

Remark 1. *The converse of Theorem 13 is not true in general: By Corollary 6, $(\mathbb{R},\tau)$ where τ is the cofinite topology on $\mathbb{R}$ is co-regular. On the other hand, it is well known that $(\mathbb{R},\tau)$ is not regular.*

Theorem 14. *Let (X,τ) be a topological space. If (X,τ^k) is regular, then (X,τ) is co-reguler.*

Proof. Let F be a closed set in (X,τ) and let $x \in X - F$. Since F is closed in (X,τ) and $\tau \subseteq \tau^k$, then F is closed in (X,τ^k). Since (X,τ^k) is regular, there exist $U, V \in \tau^k$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$. Therefore, (X,τ) is co-reguler. $\square$

Example 3. *Let $X = \mathbb{R}$, $\tau = \{X\} \cup \{U \subseteq X : 1 \notin U\}$. Then*

(a) (X,τ) *is co-reguler.*
(b) (X,τ^k) *is regular.*
(c) (X,τ) *is not regular.*
(d) (X,τ) *is co-T_2.*

Proof. (a) Let F be closed in (X,τ) and let $x \in X - F$.

Case 1. $x \neq 1$. Take $U = \{x\}$ and $V = \mathbb{R} - \{x\}$. Then $U \in \tau \subseteq \tau^k, V \in \tau^k, x \in U, F \subseteq V$ and $U \cap V = \emptyset$.

Case 2. $x = 1$. Then $F = \emptyset$. Take $U = \mathbb{R}$ and $V = \emptyset$. Then $U, V \in \tau^k, x \in U, F \subseteq V$ and $U \cap V = \emptyset$. It follows that (X,τ) is co-reguler.

(b) Let F be closed in (X,τ^k) and let $x \in X - F$. By (a), we only need to discuss the case when F is not closed in (X,τ). It is not difficult to see that $\tau^k = \tau \cup \{U \subseteq X : 1 \in U$ and $X - U$ is finite$\}$. Thus, we must have $x = 1$ and F is finite. Take $U = \mathbb{R} - F$ and $V = F$. Then $U \in \tau^k$, $V \in \tau \subseteq \tau^k$, $\{U, V\} \cap \tau = \{V\} \neq \emptyset, x \in U, y \in V$ and $U \cap V = \emptyset$. Therefore, (X,τ^k) is regular.

(c) Suppose to the contrary that (X,τ) is regular. Take $x = 3$ and $F = (0,2)$. Then F is closed in (X,τ) with $3 \in \mathbb{R} - F$ and by regularity there are $U, V \in \tau$ such that $3 \in U, F \subseteq V$ and $U \cap V = \emptyset$. Since $1 \in F$, then $1 \in V$ and so $V = \mathbb{R}$. Thus, $3 \in U \cap V$, a contradiction.

(d) Let $x, y \in X$ with $x \neq y$. Without loss of generality we may assume that there are only two cases:

Case 1. $x \neq 1$ and $y \neq 1$. Take $U = \{x\}$ and $V = \{y\}$. Then $U, V \in \tau \subseteq \tau^k$, $\{U, V\} \cap \tau = \{U, V\} \neq \emptyset, x \in U, y \in V$ and $U \cap V = \emptyset$.

Case 2. $x = 1$ and $y \neq 1$. Take $U = \mathbb{R} - \{y\}$ and $V = \{y\}$. Then $U \in \tau^k$, $V \in \tau, \{U, V\} \cap \tau = \{V\} \neq \emptyset, x \in U, y \in V$ and $U \cap V = \emptyset$.

It follows that (X,τ) is co-T_2. $\square$

Theorem 15. *A topological space (X, τ) is co-regular if and only if for every $U \in \tau$ with $x \in U$, there exists $V \in \tau^k$ such that $x \in V \subseteq cl_{\tau^k}(V) \subseteq U$.*

Proof. $\Longrightarrow$) Suppose that (X, τ) is co-regular and let $U \in \tau$ with $x \in U$. Then we have $X - U$ is closed in (X, τ) with $x \in X - (X - U)$. So by co-regularity of (X, τ) there are $V, W \in \tau^k$, such that $x \in V, X - U \subseteq W$ and $V \cap W = \emptyset$. Now $x \in V$ with

$$V \subseteq cl_{\tau^k}(V) \subseteq cl_{\tau^k}(X - W) = X - W \subseteq U.$$

$\Longleftarrow$) Let F be closed in (X, τ) and $x \in X - F$. Then $X - F \in \tau$ and by assumption there exists $U \in \tau^k$ such that $x \in U \subseteq cl_{\tau^k}(U) \subseteq X - F$. Let $V = X - cl_{\tau^k}(U)$. Then we have $U, V \in \tau^k, x \in U, F \subseteq V$ and $U \cap V = \emptyset$.
□

Theorem 16. *If (X, τ) is a co-regular topological space, then for every closed set F in (X, τ) we have*

$$F = \cap \left\{ U \in \tau^k : F \subseteq U \right\}.$$

Proof. Suppose that (X, τ) is co-regular. To see that $\cap \left\{ U \in \tau^k : F \subseteq U \right\} \subseteq F$, let $x \in X - F$. Then by co-regularity of (X, τ), there are $U, V \in \tau^k$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$. Thus we have $F \subseteq V, V \in \tau^k$ and $x \notin V$ which implies that $x \notin \cap \left\{ U \in \tau^k : F \subseteq U \right\}$. Conversely, it is obvious that $F \subseteq \cap \left\{ U \in \tau^k : F \subseteq U \right\}$. □

Theorem 17. *A closed subspace of a co-regular topological space is co-reguler.*

Proof. Let (X, τ) be a co-regular topological space and let A be a non-empty closed set of (X, τ). Let $x \in A$ and let B be a closed set in $(A, \tau|_A)$ with $x \notin B$. Since A is closed in (X, τ), then B is closed set in (X, τ). Since (X, τ) is co-regular, then there are $U, V \in \tau^k$ such that $x \in U, B \subseteq V$ and $U \cap V = \emptyset$. We have $U \cap A, V \cap A \in \tau^k|_A$ and by Lemma 2, $U \cap A, V \cap A \in (\tau|_A)^k$. Now we have $x \in U \cap A$, $B \subseteq V \cap A$ and

$$(U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \emptyset \cap A = \emptyset.$$

It follows that $(A, \tau|_A)$ is co-reguler. □

Definition 4. *A topological space (X,τ) is called s-regular if for each closed set F in (X, τ^k) and $x \in X - F$, there exist $U, V \in \tau$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$.*

Theorem 18. *Every s-regular topological space is regular.*

The following example shows that the converse of Theorem 18 is not true in general:

Example 4. *Let X be a set which contains at least three points and let $\mathcal{B}$ be a partition on X in which for some $B \in \mathcal{B}$, B has at least two points. Let τ be the topology on X having $\mathcal{B}$ is a base. Since (X, τ) is zero-dimensional, it is regular. Choose $x, y \in B$ such that $x \neq y$ and let $F = \{y\}$. Then F is closed in (X, τ^k) with $x \in X - F$. If $U, V \in \tau$ such that $x \in U$ and $F \subseteq V$, then $B \subseteq U \cap V$. Therefore, (X,τ) is not s-regular.*

Lemma 3. *[12] If (X, τ) is a T_2 topological space, then $\tau = \tau^k$.*

Theorem 19. *Let (X, τ) be a T_2 topological space. Then (X, τ) is s-regular if and only if (X, τ) is regular.*

Proof. Theorem 18 and Lemma 3 □

Corollary 7. *Every T_3 topological space is s-regular.*

Definition 5. *A topological space (X,τ) is said to be co-normal if for each disjoint closed sets A, B of (X,τ) there exist $U, V \in \tau^k$ such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.*

Theorem 20. *Every normal topological space is co-normal.*

Proof. Let (X,τ) be a normal topological space. Let A and B be two disjoint closed sets in (X,τ). Since (X,τ) is normal space then there exist $U, V \in \tau \subseteq \tau^k$ such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$. It follows that (X,τ) is co-normal. □

Theorem 21. *Let (X,τ) be a topological space. If (X,τ^k) is normal, then (X,τ) is co-normal.*

Proof. Let A, B be two disjoint closed sets in (X,τ). Since $\tau \subseteq \tau^k$, then A, B are closed sets in (X,τ^k). Since (X,τ^k) is normal, then there exist $U, V \in \tau^k$ such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$. Therefore, (X,τ) is co-normal. □

Corollary 8. *If (X,τ) is a topological space such that τ^k is the discrete topology on X, then (X,τ) is co-normal.*

The following example shows that converse of Theorem 20 is not true in general:

Example 5. *Let $X = \mathbb{R}$ and let τ be the cofinite topology. Then*

(a) (X,τ) *is co-normal.*
(b) (X,τ) *is not normal.*

Proof. (a) By Corollary 2, τ^k is the discrete topology on X. So by Corollary 8, (X,τ) is co-normal.
(b) This is well known in general topology.
□

Theorem 22. *A topological space (X,τ) is co-normal if and only if for every open set U and any closed set A in (X,τ) with $A \subseteq U$, there exists $V \in \tau^k$ such that $A \subseteq V \subseteq cl_{\tau^k}(V) \subseteq U$*

Proof. $\Longrightarrow$) Suppose that (X,τ) is a co-normal topological space. Let $U \in \tau$ and let A be closed set in (X,τ) with $A \subseteq U$. We have A and $X - U$ are disjoint closed sets in (X,τ). Thus, since (X,τ) is co-normal, then there exist $V, W \in \tau^k$ such that $A \subseteq V, X - U \subseteq W$ and $V \cap W = \emptyset$. Now

$$A \subseteq V \subseteq cl_{\tau^k}(V) \subseteq X - W \subseteq U.$$

$\Longleftarrow$) Let A and B be two disjoint closed sets in (X,τ). Put $U = X - B$. Then $U \in \tau$ with $A \subseteq U$. By assumption there exists $V \in \tau^k$ such that $A \subseteq V \subseteq cl_{\tau^k}(V) \subseteq U$. Put $W = X - cl_{\tau^k}(V)$. Then $W \in \tau^k$, $B \subseteq W$ and

$$V \cap W = V \cap (X - cl_{\tau^k}(V)) \subseteq V \cap (X - V) = \emptyset.$$

Therefore, (X,τ) is co-normal. □

Corollary 9. *A topological space (X,τ) is co-normal if for any pair of disjoint closed sets A, B, there exists $V \in \tau^k$ such that $A \subseteq V$ and $cl_{\tau^k}(V) \cap B = \emptyset$.*

Proof. Let A and B be two disjoint closed sets of (X,τ). Put $U = X - B$. Then $U \in \tau$ with $A \subseteq U$. By Theorem 22, there exists $V \in \tau^k$ such that $A \subseteq V \subseteq cl_{\tau^k}(V) \subseteq U$. Thus,

$$cl_{\tau^k}(V) \cap B \subseteq U \cap B = (X - B) \cap B = \emptyset.$$

□

Theorem 23. *A closed subspace of a co-normal topological space is co-normal.*

Proof. Let (X,τ) be a co-normal topological space and let A be a non-empty closed set of (X,τ). Let B,C be two disjoint closed sets in $(A,\tau|_A)$. Since A is closed in (X,τ), then B and C are closed in (X,τ). Since (X,τ) is co-normal, then there exist $U,V\in\tau^k$ such that $B\subseteq U, C\subseteq V$ with $U\cap V=\emptyset$. Let $U_1=U\cap A$ and $V_1=V\cap A$. Then $U_1,V_1\in\tau^k|_A$. Since A is closed, then by Lemma 2, $U_1,V_1\in(\tau|_A)^k$. Also, we have $B\subseteq U_1, C\subseteq V_1$ and

$$U_1\cap V_1=(U\cap A)\cap(V\cap A)=(U\cap V)\cap A=\emptyset.$$

□

This shows that $(A,\tau|_A)$ is co-normal.

Theorem 24. *A topological space (X,τ) is co-normal if and only if for every pair of open sets U and V in (X,τ) such that $U\cup V=X$, there exist coc-closed sets A and B in (X,τ) such that $A\subseteq U, B\subseteq V$ and $A\cup B=X$.*

Proof. $\Longrightarrow$) Suppose that (X,τ) is a co-normal topological space. Let U and V be two open sets in (X,τ) with $U\cup V=X$. Then $X-U$ and $X-V$ are disjoint closed sets of (X,τ), and since (X,τ) is co-normal, then there exist $U_1,V_1\in\tau^k$ such that $X-U\subseteq U_1$, $X-V\subseteq V_1$ and $U_1\cap V_1=\emptyset$. Let $A=X-U_1$ and $B=X-V_1$. Then A and B are coc-closed sets of (X,τ) with $A\subseteq U$, $B\subseteq V$. Moreover,

$$A\cup B=(X-U_1)\cup(X-V_1)=X-(U_1\cap V_1)=X-\emptyset=X.$$

$\Longleftarrow$) Let A and B be two disjoint closed sets in (X,τ). Set $U=X-A$ and $V=X-B$. Then U and V are open sets in (X,τ) such that $U\cup V=X$. Thus by assumption, there exist coc-closed sets A_1 and B_1 in (X,τ) such that $A_1\subseteq U, B_1\subseteq V$ and $A_1\cup B_1=X$. Let $U_1=X-A_1$ and $V_1=X-B_1$. Then U_1 and V_1 are coc-open sets in (X,τ), $A\subseteq U_1, B\subseteq V_1$ and

$$U_1\cap V_1=(X-A_1)\cap(X-B_1)=X-(A_1\cup B_1)=X-X=\emptyset.$$

□

Theorem 25. *A T_1 co-normal topological space is co-reguler.*

Proof. Let (X,τ) be T_1 and co-normal. Let $x\in X$, and F be a closed set in (X,τ) such that $x\notin F$. Since (X,τ) is T_1, then $\{x\}$ is closed. Since (X,τ) is co-normal and $\{x\}\cap F=\emptyset$, then there exist $U,V\in\tau^k$ such that $x\in\{x\}\subseteq U, F\subseteq V$ and $U\cap V=\emptyset$. □

4. Slightly Coc-Continuous Functions

Definition 6. *[21] A subset A of a topological space (X,τ) is said to be coc-clopen if both A and $X-A$ are coc-open.*

Example 6. *Let $X=\mathbb{R}$ and $\tau=\{X\}\cup\{U\subseteq X:1\notin U\}$. It is known that $\tau^k=\tau\cup\{U\subseteq X:1\in U \text{ and } X-U \text{ is finite}\}$ (see [12]). Then the set of coc-clopen sets is $\{U\subseteq X:1\in U \text{ and } X-U \text{ is finite}\}$.*

Definition 7. *[22] A function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is said to be slightly continuous if for each $x\in X$, and for each clopen set V in (Y,σ) containing $f(x)$, there exists an open set U in (X,τ) containing x such that $f(U)\subseteq V$.*

Definition 8. *A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be slightly coc-continuous if for each $x \in X$, and for each clopen set V in (Y, σ) containing $f(x)$, there exists a coc-open set U of (X, τ) containing x such that $f(U) \subseteq V$.*

Theorem 26. *Every slightly continuous function is slightly coc-continuous.*

Proof. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a slightly continuous function. Let V be a clopen set in (Y, σ) such that $f(x) \in V$. Since f is slightly continuous, then there exists $U \in \tau \subseteq \tau^k$ containing x such that $f(U) \subseteq V$. This shows that f is slightly coc-continuous. □

Theorem 27. *[22] Let (X, τ) and (Y, σ) be topological spaces. The following statements are equivalent for a function $f : (X, \tau) \longrightarrow (Y, \sigma)$:*

(a) *f is slightly continuous.*
(b) *For every clopen set $V \subseteq Y$, $f^{-1}(V)$ is open.*
(c) *For every clopen set $V \subseteq Y$, $f^{-1}(V)$ is closed.*
(d) *For every clopen set $V \subseteq Y$, $f^{-1}(V)$ is clopen.*

Theorem 28. *Let (X, τ) and (Y, σ) be topological spaces. The following statements are equivalent for a function $f : (X, \tau) \longrightarrow (Y, \sigma)$:*

(a) *f is slightly coc-continuous.*
(b) *For every clopen set $V \subseteq Y$, $f^{-1}(V)$ is coc-open.*
(c) *For every clopen set $V \subseteq Y$, $f^{-1}(V)$ is coc-closed.*
(d) *For every clopen set $V \subseteq Y$, $f^{-1}(V)$ is coc-clopen.*

Proof.

(a) $\Longrightarrow$ (b): Let V be a clopen set in (Y, σ). Since f is slightly coc-continuous, then for every $x \in f^{-1}(V)$ there exists $U_x \in \tau^k$ containing x, such that $f(U_x) \subseteq V$. It is easy to check that $f^{-1}(V) = \cup \{U_x : x \in f^{-1}(V)\}$. Hence $f^{-1}(V)$ is coc-open.

(b) $\Longrightarrow$ (c): Let V be a clopen set in (Y, σ). Then $Y - V$ is clopen and by (b) $f^{-1}(Y - V)$ is coc-open. Moreover, we have

$$f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V).$$

It follows that $f^{-1}(V)$ is coc-closed.

(c) $\Longrightarrow$ (d): Let V be a clopen set in (Y, σ). Then $Y - V$ is also clopen and by (c), $f^{-1}(V)$ and $f^{-1}(Y - V) = X - f^{-1}(V)$ are coc-closed. It follows that $f^{-1}(V)$ is coc-clopen.

(d) $\Longrightarrow$ (a): Let $x \in X$ and V be a clopen set in (Y, σ) containing $f(x)$. By (d), $f^{-1}(V)$ is coc-clopen. Take $U = f^{-1}(V)$. Then U is coc-open with $f(U) \subseteq V$. It follows that f is slightly coc-continuous.

□

The converse of Theorem 26 is not true in general as the following two examples show:

Example 7. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $\sigma = \{X, \emptyset, \{c\}\}$. Let $f : (X, \sigma) \to (X, \tau)$ be the identity function. Then*

(a) *f is slightly coc-continuous.*
(b) *f is not slightly continuous.*

Proof. (a) Let U be a clopen set in (X, τ). Since clearly σ^k is the discrete topology, then $f^{-1}(U) = U$ is coc-open in (X, σ). So f is slightly coc-continuous.

(b) Let $U = \{b, c\}$. Then U is clopen in (X, τ) but $f^{-1}(U) = \{b, c\}$ which is not open in (X, σ). Therefore, f is not slightly continuous.
□

Example 8. *Let $\mathbb{R}$ be the real numbers. Take two topologies on $\mathbb{R}$, τ and σ where τ is cofinite topology and σ is discrete topology. Let $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \sigma)$ be an identity function. By Corollary 2, τ^k is the discrete topology and hence, f is slightly coc-continuous. On the other hand, since $\{0\}$ is clopen in $(\mathbb{R}, \sigma)$ but $f^{-1}(\{0\}) = \{0\}$ is not open in $(\mathbb{R}, \tau)$, f is not slightly continuous.*

Theorem 29. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a slightly coc-continuous function and A is a closed set in (X, τ), then the restriction $f_{|A} : (A, \tau_{|A}) \longrightarrow (Y, \sigma)$ is slightly coc-continuous.*

Proof. Let V be a clopen set in (Y, σ). By Lemma 2, $(f_{|A})^{-1}(V) = f^{-1}(V) \cap A$ is coc-open in $(A, \tau_{|A})$. Thus $f_{|A}$ is slightly coc-continuous. □

Definition 9. *A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be*

(a) coc-irresolute [13], if for every coc-open subset U of (Y, σ), $f^{-1}(U)$ is coc-open in (X, τ).
(b) coc-open [13], if for every coc-open subset A of (X, τ), $f(A)$ is coc-open in (Y, σ).
(c) coc-continuous [12], if for each point $x \in X$ and each open subset V in (Y, σ) containing $f(x)$, there exists an coc-open subset U in (X, τ) containing x such that $f(U) \subseteq V$.
(d) weakly coc-continuous, if for each point $x \in X$ and each open subset V in (Y, σ) containing $f(x)$, there exists an coc-open subset U in (X, τ) containing x such that $f(U) \subseteq Cl_\sigma(V)$,
(e) contra coc-continuous, if $f^{-1}(F)$ is coc-open in (X, τ) for each closed set F in (Y, σ).

Theorem 30. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \delta)$ be functions. Then the following properties hold:*

(a) If f is coc-irresolute and g is slightly coc-continuous, then $g \circ f : (X, \tau) \longrightarrow (Z, \delta)$ is slightly coc-continuous.
(b) If f is slightly coc-continuous and g is slightly continuous, then $g \circ f : (X, \tau) \longrightarrow (Z, \delta)$ is slightly coc-continuous.

Proof. (a) Let V be any clopen set in (Z, δ). Since g is slightly coc-continuous, $g^{-1}(V)$ is coc-open set in (Y, σ). Since f is coc-irresolute, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is coc-open. Therefore, $g \circ f$ is slightly coc-continuous.
(b) Let V be any clopen set in (Z, δ). Since g is slightly continuous, then $g^{-1}(V)$ is clopen. Since f is slightly coc-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is coc-open. Therefore, $g \circ f$ is slightly coc-continuous.
□

Corollary 10. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \delta)$ be functions. Then, the following properties hold:*

(a) If f is coc-irresolute and g is coc-continuous, then $g \circ f$ is slightly coc-continuous.
(b) If f is coc-irresolute and g is slightly continuous, then $g \circ f$ is slightly coc-continuous.
(c) If f is coc-continuous and g is slightly continuous, then $g \circ f$ is slightly coc-continuous.
(d) If f is slightly coc-continuous and g is continuous, then $g \circ f$ is slightly coc-continuous.

Theorem 31. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be surjective, coc-irresolute and coc-open and $g : (Y, \sigma) \longrightarrow (Z, \delta)$ be a function. Then $g \circ f : (X, \tau) \longrightarrow (Z, \delta)$ is slightly coc-continuous if and only if g is slightly coc-continuous.*

Proof. $\Longleftarrow$) Let $g \circ f$ be slightly coc-continuous and V be clopen set in (Z, δ). Then $(g \circ f)^{-1}(V)$ is coc-open in (X, τ). Since f is coc-open, then $f((g \circ f)^{-1}(V))$ is coc-open in (Y, σ). Since f is surjective, then $f((g \circ f)^{-1}(V)) = g^{-1}(V)$. It follows that g is slightly coc-continuous.

$\Longrightarrow$) If g is slightly coc-continuous, then by Theorem 30 , $g \circ f$ is slightly coc-continuous.
□

Definition 10. *[21] A topological space (X, τ) is said to be coc-connected if X can not be written as a union of two disjoint non-empty coc-open sets. A topological space (X, τ) is said to be coc-disconnected if it is not coc-connected.*

It is clear that a topological space (X, τ) is coc-connected if and only if (X, τ^k) is connected. Therefore, $(\mathbb{R}, \tau)$ where τ is the co-countable topology on $\mathbb{R}$ is an example of a coc-connected topological space.

Also, it is clear that coc-connected topological spaces are connected, however $(\mathbb{R}, \tau)$ where τ is the cofinite topology on $\mathbb{R}$ is an example of a connected topological space that is coc-disconnected.

Theorem 32. *Let $f : (X, \tau) \to (Y, \sigma)$ be slightly coc-continuous and surjective function. If (X, τ) is coc-connected, then (Y, σ) is connected.*

Proof. Suppose to the contrary that (Y, σ) is disconnected. Then there exist non-empty disjoint open sets U and V such that $Y = U \cup V$. It is clear that U and V are clopen sets in (Y, σ). Since f is slightly coc-continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are coc-open in (X, τ). Also,

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset.$$

and

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(Y) = X.$$

□

Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Therefore, (X, τ) is coc-disconnected space. This is a contradiction.

Corollary 11. *The inverse image of a disconnected topological space under a surjective slightly coc-continuous function is coc-disconnected.*

Recall that a topological space (X, τ) is called extremally disconnected if the closure of each open set is open.

Theorem 33. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is slightly coc-continuous such that (Y, σ) is extremally disconnected, then f is weakly coc-continuous.*

Proof. Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Since (Y, σ) is extremally disconnected, then $Cl_\sigma(V)$ is open and hence clopen. Since f is slightly coc-continuous, then there exists a coc-open set $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq Cl_\sigma(V)$. It follows that f is weakly coc-continuous. □

Recall that a topological space (X, τ) is called locally indiscrete if every open set is closed.

Theorem 34. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is slightly coc-continuous such that (Y, σ) is locally indiscrete, then f is coc-continuous and contra coc-continuous.*

Proof. Let V be an open set of Y. Since (Y,σ) is locally indiscrete, then V is clopen. Since f is slightly coc-continuous, then $f^{-1}(V)$ is coc-clopen set in X. Therefore, f is coc-continuous and contra coc-continuous. □

Recall that a topological space (X,τ) is called zero-dimensional if τ has a base consists of clopen sets.

Theorem 35. *If $f:(X,\tau)\to(Y,\sigma)$ is slightly coc-continuous such that (Y,σ) is zero-dimensional, then f is coc-continuous.*

Proof. Let $x\in X$ and let $V\in\sigma$ such that $f(x)\in\sigma$. Since (Y,σ) is zero-dimensional, there exists a clopen set B such that $f(x)\in B\subseteq V$. Since f is slightly coc-continuous, then there exists a coc-open set W in (X,τ) such that $f(x)\in f(W)\subseteq U\subseteq V$. Therefore, f is coc-continuous. □

Definition 11. *[23]A topological space (X,τ) is said to be:*

(a) ClopenT_2 (Clopen Hausdorff or Ultra-Hausdorff) if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x\in U$ and $y\in V$.
(b) Clopen regular, if for each clopen set F and each point $x\notin F$, there exist disjoint open sets U and V such that $F\subseteq U$ and $x\in V$.

Theorem 36. *If $f:(X,\tau)\to(Y,\sigma)$ is slightly coc-continuous and injective function and (Y,σ) is clopen T_2, then (X,τ^k) is T_2.*

Proof. Let $x,y\in X$, with $x\neq y$. Since f is injective, then $f(x)\neq f(y)$. Since (Y,σ) is clopen T_2, then there exist two disjoint clopen sets U and V in (Y,σ) such that $f(x)\in U$ and $f(y)\in V$. Since f is slightly coc-continuous, $f^{-1}(U),f^{-1}(V)\in\tau^k$. Now $x\in f^{-1}(U),y\in f^{-1}(V)$ and $f^{-1}(U)\cap f^{-1}(V)=f^{-1}(U\cap V)=f^{-1}(\emptyset)=\emptyset$. Therefore, (X,τ^k) is a T_2 topological space. □

Theorem 37. *If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly coc-continuous injective open function from an s-regular topological space (X,τ) onto a topological space (Y,σ), then (Y,σ) is clopen regular.*

Proof. Let F be a clopen set in Y and let $y\in Y$ such that $y\notin F$. Since f is onto then there is $x\in X$ such that $f(x)=y$. Since f is slightly coc-continuous, then by Theorem 28 (c), $f^{-1}(F)$ is a coc-closed set. Since (X,τ) is s-regular, and $x\notin f^{-1}(F)$, there exist two disjoint open sets U and V such that $f^{-1}(F)\subseteq U$ and $x\in V$. Since f is onto, $F=f(f^{-1}(F))\subseteq f(U)$. Also, $y=f(x)\in f(V)$. Since f is injective

$$f(U)\cap f(V)=f(U\cap V)=f(\emptyset)=\emptyset.$$

Therefore, (Y,σ) is clopen regular. □

Theorem 38. *If $f:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly continuous function and $g:(X,\tau)\longrightarrow(Y,\sigma)$ is slightly coc-continuous function and (Y,σ) is clopen Hausdorff, then $E=\{x\in X: f(x)=g(x)\}$ is coc-closed in (X,τ).*

Proof. Let $x\in X-E$. Then $f(x)\neq g(x)$. Since (Y,σ) is clopen Hausdorff, then there exists two clopen sets $V,W\subseteq Y$ such that $f(x)\in V$, $g(x)\in W$ and $V\cap W=\emptyset$. Since f is slightly continuous and g is slightly coc-continuous, then $f^{-1}(V)$ is open and $g^{-1}(W)$ is coc-open in (X,τ) with $x\in f^{-1}(V)$ and $x\in g^{-1}(W)$. Set $O=f^{-1}(V)\cap g^{-1}(W)$. Then O is coc-open. with $x\in O\subseteq X-E$. It follows that $X-E$ is coc-open and E is coc-closed. □

Theorem 39. *Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be slightly coc-continuous and injective function. If (Y,σ) is zero-dimensional and T_2, then (X,τ^k) is T_2.*

Proof. Let $x, y \in X$ with $x \neq y$. Since f is injective, then $f(x) \neq f(y)$. Since (Y, σ) is T_2, then there exist two open sets U and V in Y such that $f(x) \in U, f(y) \in V$ and $U \cap V = \emptyset$. Since (Y, σ) is zero-dimensional, then there exist two clopen sets $U_1, V_1 \subseteq Y$ such that $f(x) \in U_1 \subseteq U$ and $f(y) \in V_1 \subseteq V$. Since f is slightly coc-continuous, then $f^{-1}(U_1)$ and $f^{-1}(V_1)$ are coc-open sets. We have $x \in f^{-1}(U_1), y \in f^{-1}(V_1)$ and $f^{-1}(U_1) \cap f^{-1}(V_1) = f^{-1}(U_1 \cap V_1) \subseteq f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. This shows that (X, τ^k) is T_2. □

Theorem 40. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be slightly coc-continuous, open and injective function. If (Y, σ) is zero-dimensional, then (X, τ) is co-regular.*

Proof. Let $x \in X$ and U be an open set containing x. Since f is an open function, then $f(U)$ is open in(Y, σ) with $f(x) \in f(U)$. Since (Y, σ) is zero-dimensional, then there exist a clopen set $V \subseteq Y$ such that $f(x) \in V \subseteq f(U)$. Since f is slightly coc-continuous, then $f^{-1}(V)$ is coc-clopen set in X with $x \in f^{-1}(V) = Cl_{\tau^k}(f^{-1}(V) \subseteq U$. This implies that (X, τ) is co-regular. □

Theorem 41. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be slightly coc-continuous, closed and injective function. If (Y, σ) is zero-dimensional, then (X, τ) is co-regular.*

Proof. Let $x \in X$ and F be a closed set of X such that $x \notin F$. Since f is a closed function, then $f(F)$ is closed in (Y, σ). Since f is injective, then $f(x) \notin f(F)$ and so $f(x) \in Y - f(F) \in \sigma$. Since$(Y, \sigma)$ is zero-dimensional, then there exists a clopen set V in Y such that $f(x) \in V \subseteq Y - f(F)$. Since f is slightly coc-continuous, then $f^{-1}(V)$ is a coc-clopen set in (X, τ). Moreover, $x \in f^{-1}(V)$ and $F \subseteq X - f^{-1}(V)$ which is also coc-clopen set of (X, τ). Therefore (X, τ) is co-regular. □

Theorem 42. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be slightly coc-continuous, closed and injective function. If (Y, σ) is normal, then (X, τ) is co-normal.*

Proof. Let A and B be any two non-empty disjoint closed sets in (X, τ). Since f is closed and injective, we have $f(A)$ and $f(B)$ are two disjoint closed sets in Y. Since (Y, σ) is normal, then there exist two open sets U and V in Y such that $f(A) \subseteq U$, $f(B) \subseteq V$ and $U \cap V = \emptyset$. For each $y \in f(A)$, $y \in U$ and since (Y, σ) is zero-dimensional there exists a clopen set U_y such that $y \in U_y \subseteq U$. Thus, $f(A) \subseteq \cup\{U_y : y \in f(A)\} \subseteq U$. Put $G = \cup\{f^{-1}(U_y) : y \in f(A)\}$. Then $A \subseteq G \subseteq f^{-1}(U)$. Since f is slightly coc-continuous, $f^{-1}(U_y)$ is coc-open for each $y \in f(A)$ and so $G = \cup\{f^{-1}(U_y) : y \in f(A)\}$ is coc-open in X. Similarly, there exists a coc-open set H in X such that $B \subseteq H \subseteq f^{-1}(V)$ and $G \cap H \subseteq f^{-1}(U \cap V) = \emptyset$. This shows that (X, τ) is co-normal. □

5. Conclusions

This paper deals with the axioms of separation of points and sets and a type of continuity (Slight coc-continuity). The results are well related to the classical properties. The new concepts are justified by their dependence and independence through examples. In the final part we carry out external characterizations of separation, via slightly coc-continuous functions. The relationship is confronted with zero-dimensional spaces. In future studies, the following topics could be considered: (1) To define other types of sepaeration axioms that are related to co-compact open sets (2) To study the uniform structures that are related to co-compact open sets.

Author Contributions: Formal analysis, investigation, and writing-original draft preparation S.A.G. and E.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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