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Identities and Computation Formulas for Combinatorial Numbers Including Negative Order Changhee Polynomials

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Abstract: The purpose of this paper is to construct generating functions for negative order Changhee numbers and polynomials. Using these generating functions with their functional equation, we prove computation formulas for combinatorial numbers and polynomials. These formulas include Euler numbers and polynomials of higher order, Stirling numbers, and negative order Changhee numbers and polynomials. We also give some properties of these numbers and polynomials with their generating functions. Moreover, we give relations among Changhee numbers and polynomials of negative order, combinatorial numbers and polynomials and Bernoulli numbers of the second kind. Finally, we give a partial derivative of an equation for generating functions for Changhee numbers and polynomials of negative order. Using these differential equations, we derive recurrence relations, differential and integral formulas for these numbers and polynomials. We also give p-adic integrals representations for negative order Changhee polynomials including Changhee numbers and Deahee numbers.

Keywords: generating function; Bernoulli numbers and polynomials of the second kind; Euler numbers and polynomials; Stirling numbers; Combinatorial numbers and polynomials; Changhee numbers and polynomials; p-adic integrals

MSC: 05A15; 05A10; 11B83; 26C05; 11S80

1. Introduction

The finite sums of powers of binomial coefficients with combinatorial numbers and polynomials have been used in almost all areas of mathematics, probability theory, statistics, physics, computer science and the other applied sciences. These sums are also used to construct mathematical models. Recently, generating functions including combinatorial numbers and polynomials, and also the finite sums of powers of binomial coefficients in terms of the hypergeometric functions, were given by Simsek [1]. By using these functions, many computation formulas and relations including these sums and various kinds of special numbers and polynomials have been given (*cf*. References [1–9]).

In recent years, using different methods and techniques, negative order special numbers and polynomials, which are negative order Bernoulli polynomials and negative order Euler polynomials, have been studied by many mathematicians. In this paper, we investigate and study generating functions for negative order Changhee polynomials and numbers. By using these functions with combinatorial numbers, we give many new formulas and identities including Bernoulli numbers and polynomials of the second kind, Euler numbers and polynomials, Stirling numbers and combinatorial numbers and polynomials. The goal of this paper is to give computation formulas for negative order Changhee polynomials and numbers including the finite sums of powers of binomial sums.

In this paper we use the following notations and definitions:

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, which denotes set of natural numbers. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively sets of integer numbers, real numbers, complex numbers, respectively.

$$(x)_v = x(x-1)\cdots(x-v+1),$$

$(x)_0 = 1$ and

$$\binom{x}{v} = \frac{(x)_v}{v!},$$

where $v \in \mathbb{N}_0$ and

$$(z)^v = (-1)^v(-z)_v = z(z+1)\cdots(z+v-1)$$

(*cf*. References [1,3–10]).

In order to give the results of this paper, we need the following generating functions for special polynomials and numbers.

The Euler polynomials of order k are defined by

$$F_E(t,x;k) = \left(\frac{2}{e^t+1}\right)^k e^{tx} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}. \tag{1}$$

(*cf*. References [1,3–10]).

Substituting $k = 0$ into (1), we have

$$E_n^{(0)}(x) = x^n.$$

Substituting $x = 0$ into (1), we have the Euler numbers of order k:

$$E_n^{(k)} = E_n^{(k)}(0),$$

and substituting $x = 0$ and $k = 1$ into (1), we have the Euler numbers:

$$E_n = E_n^{(1)}$$

(*cf*. References [1,3–10]).

The Stirling numbers of the first kind are defined by

$$F_{S_1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!}. \tag{2}$$

From the above function, if $k > n$, then we have

$$S_1(n,k) = 0.. \tag{3}$$

These numbers can also be written as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \tag{4}$$

(*cf*. References [1–43]).

The λ-Stirling numbers are defined by

$$F_{S_2}(t,k;\lambda) = \frac{(\lambda e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k;\lambda)\frac{t^n}{n!}, \tag{5}$$

(*cf*. References [31,35,43]).

Setting $\lambda = 1$ in (5), we have the Stirling numbers of the second kind:

$$S_2(n,k) = S_2(n,k;1).$$

If $k > n$, then we have

$$S_2(n,k) = 0 \tag{6}$$

(*cf*. References [1–9,11,13–46]).

The Bernoulli polynomials of the second kind are defined by

$$F_{B_2}(t,x) = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \tag{7}$$

and also

$$b_n(x) = \int_x^{x+1} (z)_n dz \tag{8}$$

(*cf*. see, for detail, References ([33], pp. 113–117)). When $x = 0$, we have the Bernoulli numbers of the second kind (*cf*. References [13,14], ([33], pp. 113–117)).

The Peters polynomials are defined by

$$F_P(t,x;\lambda,\mu) = \frac{(1+t)^x}{\left(1+(1+t)^\lambda\right)^\mu} = \sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\frac{t^n}{n!}, \tag{9}$$

where $x, t \in \mathbb{C}$ (*cf*. References [2,16–23]).

The Peters polynomials are including some well-known families of special polynomials and numbers such as the Boole polynomials and numbers, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, the Stirling numbers, the Changhee polynomials and numbers and other combinatorial polynomials and numbers.

Substituting $\mu = 1$ into (9), we have $\xi_n(x) = s_n(x;\lambda,1)$ denotes the Boole polynomials (*cf*. References [14,33]).

Let m be any integer. The Changhee polynomials of order m are given by the following generating function:

$$F(t,x,m) = \frac{2^m(1+t)^x}{(2+t)^m} = \sum_{n=0}^{\infty} Ch_n^{(m)}(x)\frac{t^n}{n!}. \tag{10}$$

Let us examine the generating function in the Equation (10) for some special cases of integer m as follows:

(1) Let $m = d$ and $d \in \mathbb{N}$. Then, we have Changhee polynomials of order d (*cf*. Reference [19]). Setting $x = 0$ into Equation (10), we have Changhee numbers of order d, which defined by means of the following generating function:

$$\frac{2^d}{(2+t)^d} = \sum_{n=0}^{\infty} Ch_n^{(d)}\frac{t^n}{n!}, \tag{11}$$

which, for $d = 1$, yields $Ch_n(x) = Ch_n^{(1)}(x)$, which denote Changhee polynomials and $Ch_n = Ch_n(0)$, which denotes Changhee numbers. These numbers are given by

$$Ch_n = \left(-\frac{1}{2}\right)^n n! \tag{12}$$

(*cf*. References [17,19]).

(2) Let $m = 0$. By using Equation (10), we get the following generating function:

$$(1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(0)}(x)\frac{t^n}{n!}. \tag{13}$$

(3) Let $m = -k$ and $k \in \mathbb{N}$. We modified Equation (10). Thus, we define Changhee polynomials of order $-k$, $Ch_n^{(-k)}(x)$, by means of the following generating function:

$$H(t, x, -k) = \frac{(1+t)^x (2+t)^k}{2^k} = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x)\frac{t^n}{n!}. \tag{14}$$

Here $Ch_n^{(-k)}(x)$ are called negative order Changhee polynomials.

Setting $x = 0$ into Equation (14), we get Changhee numbers of order $-k$. These numbers are given by the following generating function:

$$K(t, -k) = \frac{(2+t)^k}{2^k} = \sum_{n=0}^{\infty} Ch_n^{(-k)}\frac{t^n}{n!}. \tag{15}$$

In next Sections, we investigate some properties of the negative order Changhee polynomials. These polynomials and numbers have various relations with many well-known families of special numbers and polynomial. In particular, their relationships to combinatoric numbers are very interesting and they have very important results. These are discussed in detail in the following sections.

By using Equation (13), we have

$$Ch_n^{(0)}(x) = x(x-1)\cdots(x-n+1).$$

Few values of the polynomials $Ch_n^{(0)}(x)$ are given as follows:

$$\begin{aligned}
Ch_0^{(0)}(x) &= 1, \\
Ch_1^{(0)}(x) &= x, \\
Ch_2^{(0)}(x) &= x^2 - x, \\
Ch_3^{(0)}(x) &= x^3 - 3x^2 - 2x, \ldots
\end{aligned}$$

Therefore, with help of Equation (4), we obtain a relation between the polynomials $Ch_n^{(0)}(x)$ and the Stirling numbers of the first kind by the following formula:

$$Ch_n^{(0)}(x) = \sum_{l=0}^{n} S_1(n, l)x^l. \tag{16}$$

Combinatorial Numbers

Here, we give some well-known combinatorial numbers and their generating functions. Some formulas and relations between these numbers and negative order Changhee polynomials and numbers are given in the following sections.

The numbers $y_1(n,k;\lambda)$ is defined by

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!}(\lambda e^t + 1)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}. \tag{17}$$

(*cf*. Reference [8]).

By Equation (8) in [8], we have

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{n}\binom{k}{j} j^n \lambda^j, \ \ (n \in \mathbb{N}_0).$$

(*cf*. Reference [8], Theorem 1).

Also we have

$$y_1(n,k;\lambda) = \sum_{j=0}^{n}\binom{k}{j} j! S_2(n,j)\lambda^j(\lambda+1)^{k-j}.$$

(*cf*. Reference [47], Equation (16)).

Substituting $\lambda = 1$ into the aforementioned equation, we have

$$B(n,k) = k! y_1(n,k;1) = \frac{\partial^m}{\partial t^m}(e^t - 1)^k|_{t=0}. \tag{18}$$

(*cf*. References [1,8,15]).

The numbers $y_3(n,k;\lambda,a,b)$ is defined by

$$F_{y_3}(t,k;\lambda,a,b) = \frac{e^{bkt}}{k!}\left(\lambda e^{(a-b)t} + 1\right)^k = \sum_{n=0}^{\infty} y_3(n,k;\lambda,a,b)\frac{t^n}{n!}. \tag{19}$$

(*cf*. Reference [9]).

By Equation (19), we have

$$y_3(n,k;\lambda,a,b) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j(bk + j(a-b))^n.$$

(*cf*. Reference [9]).

The numbers $y_4(n,k;\lambda,a,b)$ are defined by

$$F_{y_4}(t,k;\lambda,a,b) = \frac{e^{btk}}{(a+b+1)^k}(e^{(a-b)t} + \lambda)^k = \sum_{n=0}^{\infty} y_4(n,k;\lambda,a,b)\frac{t^n}{n!}. \tag{20}$$

(*cf*. Reference [39]).

By using (20), we have

$$y_4(n,k;\lambda,a,b) = \frac{1}{(a+b+1)^k}\sum_{j=0}^{k}\binom{k}{j}\lambda^j(bk + j(a-b))^n$$

and

$$y_4(n,k;\lambda,a,b) = \frac{k!\lambda^k}{(a+b+1)^k} y_3(n,k;\lambda^{-1},a,b)$$

(*cf*. Reference [39]).

Results of this paper are briefly summarized below.

In Section 2, with help of generating functions and their functional equations, some properties of Changhee numbers and polynomials of negative order are given. By using these functional equations, we derive computation formulas for negative order Changhee polynomials.

In Section 3, we give partial derivative equations for the generating function of the negative order Changhee polynomials. By using these partial differential equations, we derive differential formulas for the negative order Changhee polynomials.

In Section 4, we give integral representations of negative order Changhee polynomials.

In Section 5, we give some identities and relations including Bernoulli numbers and polynomials of the second kind, Euler numbers and polynomials, Stirling numbers, negative order Changhee numbers and polynomials and combinatorial numbers and polynomials such as the special numbers $y_1(n,k;\lambda)$, $y_3(n,k;\lambda,a,b)$, $y_4(n,k;\lambda,a,b)$, and $B(n,k)$.

2. Changhee Polynomials of Negative Order

In this section, we investigate some properties of negative order Changhee numbers and polynomials.

With the help of functional equations including generating functions for special numbers and polynomials, in this section we give relations between negative order Changhee polynomials, negative order Euler polynomials, and combinatorial numbers including the Stirling numbers of the first kind, and combinatorial numbers such as the numbers $y_1(n,k;\lambda)$, $y_3(n,k;\lambda,a,b)$, $y_4(n,k;\lambda,a,b)$, and $B(n,k)$.

Combining (14) and (15), we get

$$H(t,x,-k) = (1+t)^x K(t,-k).$$

By using the above relation, we obtain

$$\sum_{n=0}^{\infty} Ch_n^{(-k)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!} \sum_{n=0}^{\infty} Ch_n^{(-k)} \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Ch_n^{(-k)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (x)_j Ch_{n-j}^{(-k)} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we arrive at the following theorem.

Theorem 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)}(x) = \sum_{j=0}^{n} \binom{n}{j} (x)_j Ch_{n-j}^{(-k)}. \tag{21}$$

We note that Equation (14) is related to some well-known special numbers and polynomials. That is, replace t by $e^t - 1$ into (14), we have the following functional equation:

$$H(e^t-1,x,-k) = F_E(t,x;-k).$$

Combining the above functional equation with (1) and (5), we obtain

$$\sum_{m=0}^{\infty} E_m^{(-k)}(x)\frac{t^m}{m!} = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x) F_{S_2}(t,n;1).$$

Combining the left side of the above equation with Equation (5) and after the necessary algebraic computations, we get the following result:

$$\sum_{m=0}^{\infty} E_m^{(-k)}(x)\frac{t^m}{m!} = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x)\sum_{m=0}^{\infty} S_2(m,n)\frac{t^m}{m!}.$$

Since $S_2(m,n) = 0$ for $n > m$, the aforementioned equation reduces to the following equation:

$$\sum_{m=0}^{\infty} E_m^{(-k)}(x)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} Ch_n^{(-k)}(x)S_2(m,n)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we arrive at the following theorem.

Theorem 2. *Let $m \in \mathbb{N}_0$. Then we have*

$$E_m^{(-k)}(x) = \sum_{n=0}^{m} Ch_n^{(-k)}(x)S_2(m,n). \tag{22}$$

In Reference [8], Simsek gave the following formula:

$$E_m^{(-k)} = 2^{-k}B(m,k). \tag{23}$$

Combining (22) with (23), we arrive at the following theorem:

Theorem 3. *Let $m \in \mathbb{N}_0$. Then we have*

$$B(m,k) = 2^k \sum_{n=0}^{m} Ch_n^{(-k)}S_2(m,n).$$

Computation Formula for Changhee Polynomials of Negative Order

Here we give some computation formulas for Changhee polynomials of negative order from (14). By using Equation (14), we get

$$\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(1+t)^{j+x} = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x)\frac{t^n}{n!}.$$

We assume that $|t| < 1$. Thus, we get

$$\sum_{n=0}^{\infty}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x+j)_n\frac{t^n}{n!} = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x)\frac{t^n}{n!}. \tag{24}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation and using Vandermonde's identity, we get the following theorem.

Theorem 4. *Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)}(x) = \frac{1}{2^k}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{k}{j}\binom{n}{l}(j)_{n-l}(x)_l. \tag{25}$$

Computation formula for the negative order Changhee numbers is given by the following theorem.

Theorem 5. *Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)} = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} (j)_n. \tag{26}$$

Proof. By applying binomial theorem to Equation (15) with $|t| < 1$, we get

$$\frac{1}{2^k} \sum_{n=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n} t^n = \sum_{n=0}^{\infty} Ch_n^{(-k)} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we arrive at the desired result. □

Combining (25) with (26), we get the following corollary:

Corollary 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)}(x) = \sum_{l=0}^{n} \binom{n}{l} (x)_l Ch_{n-l}^{(-k)}. \tag{27}$$

We now give some alternative computation formula for the Changhee numbers of order $-k$. Combining (4) with (26), we get the following corollary:

Corollary 2. *Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)} = \frac{1}{2^k} \sum_{j=0}^{k} \sum_{m=0}^{n} \binom{k}{j} S_1(n,m) j^m. \tag{28}$$

Combining (18) with (28), we get the following corollary:

Corollary 3. *Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)} = \frac{1}{2^k} \sum_{m=0}^{n} \binom{k}{j} S_1(n,m) B(m,k). \tag{29}$$

By using (24), we have the following result:

Corollary 4. *Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)}(x) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} (x+j)_n. \tag{30}$$

Combining (24) with (4), we have the following result:

Corollary 5.

$$Ch_n^{(-k)}(x) = \frac{1}{2^k} \sum_{j=0}^{k} \sum_{l=0}^{n} \binom{k}{j} S_1(n,l) \, (x+j)^l \,. \tag{31}$$

By using (28) or (29) with help of Equation (1) in the work of Simsek [1], a few values of the Changhee numbers and polynomials of order $-k$ for are $n = 0, 1, 2, 3, 4$ given in Table 1.

Table 1. A few values of negative order Changhee numbers and polynomials for $n = 0, 1, 2, 3, 4$.

	$Ch_n^{(-k)}$	$Ch_n^{(-k)}(x)$
0	1	1
1	$\frac{k}{2}$	$x + \frac{k}{2}$
2	$\frac{k^2-k}{4}$	$x^2 + (k-1)x + \frac{k^2-k}{4}$
3	$\frac{k^3-3k^2+4k}{8}$	$x^3 + 3\left(\frac{k}{2}-1\right)x^2 + \left(\frac{3k^2-9k+8}{4}\right)x + \frac{k^3-3k^2+4k}{8}$
4	$\frac{k^4-6k^3+19k^2-28k}{16}$	$x^4 + (2k-6)x^3 + \left(\frac{3}{2}k^2 - 6k + 11\right)x^2 + \left(\frac{k^3-3k^2+4k}{2}\right)x + \left(\frac{k^4-6k^3+19k^2-28k}{16}.\right)$

3. Partial Derivative of the Generating Function $H(t, x, -k)$

In this section, we consider partial derivative of Equation (14) with respect to x and t, we get two partial differential equations for the function $H(t, x, -k)$. By using these equations, we give derivative formulas and a recurrence relation for the polynomials $Ch_n^{(-k)}(x)$.

By applying partial derivative operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ to Equation (14), we get the following partial derivative equations, respectively:

$$\frac{\partial^m}{\partial x^m}\{H(t, x; -k)\} = m!H(t, x, -k)F_{S_1}(t, m) \tag{32}$$

and

$$\frac{\partial}{\partial t}\{H(t, x; -k)\} = 2kH(t, x, 1-k) + xH(t, x-1, -k), \tag{33}$$

or

$$\frac{\partial}{\partial t}\{H(t, x; -k)\} = \frac{k}{2}F(t, 0, 1)H(t, x, -k) + xH(t, x-1, -k). \tag{34}$$

By combining (32) with (2), we get

$$\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{Ch_n^{(-k)}(x)\right\}\frac{t^n}{n!} = m!\sum_{n=0}^{\infty} Ch_n^{(-k)}(x)\frac{t^n}{n!}\sum_{j=0}^{\infty} S_1(n, m)\frac{t^j}{j!}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m}\left\{Ch_n^{(-k)}(x)\right\}\frac{t^n}{n!} = m!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}Ch_j^{(-k)}(x)S_1(n-j, m)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we arrive at the following theorem.

Theorem 6. *Let $m \in \mathbb{N}$. Then we have*

$$\frac{\partial^m}{\partial x^m}\left\{Ch_n^{(-k)}(x)\right\} = m!\sum_{j=0}^{n}\binom{n}{j}Ch_j^{(-k)}(x)S_1(n-j, m). \tag{35}$$

By combining (33) with (14), we obtain

$$\sum_{n=1}^{\infty} Ch_n^{(-k)}(x)\frac{t^{n-1}}{(n-1)!} = 2k\sum_{n=0}^{\infty} Ch_n^{(1-k)}(x)\frac{t^n}{n!} + x\sum_{n=0}^{\infty} Ch_n^{(-k)}(x-1)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Ch_{n+1}^{(-k)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{2kCh_n^{(1-k)}(x) + xCh_n^{(-k)}(x-1)\right\}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we arrive at the following theorem.

Theorem 7. *Let $m \in \mathbb{N}$. Then we have*

$$Ch_{n+1}^{(-k)}(x) = 2kCh_n^{(1-k)}(x) + xCh_n^{(-k)}(x-1). \tag{36}$$

By combining (34) with (10) and (14), we get

$$\sum_{n=1}^{\infty} Ch_n^{(-k)}(x)\frac{t^{n-1}}{(n-1)!} = \frac{k}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}Ch_{n-j}Ch_n^{(-k)}(x)\frac{t^n}{n!} + x\sum_{n=0}^{\infty}Ch_n^{(-k)}(x-1)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we arrive at the following theorem.

Theorem 8. *Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_{n+1}^{(-k)}(x) = k\sum_{j=0}^{n}(-1)^{n-j}\frac{n!}{j!2^{n-j+1}}Ch_j^{(-k)}(x) + xCh_n^{(-k)}(x-1).$$

4. Integral Representations for Negative Order Changhee Polynomials

In this section, we give p-adic integrals and the Riemann integral representations for negative order Changhee polynomials.

4.1. Riemann Integral Representation for Negative Order Changhee Polynomials

We integrate the Equation (14) over x in order to get an integral equation for the function $H(t,x,-k)$. By using these equations, we give integral formulas and recurrence relations for the polynomials $Ch_n^{(-k)}(x)$.

Theorem 9. *Let $n \in \mathbb{N}$. Then we have*

$$\int_u^{u+1} Ch_{n-1}^{(-k)}(x)dx = \sum_{j=0}^{n}\binom{n}{j}\frac{b_{n-j}(0)\left(Ch_j^{(-k)}(u+1) - Ch_j^{(-k)}(u)\right)}{n}. \tag{37}$$

Proof. Integrating both side of Equation (10) from u to $u+1$, we get

$$t\int_u^{u+1} H(t,x,-k)dx = F_{b_2}(t,0)\left(H(t,u+1,-k) - H(t,u,-k)\right).$$

We assume that this series is uniformly convergent. In this case, we can rearrange the summation and integration.

$$\sum_{n=0}^{\infty} n\int_u^{u+1} Ch_{n-1}^{(-k)}(x)dx\frac{t^n}{n!} = \sum_{n=0}^{\infty} b_n(0)\frac{t^n}{n!}\sum_{m=0}^{\infty}\left(Ch_j^{(-k)}(u+1) - Ch_j^{(-k)}(u)\right)\frac{t^m}{m!}.$$

By using the Cauchy product rule, we get

$$\sum_{n=0}^{\infty} n\int_u^{u+1} Ch_{n-1}^{(-k)}(x)dx\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}b_{n-j}(0)\left(Ch_j^{(-k)}(u+1) - Ch_j^{(-k)}(u)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we arrive arrive at the desired result. □

By using (8) and (25), we get

$$\int_{u}^{u+1} Ch_{n-1}^{(-k)}(x)dx = \frac{1}{2^k}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{k}{j}\binom{n}{l}(j)_{n-1}b_l(u). \tag{38}$$

By using (8) and (27), we have

$$\int_{u}^{u+1} Ch_{n-1}^{(-k)}(x)dx = \sum_{l=0}^{n}\binom{n}{l}b_l(u)Ch_{n-l}^{(-k)}. \tag{39}$$

4.2. *p-Adic Integral Representations for Negative Order Changhee Polynomials*

Let $\mathbb{Z}_p$ denote the set of p-adic integers. The Volkenborn integral is defined by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x), \tag{40}$$

where $\mu_1(x)$ is given as follows:

$$\mu_1(x) = \frac{1}{p^N}$$

(*cf*. References [34,40,48]; see also the references cited in each of these earlier works).

The Daehee numbers D_n are defined by

$$D_n = \int_{\mathbb{Z}_p}(x)_n d\mu_1(x) = \frac{(-1)^n n!}{n+1}, \tag{41}$$

where $n \in \mathbb{N}_0$ (*cf*. References [18,34,40]). By (41), it easily see that

$$\int_{\mathbb{Z}_p}(x)_n d\mu_1(x) = \sum_{l=0}^{n} S_1(n,l)B_l, \tag{42}$$

where B_l denotes the Bernoulli numbers and $n \in \mathbb{N}_0$ (*cf*. Reference [18]).

The fermionic p-adic integral is defined by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1}(-1)^x f(x), \tag{43}$$

where $\mu_{-1}(x)$ is given as follows:

$$\mu_{-1}(x) = (-1)^x$$

(*cf*. [26], see also References [24,40]; see also the references cited in each of these earlier works).

The Changhee numbers Ch_n are defined by

$$Ch_n = \int_{\mathbb{Z}_p}(x)_n d\mu_{-1}(x) = (-1)^n 2^{-n} n!, \tag{44}$$

where $n \in \mathbb{N}_0$ (*cf*. Reference [17]).

By applying the Volkenborn integral to (21), we have

$$\int_{\mathbb{Z}_p} Ch_n^{(-k)}(x)d\mu_1(x) = \sum_{j=0}^{n}\binom{n}{j}Ch_{n-j}^{(-k)}\int_{\mathbb{Z}_p}(x)_j d\mu_1(x).$$

Combining the aforementioned equation with (41), we get the following Volkenborn integral representation for negative order Changhee polynomials:

$$\int_{\mathbb{Z}_p} Ch_n^{(-k)}(x)d\mu_1(x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{j!Ch_{n-j}^{(-k)}}{j+1} \tag{45}$$

and

$$\int_{\mathbb{Z}_p} Ch_n^{(-k)}(x)d\mu_1(x) = \sum_{j=0}^{n}\binom{n}{j}D_jCh_{n-j}^{(-k)}. \tag{46}$$

By applying fermionic p-adic integral to (21), we have

$$\int_{\mathbb{Z}_p} Ch_n^{(-k)}(x)d\mu_{-1}(x) = \sum_{j=0}^{n}\binom{n}{j}Ch_{n-j}^{(-k)}\int_{\mathbb{Z}_p}(x)_jd\mu_{-1}(x). \tag{47}$$

By (41) and (47), we obtain the following fermionic p-adic integral representation for negative order Changhee polynomials:

$$\int_{\mathbb{Z}_p} Ch_n^{(-k)}(x)d\mu_{-1}(x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{j!Ch_{n-j}^{(-k)}}{2^j} \tag{48}$$

and

$$\int_{\mathbb{Z}_p} Ch_n^{(-k)}(x)d\mu_{-1}(x) = \sum_{j=0}^{n}\binom{n}{j}Ch_jCh_{n-j}^{(-k)}. \tag{49}$$

By using (14), we have

$$\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}2^{2k-j}(n)_jCh_{n-j}^{(-k)}(x)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we get

$$(x)_n = \sum_{j=0}^{k}\binom{k}{j}2^{2k-j}(n)_jCh_{n-j}^{(-k)}(x). \tag{50}$$

By applying the Volkenborn integral to the aforementioned equation, we obtain

$$\int_{\mathbb{Z}_p}(x)_nd\mu_1(x) = \sum_{j=0}^{k}\binom{k}{j}2^{2k-j}(n)_j\int_{\mathbb{Z}_p} Ch_{n-j}^{(-k)}(x)d\mu_1(x).$$

Combining the aforementioned equation with (41), we obtain

$$\sum_{j=0}^{k}\binom{k}{j}2^{2k-j}(n)_j\int_{\mathbb{Z}_p} Ch_{n-j}^{(-k)}(x)d\mu_1(x) = \frac{(-1)^nn!}{n+1}. \tag{51}$$

By applying fermionic p-adic integral to Equation (50), we get

$$\int_{\mathbb{Z}_p}(x)_nd\mu_{-1}(x) = \sum_{j=0}^{k}\binom{k}{j}2^{2k-j}(n)_j\int_{\mathbb{Z}_p} Ch_{n-j}^{(-k)}(x)d\mu_{-1}(x).$$

Combining the aforementioned equation with (44), we obtain

$$\sum_{j=0}^{k}\binom{k}{j}2^{2k-j}(n)_j\int\limits_{\mathbb{Z}_p}Ch_{n-j}^{(-k)}(x)d\mu_{-1}(x)=(-1)^n\frac{n!}{2^n}. \tag{52}$$

5. Identities and Relations

In this section, by using generating functions and their functional equations, we give many interesting and novel identities and relations including Bernoulli numbers and polynomials of the second kind, Euler numbers and polynomials, Stirling numbers, negative order Changhee numbers, and combinatorial numbers such as $y_1(n,k;\lambda)$, $y_3(n,k;\lambda,a,b)$, and $B(n,k)$.

Substituting $t=\lambda e^z-1$ into (15), we get the following functional equation:

$$\frac{k!}{2^k}F_{y_1}(z,k;\lambda)=K(\lambda e^z-1,-k).$$

Combining the aforementioned equation with Equation (5), we get

$$\frac{k!}{2^k}\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}Ch_n^{(-k)}\sum_{m=0}^{\infty}S_2(m,n:\lambda)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the aforementioned equation, we get the following theorem.

Theorem 10. *Let $n\in\mathbb{N}_0$. Then we have*

$$y_1(n,k;\lambda)=\frac{2^k}{k!}\sum_{n=0}^{\infty}Ch_n^{(-k)}S_2(m,n:\lambda). \tag{53}$$

Substituting $\lambda=1$ into (53) and using (6), we get the following corollary:

Corollary 6. *Let $n\in\mathbb{N}_0$. Then we have*

$$y_1(n,k;1)=\frac{2^k}{k!}\sum_{n=0}^{m}Ch_n^{(-k)}S_2(m,n).$$

Setting $t=\lambda e^t-1$ into Equation (14), we get the following functional equation:

$$k!\lambda^kF_{y_3}(t,k;\lambda;2,1)=2^kH(\lambda e^t-1,k;-k).$$

Combining the aforementioned equation with (5), (19) and (14), we get

$$\frac{k!\lambda^k}{2^k}\sum_{m=0}^{\infty}y_3(m,k;\lambda;2,1)\frac{t^m}{m!}=\sum_{n=0}^{\infty}Ch_n^{(-k)}(k)\sum_{m=0}^{\infty}S_2(m,n;\lambda)\frac{t^m}{m!}.$$

Therefore, after comparing the coefficients of $\frac{t^m}{m!}$ on the both sides of the aforementioned equation and some calculation, we arrive at the following theorem:

Theorem 11. *Let $m\in\mathbb{N}_0$. Then we have*

$$y_3(m,k;\lambda;2,1)=\frac{2^k}{k!\lambda^k}\sum_{n=0}^{\infty}Ch_n^{(-k)}(k)S_2(m,n;\lambda). \tag{54}$$

Substituting $\lambda=1$ into (54), and using (6), we get the following corollary:

Corollary 7. *Let $m \in \mathbb{N}_0$. Then we have*

$$y_3(m,k;1;2;1) = \frac{2^k}{k!} \sum_{n=0}^{m} Ch_n^{(-k)}(k) S_2(m,n). \tag{55}$$

Combining (20) with (55), we obtain the following corollary:

Corollary 8. *Let $m \in \mathbb{N}_0$. Then we have*

$$y_4(m,k;\lambda^{-1};2,1) = 2^{-k} \sum_{n=0}^{\infty} Ch_n^{(-k)}(k) S_2(m,n;\lambda^{-1}).$$

Combining (37) and (38), we arrive at the following theorem:

Theorem 12. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} b_{n-j}(0) \left(Ch_j^{(-k)}(u+1) - Ch_j^{(-k)}(u) \right) = \frac{n}{2^k} \sum_{j=0}^{k} \sum_{l=0}^{n} \binom{k}{j} \binom{n}{l} (j)_{n-1} b_l(u).$$

Combining (37) with (39), we arrive at the following theorem:

Theorem 13. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} b_{n-j}(0) \left(Ch_j^{(-k)}(u+1) - Ch_j^{(-k)}(u) \right) = n \sum_{l=0}^{n} \binom{n}{l} b_l(u) Ch_{n-l}^{(-k)}.$$

Combining (52) with (48), we get the following theorem.

Theorem 14. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{k} \binom{k}{j} 2^{2k-j} (n)_j \sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} \frac{l! Ch_{n-j-l}^{(-k)}}{2^l} = (-1)^n \frac{n!}{2^n}.$$

Combining (51) with (45), we get the following theorem.

Theorem 15. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{k} \binom{k}{j} 2^{2k-j} (n)_j \sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} \frac{l! Ch_{n-j-l}^{(-k)}}{l+1} = \frac{(-1)^n n!}{n+1}.$$

6. Conclusions

Although many books, papers and other research theses about special functions, special numbers and polynomials have been written in recent years, active, productive and applied studies are still continuing in these fields. For this reason, generating functions for new families of special numbers and polynomials involving Changhee numbers and polynomials of negative order and combinatorial numbers are constructed. By considering these generating functions with their functional equations, integral and differential equations, various properties for negative order Changhee numbers and polynomials and some combinatorial numbers are obtained and studied. By using these equations, we derive many new and novel identities and formulas for the Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, combinatorial numbers and polynomials and Changhee numbers and polynomials. As a result, formulas, identities and relations of this paper may potentially be used, not only in mathematics, but also in mathematical physics, computer sciences, engineering, and so forth.

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