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Two-Variable Quantum Integral Inequalities of Simpson-Type Based on Higher-Order Generalized Strongly Preinvex and Quasi-Preinvex Functions

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Abstract: In this paper, we present a new definition of higher-order generalized strongly preinvex functions. Moreover, it is observed that the new class of higher-order generalized strongly preinvex functions characterize various new classes as special cases. We acquire a new q_1q_2-integral identity, then employing this identity, we establish several two-variable q_1q_2-integral inequalities of Simpson-type within a class of higher-order generalized strongly preinvex and quasi-preinvex functions. Finally, the utilities of our numerical approximations have concrete applications.

Keywords: quantum calculus; Simpson-type inequalities; strongly preinvex functions; co-ordinated higher-order generalized strongly preinvex functions; co-ordinated higher-order generalized strongly quasi-preinvex functions

1. Introduction

Quantum calculus or q-calculus is regularly known as "calculus with no limits", and was first expounded by Jackson in the early the twentieth century, although the historical backdrop of quantum calculus can be traced back to some much earlier work done by Euler and Jacobi et al. (see [1]). Numerous problems require utilizing quantum analytics which incorporates both q-derivatives and q-integrals. Over the ongoing decade, the examination of q-calculus has captivated in light of a legitimate concern from some analylsts, since it has been found to have plenty of utilities in mathematics and physics. The precept goal of q-calculus is that it acts as a bridge between mathematics and physics, and it is a significant tool for researchers working in analytic number theory, special functions, quantum mechanics or mathematical inequalities. In q-calculus, we obtain the q-analogues of mathematical objects which can be recaptured as $q \to 1^-$. q-calculus has potential applications in pure and applied mathematics. In pure mathematics, q-calculus has been implemented in mathematical inequalities to unify q-derivative and q-integral versions of inequalities. For certain examinations on q-calculus see [2–8].

The concept of convexity has been extended in several directions, since these generalized versions have significant applications in different fields of pure and applied sciences. One of the convincing examples on extensions of convexity is the introduction of invex function, which was introduced by Hanson [9]. Weir et al. [10] proposed the idea of preinvex functions and implemented it to the establishment of sufficient optimality conditions and duality in nonlinear programming. Mohan et al. [11] introduced the well-known condition C.

Due to recent advancements in convexity, Polyak [12], introduced the generalization of convex functions, the so-called strongly convex functions. It is play a crucial role in optimization theory and other fields. For example, Karmardian [13] employed strongly convex functions to discuss the unique existence of a solution of the nonlinear complementarity problems. Strongly convex functions also have significant contribution in the convergence analysis of the iterative methods for solving variational inequalities and equilibrium problems; see Zu and Marcotte [14]. Nikodem and Pales [15] investigated the characterization of the inner product spaces using strongly convex functions, which can be viewed as a novel and innovative application. Qu and Li [16] investigated the exponential stability of primal-dual gradient dynamics using the concept of strongly convex functions. Rashid et al. [17] have derived Hermite–Hadamard type inequalities for various classes of strongly convex functions, which provide upper and lower estimates for the integrand. For moreapplications in the real world and antimatroids, see References [18–27] and the references therein.

The classical Simpson inequality is described as follows: The function $\Psi : [\xi_1, \xi_2] \to \mathbb{R}$ is a four times continuously differentiable, and $||\Psi^{(4)}||_\infty = \sup_{z\in(\xi_1,\xi_2)} \left|\Psi^{(4)}(z)\right| < \infty$. Then, one has following inequality:

$$\left|\frac{1}{3}\left[\frac{\Psi(\xi_1)+\Psi(\xi_2)}{2} + 2\Psi\left(\frac{\xi_1+\xi_2}{2}\right) - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Psi(z)dz\right]\right| \leq \frac{(\xi_2-\xi_1)^4}{2880}\left|\left|\Psi^{(4)}\right|\right|_\infty. \tag{1}$$

For more details on inequalities, we refer the interested reader to [28–45] and the references cited therein.

The main idea of this research is to introduce several q-integral inequalities of Simpson-type within a class of higher-order generalized strongly preinvex functions on co-ordinates. The quantum integral Simpson type inequality for convex function on co-ordinates is presented by Humaira et al. in [25] and is described as follows

Lemma 1. *If a function $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a mixed partial q_1q_2-differentiable function over Λ° (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(z,w)}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}$ being continuous and integrable on $[\xi_1,\xi_2]\times[\xi_3,\xi_4] \subset \Lambda^\circ$ with $0 < q_k < 1$ and $1 \leq k \leq 2$, then the one has equality:*

$$\begin{aligned}
&\frac{\Psi\left(\xi_1,\frac{\xi_3+\xi_4}{2}\right)+\Psi\left(\xi_2,\frac{\xi_3+\xi_4}{2}\right)+4\Psi\left(\frac{\xi_1+\xi_2}{2},\frac{\xi_3+\xi_4}{2}\right)+\Psi\left(\frac{\xi_1+\xi_2}{2},\xi_3\right)+\Psi\left(\frac{\xi_1+\xi_2}{2},\xi_4\right)}{9}\\
&+\frac{\Psi(\xi_1,\xi_3)+\Psi(\xi_2,\xi_3)+\Psi(\xi_1,\xi_4)+\Psi(\xi_2,\xi_4)}{36}\\
&-\frac{1}{6(\xi_2-\xi_1)}\int_{\xi_1}^{\xi_2}\left[\Psi(x,\xi_3)+4\Psi\left(x,\frac{\xi_3+\xi_4}{2}\right)+\Psi(x,\xi_4)\right]{}_0d_{q_1}x \qquad (2)\\
&-\frac{1}{6(\xi_4-\xi_3)}\int_{\xi_3}^{\xi_4}\left[\Psi(\xi_1,y)+4\Psi\left(\frac{\xi_1+\xi_2}{2},y\right)+\Psi(\xi_2,y)\right]{}_0d_{q_2}y\\
&+\frac{1}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4}\Psi(x,y)_0d_{q_2}y_0d_{q_1}x
\end{aligned}$$

$$= (\xi_2-\xi_1)(\xi_4-\xi_3)\int_0^1\int_0^1 \mathcal{P}(z,q_1)\mathcal{T}(w,q_2)\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi((1-z)\xi_1+z\xi_2,(1-w)\xi_3+w\xi_4)}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}{}_0d_{q_1}z_0d_{q_2}w,$$

where

$$\mathcal{P}(z,q_1) = \begin{cases} q_1z-\frac{1}{6}, & z\in\left[0,\frac{1}{2}\right), \\ q_1z-\frac{5}{6}, & z\in[\frac{1}{2},1), \end{cases}$$

and

$$\mathcal{T}(w, q_2) = \begin{cases} q_2 w - \frac{1}{6}, & w \in [0, \frac{1}{2}), \\ q_2 w - \frac{5}{6}, & w \in [\frac{1}{2}, 1). \end{cases}$$

In this study, a new concept of higher-order generalized strongly preinvex and quasi-preinvex functions are introduced in this paper. These new concepts take into account the q-calculus. These novelties are a combination of an auxiliary result based on identity which correlates with the q_1q_2-integral. New results are presented and new theorems are established. In addition to this the numerical approximations for the new Definitions 6 and 7 in q-calculus are presented. The newly introduced numerical approximation is used to solve problems in fluid mechanics, aerodynamics, and antimatroids. The new definition could open new doors of investigation toward preinvexity and q-calculus.

2. Formulations and Basic Facts

Let us recall the formulations and basic facts which are firmly concerned to this paper.

Mititelu [30] defined the notion of invex sets as follows:

Definition 1 ([30])**.** *If $\Omega_\eta \subset \mathbb{R}^n$ and $\eta(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous bifunction, then $\Omega_\eta \subset \mathbb{R}^n$ is said to be invex set*

$$\xi_1 + \tau\eta(\xi_2, \xi_1) \in \Omega_\eta, \quad \forall \xi_1, \xi_2 \in \Omega_\eta, \ \tau \in [0, 1].$$

The invex set Ω_η is also known as the η-connected set. Note that, if $\eta(\xi_1, \xi_2) = \xi_2 - \xi_1$, this means that every convex set is an invex set, but the converse is not true.

The concept of preinvex functions was introduced by Weir and Mond [10] as follows:

Definition 2 ([10])**.** *A function $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is said to be preinvex if*

$$\Psi\left(\xi_1 + \tau\eta(\xi_2, \xi_1)\right) \leq (1 - \tau)\Psi(\xi_1) + \tau\Psi(\xi_2)$$

for all $\xi_1, \xi_2 \in \Omega_\eta, \ \tau \in [0, 1]$.

For current research on preinvex functions, concerned readers are referred to [4,8–11,30,46].

The notion of strongly preinvex functions was introduced by Noor et al. [47].

Definition 3. *A function $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a strongly preinvex for modulus $\mu > 0$ if*

$$\Psi\left(\xi_1 + \tau\eta(\xi_2, \xi_1)\right) \leq (1 - \tau)\Psi(\xi_1) + \tau\Psi(y) - \mu\tau(1 - \tau)\|\eta(\xi_2, \xi_1)\|^2$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $\tau \in [0, 1]$.

Here, we introduce a new definition which combines the preinvex functions and the strongly preinvex functions given above.

Definition 4. *A function $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a higher-order generalized strongly preinvex for modulus $\mu \geq 0$ with order $\theta > 0$ if*

$$\Psi\left(\xi_1 + \tau\eta(\xi_2, \xi_1)\right) \leq (1 - \tau)\Psi(\xi_1) + \tau\Psi(y) - \mu\tau(1 - \tau)\|\eta(\xi_2, \xi_1)\|^\theta$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $\tau \in [0, 1]$.

We now discuss some special cases.
(I) Choosing $\mu = 0$, then the class of generalized strongly preinvex functions reduces to the class of preinvex functions as defined in Definition 2.
(II) Choosing $\theta = 2$, then the generalized higher-order strongly preinvex function becomes generalized strongly preinvex functions, that is,

Definition 5. *A function $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a generalized strongly preinvex for modulus $\mu \geq 0$ if*

$$\Psi\left(\xi_1 + \tau\eta(\xi_2, \xi_1)\right) \leq (1-\tau)\Psi(\xi_1) + \tau\Psi(y) - \mu\tau(1-\tau)\|\eta(\xi_2, \xi_1)\|^2$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $\tau \in [0,1]$.

(III) Choosing $\eta(\xi_2, \xi_1) = \xi_2 - \xi_1$, then we obtain the higher-order generalized strongly convex function

Definition 6. *A function $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a higher-order generalized strongly convex for modulus $\mu \geq 0$ with order $\theta > 0$ if*

$$\Psi\left((1-\tau)\xi_1 + \tau\xi_2\right) \leq (1-\tau)\Psi(\xi_1) + \tau\Psi(y) - \mu\tau(1-\tau)\|\xi_2 - \xi_1\|^\theta$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $\tau \in [0,1]$.

Definition 7. *A function $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a higher-order generalized strongly quasi-preinvex for modulus $\mu \geq 0$ with order $\theta > 0$ if*

$$\Psi\left(\xi_1 + \tau\eta(\xi_2, \xi_1)\right) \leq \max(\Psi(\xi_1), \Psi(\xi_2)) - \mu\tau(1-\tau)\|\eta(\xi_2, \xi_1)\|^\theta$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $\tau \in [0,1]$.

For appropriate and suitable choice of the bifunction $\eta(\xi_2, \xi_1)$, θ and μ one can obtain various new and known classes of higher-order generalized strongly preinvex and quasi-preinvex functions. This shows that the higher-order generalized strongly preinvex and quasi-preinvex functions involving the bifunction $\eta(\xi_2, \xi_1)$ is quite a general and unifying one. One can explore the applications of higher-order generalized strongly preinvex and quasi-preinvex function; however, this is another direction for further research.

Recall some basic definitions and properties on q-analogue for single and double variables. Let $\mathcal{V} = [\xi_1, \xi_2] \subseteq \mathbb{R}$ with constant $0 < q < 1$ and let $\mathcal{U} = [\xi_1, \xi_2] \times [\xi_3, \xi_4] \subseteq \mathbb{R}^2$ with constants $0 < q_k < 1, 1 \leq k \leq 2$.

Tariboon et al. [2,3] introduced the formula of q-derivative, q-integral and related properties for one variable function, as follows:

Definition 8. *Assume that a mapping $\Psi : \mathcal{V} \to \mathbb{R}$ is continuous and $s \in \mathcal{V}$. Then one has q-derivative of Ψ on $\mathcal{V}$ at s is defined as*

$$_{\xi_1}D_q\Psi(s) = \frac{\Psi(s) - \Psi(qs + (1-q)\xi_1)}{(1-q)(s-\xi_1)}, \quad s \neq \xi_1. \tag{3}$$

It is obvious that

$$\lim_{s \to \xi_1} {}_{\xi_1}D_q\Psi(s) = {}_{\xi_1}D_q\Psi(\xi_1).$$

we say that Ψ is said to be q-differentiable over $\mathcal{V}$, moreover $_{\xi_1}D_q\Psi(s)$ exists $\forall\ s \in \mathcal{V}$.

Note that if $\xi_1 = 0$ in (3), then ${}_0D_q\Psi = D_q\Psi$, where $D_q\Psi$ is a well-defined q-derivative of $\Psi(s)$, that is explained as

$$D_q\Psi(s) = \frac{\Psi(s) - \Psi(qs)}{(1-q)(s)}.$$

Definition 9. *Suppose that a continuous mapping is $\Psi : \mathcal{V} \to \mathbb{R}$ is q-differentiable and is denoted by ${}_{\xi_1}D_q^2\Psi$, if*

$${}_{\xi_1}D_q^2\Psi = {}_{\xi_1}D_q\left({}_{\xi_1}D_q\Psi\right).$$

Similarly, a higher-order q-differentiable is defined as ${}_{\xi_1}D_q^n\Psi$: $\mathcal{V} \to \mathbb{R}$.

Definition 10. *Assume that a mapping $\Psi : \mathcal{V} \to \mathbb{R}$ is continuous. Then the q-integral on $\mathcal{V}$ is defined by*

$$\int_{\xi_1}^{s} \Psi(z)_{\xi_1}d_qz = (1-q)(s-\xi_1)\sum_{n=0}^{\infty} q^n\Psi(q^ns + (1-q^n)\xi_1) \tag{4}$$

for $s \in V$.

Moreover if $\xi_1 = 0$ into (4), then we get the following formula of the q-integral, which is denoted as

$$\int_0^s \Psi(z)_0d_qz = (1-q)s\sum_{n=0}^{\infty} q^n\Psi(q^ns)$$

Theorem 1. *Assume that a mapping $\Psi : \mathcal{V} \to \mathbb{R}$ is continuous, then the following properties hold:*

$$(i)\quad {}_{\xi_1}D_q\int_{\xi_1}^{s} \Psi(z)_{\xi_1}d_qz = \Psi(s);$$

$$(ii)\quad \int_{\xi_1}^{s} {}_{\xi_1}D_q\Psi(z)_{\xi_1}d_qz = \Psi(s);$$

$$(iii)\quad \int_{\xi_2}^{s} {}_{\xi_1}D_q\Psi(z)_{\xi_1}d_qz = \Psi(s) - \Psi(\xi_2), \quad \xi_2 \in (\xi_1, s).$$

Theorem 2. *Assuming that a mapping $\Psi : \mathcal{V} \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$, we have the following properties:*

$$(i)\quad \int_{\xi_1}^{s} [\Psi_1(z) + \Psi_2(z)]\,{}_{\xi_1}d_qz = \int_{\xi_1}^{s} \Psi_1(z)_{\xi_1}d_qz + \int_{\xi_1}^{s} \Psi_2(z)_{\xi_1}d_qz$$

$$(ii)\quad \int_{\xi_1}^{s} (a\Psi_1(z))_{\xi_1}d_qz = a\int_{\xi_1}^{s} \Psi_1(z)_{\xi_1}d_qz.$$

Humaira et al. [45] developed the theory of quantum integral inequalities for two-variables functions and introduced q_1q_2-Simpson-type form inequalities for two-variables functions over finite rectangles.

Definition 11. *Assume that a mapping of two variables $\Psi : \mathcal{U} \to \mathbb{R}$ is continuous. Then partial q_1-derivative, q_2 -derivative and q_1q_2-derivative at $(z,w) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$ are, respectively, defined as:*

$$\begin{aligned}
\frac{{}_{\xi_1}\partial_{q_1}\Psi(z,w)}{{}_{\xi_1}\partial_{q_1}z} &= \frac{\Psi(z,w) - \Psi(q_1z + (1-q_1)\xi_1, w)}{(1-q_1)(z-\xi_1)}, \quad z \neq \xi_1, \\
\frac{{}_{\xi_3}\partial_{q_2}\Psi(z,w)}{{}_{\xi_3}\partial_{q_2}w} &= \frac{\Psi(z,w) - \Psi(z, q_2w + (1-q_2)\xi_3)}{(1-q_2)(w-\xi_3)}, \quad w \neq \xi_3, \\
\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(z,w)}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w} &= \frac{1}{(1-q_1)(1-q_2)(z-\xi_1)(w-\xi_3)} \\
&\times \Big[\Psi(q_1z + (1-q_1)\xi_1, q_2w + (1-q_2)\xi_3) - \Psi(q_1z + (1-q_1)\xi_1, w) \\
&- \Psi(z, q_2w + (1-q_2)\xi_3) + \Psi(z,w)\Big], \quad z \neq \xi_1, \quad w \neq \xi_3.
\end{aligned}$$

The function Ψ: $\mathcal{U} \to \mathbb{R}$ is called partially q_1- q_2- and q_1q_2-differentiable on $[\xi_1, \xi_2] \times [\xi_3, \xi_4]$ if $\frac{{}_{\xi_1}\partial_{q_1}\Psi(z,w)}{{}_{\xi_1}\partial_{q_1}z}$, $\frac{{}_{\xi_3}\partial_{q_2}\Psi(z,w)}{{}_{\xi_3}\partial_{q_2}w}$ and $\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1q_2}\Psi(z,w)}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}$ exist for all $(z,w) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$.

Definition 12. *Assume that a mapping of two variables $\Psi : \mathcal{U} \to \mathbb{R}$ is continuous. Then the definite q_1q_2-integral on $[\xi_1, \xi_2] \times [\xi_3, \xi_4]$ are described as*

$$\begin{aligned}
&\int_{\xi_3}^{t}\int_{\xi_1}^{s} \Psi(z,w)\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_3}d_{q_2}w = (1-q_1)(1-q_2)(s-\xi_1)(t-\xi_3) \\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m \Psi\left(q_1^n s + (1-q_1^n)\,\xi_1, q_2^m t + (1-q_2^m)\,\xi_3\right)
\end{aligned}$$

for $(s,t) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$.

Theorem 3. *Assume that a mapping of two variables $\Psi : \mathcal{U} \to \mathbb{R}$ is continuous, then the following properties hold:*

$$\begin{aligned}
&(i) \quad \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}s\,{}_{\xi_3}\partial_{q_2}t}\int_{\xi_4}^{t}\int_{\xi_1}^{s} \Psi(z,w)\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_3}d_{q_2}w = \Psi(s,t) \\
&(ii) \quad \int_{\xi_3}^{t}\int_{\xi_1}^{s} \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(z,w)}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_3}d_{q_2}w = \Psi(s,t)
\end{aligned}$$

$$\begin{aligned}
&(iii) \quad \int_{t_1}^{t}\int_{s_1}^{s} \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(z,w)}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_3}d_{q_2}w \\
&= \Psi(s,t) - \Psi(s,t_1) - \Psi(s_1,t) + \Psi(s_1,t_1), \quad (s_1,t_1) \in (\xi_1, s) \times (\xi_4, t).
\end{aligned}$$

Theorem 4. *Assume that $\Psi_1, \Psi_2 : \mathcal{U} \to \mathbb{R}$ are continuous mappings of two variables. Then the following properties hold for $(s,t) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$,*

$$\begin{aligned}
&(i) \quad \int_{\xi_3}^{t}\int_{\xi_1}^{s} \left[\Psi_1(z,w) + \Psi_2(z,w)\right]\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_4}d_{q_2}w \\
&= \int_{\xi_3}^{t}\int_{\xi_1}^{s} \Psi_1(z,w)\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_3}d_{q_2}w + \int_{\xi_3}^{t}\int_{\xi_1}^{s} \Psi_2(z,w)\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_3}d_{q_2}w. \\
&(ii) \quad \int_{\xi_3}^{t}\int_{\xi_1}^{s} a\Psi(z,w)\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_3}d_{q_2}w = a\int_{\xi_3}^{t}\int_{\xi_1}^{s} \Psi(z,w)\,{}_{\xi_1}d_{q_1}z\,{}_{\xi_3}d_{q_2}w.
\end{aligned}$$

3. Auxiliary Result

Lemma 2. *Assume that $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a mixed partial q_1q_2-differentiable function on Λ° (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}$ being continuous and integrable on $[\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)] \subset \Lambda^{\circ}$ for $\eta_1(\xi_2, \xi_1), \eta_2(\xi_4, \xi_3) > 0$ and $q_1, q_2 \in (0, 1)$, then one has the following equality:*

$$\Gamma_{\Psi}(\xi_1, \xi_2, \xi_3, \xi_4; q_1, q_2) = \frac{1}{9}\left[\begin{array}{c} \Psi\left(\xi_1, \frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right) + \Psi\left(\xi_1 + \eta_1(\xi_2, \xi_1), \frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right) \\ +4\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2}, \frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right) + \Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2}, \xi_3\right) \\ +\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2}, \xi_3 + \eta_2(\xi_4, \xi_3)\right) \end{array}\right]$$

$$+\frac{1}{36}\left[\begin{array}{c} \Psi(\xi_1, \xi_3) + \Psi(\xi_1 + \eta_1(\xi_2, \xi_1), \xi_3) + \Psi(\xi_1, \xi_3 + \eta_2(\xi_4, \xi_3)) \\ +\Psi(\xi_1 + \eta_1(\xi_2, \xi_1), \xi_3 + \eta_2(\xi_4, \xi_3)) \end{array}\right]$$

$$\begin{aligned}
&-\frac{1}{6\eta_1(\xi_2,\xi_1)}\int_{\xi_1}^{\xi_1+\eta_1(\xi_2,\xi_1)}\left[\Psi(x,\xi_3) + 4\Psi\left(x, \frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right) + \Psi(x, \xi_3 + \eta_2(\xi_4,\xi_3))\right] {}_0d_{q_1}x \\
&-\frac{1}{6\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\xi_4,\xi_3)}\left[\Psi(\xi_1,y) + 4\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2}, y\right) + \Psi(\xi_1 + \eta_1(\xi_2,\xi_1), y)\right] {}_0d_{q_2}y \\
&+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\xi_2,\xi_1)}\int_{\xi_3}^{\xi_3+\eta_2(\xi_4,\xi_3)}\Psi(x,y)\,{}_0d_{q_2}y\,{}_0d_{q_1}x \\
&= \mathcal{K}\int_0^1\int_0^1 \Pi_1(\varrho,q_1)\Pi_2(\rho,q_2)\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1), \xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\,{}_0d_{q_1}\varrho\,{}_0d_{q_2}\rho, \qquad (5)
\end{aligned}$$

where

$$\Pi_1(\varrho, q_1) = \begin{cases} q_1\varrho - \frac{1}{6}, & if \quad 0 \le \varrho < \frac{1}{2}, \\ q_1\varrho - \frac{5}{6}, & if \quad \frac{1}{2} \le \varrho \le 1, \end{cases}$$

$$\Pi_2(\rho, q_2) = \begin{cases} q_2\rho - \frac{1}{6}, & if \quad 0 \le \rho < \frac{1}{2}, \\ q_2\rho - \frac{5}{6}, & if \quad \frac{1}{2} \le \rho \le 1, \end{cases}$$

and $\mathcal{K} = \eta_1(\xi_2, \xi_1)\eta_2(\xi_4, \xi_3)$.

Proof. Now, we consider

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(q_1\varrho - \frac{1}{6}\right)\left(q_2\rho - \frac{1}{6}\right)\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1), \xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\,{}_0d_{q_1}\varrho\,{}_0d_{q_2}\rho \\
&+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left(q_1\varrho - \frac{1}{6}\right)\left(q_2\rho - \frac{5}{6}\right)\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1), \xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\,{}_0d_{q_1}\varrho\,{}_0d_{q_2}\rho \\
&+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left(q_1\varrho - \frac{5}{6}\right)\left(q_2\rho - \frac{1}{6}\right)\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta(\xi_2,\xi_1), \xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\,{}_0d_{q_1}\varrho\,{}_0d_{q_2}\rho \qquad (6) \\
&+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left(q_1\varrho - \frac{5}{6}\right)\left(q_2\rho - \frac{5}{6}\right)\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta(\xi_2,\xi_1), \xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\,{}_0d_{q_1}\varrho\,{}_0d_{q_2}\rho.
\end{aligned}$$

By the definition of partial q_1q_2-derivatives and definite q_1q_2-integrals, we have

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(q_1\varrho-\frac{1}{6}\right)\left(q_2\rho-\frac{1}{6}\right)\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1),\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}{}_0d_{q_1}\varrho_0d_{q_2}\rho$$

$$=\frac{1}{(1-q_1)(1-q_2)\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{\left(q_1\varrho-\frac{1}{6}\right)\left(q_2\rho-\frac{1}{6}\right)}{\varrho\rho}$$

$$\times\left(\Psi(\xi_1+\varrho q_1\eta_1(\xi_2,\xi_1),\xi_3+\rho q_2\eta_2(\xi_4,\xi_3))-\Psi(\xi_1+\varrho q_1\eta_1(\xi_2,\xi_1),\rho)\right.$$

$$\left.-\Psi(\varrho,\xi_3+\rho q_2\eta_2(\xi_4,\xi_3))+\Psi(\varrho,\rho)\right){}_0d_{q_1}\varrho_0d_{q_2}\rho.$$

We observe that

$$\begin{aligned}
&\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&=-\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&-\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&-\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty}q_2^m\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&-\frac{q_2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&=\frac{q_2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty}q_2^m\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&-\frac{q_2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&-\frac{q_1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&=\frac{q_1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&-\frac{q_1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&\frac{q_1q_2}{\eta_2(\xi_4,\xi_3)\eta_1(\xi_2,\xi_1)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&-\frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3\right)\\
&- \frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3\right)\\
&- \frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&\frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&= -\frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&+ \frac{1}{6\eta_1(\xi_2,\xi_1)(\xi_4-\xi_3)}\sum_{n=0}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&- \frac{q_2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&+ \frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&\frac{q_1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&= \frac{q_1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\left[-\sum_{n=0}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\right.\\
&\left.+\sum_{n=0}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3\right)\right]\\
&- \frac{q_1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&- \frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&= \frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Psi\left(\xi_1,\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&- \frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty} q_2^m\Psi\left(\xi_1,\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&- \frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&\frac{q_2}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&= -\frac{q_2}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\left[-\sum_{m=0}^{\infty} q_2^m\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\right.\\
&\left.+\sum_{m=0}^{\infty} q_2^m\Psi\left(\xi_1,\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\right]\\
&+ \frac{q_2}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&\frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&+\frac{1}{6\eta_1(\xi_2,\xi_1)(\xi_4-\xi_3)}\sum_{m=0}^{\infty}q_2^m\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&+\frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\frac{q_2^m}{2}\xi_4+\left(1-\frac{q_2^m}{2}\right)\xi_3\right),\\
&-\frac{q_2}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&=\frac{\Psi(\xi_1,\xi_3)}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\\
&+\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&-\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&= -\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Psi\left(\xi_1,\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&-\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&-\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&= -\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3\right)\\
&-\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right),\\
&\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right)\\
&=\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)\\
&+\frac{1}{36\eta_1(\xi_2,\xi_1)(\xi_4-\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Psi\left(\xi_1+\frac{q_1^n}{2}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{2}\eta_2(\xi_4,\xi_3)\right).
\end{aligned}
$$

Similarly, in the same way we can compute the outcomes of the rest of the three q_1q_2-integrals, respectively, and by adding all of the q_1q_2-integrals we get the following result:

$$
\begin{aligned}
&\int_0^1\int_0^1 \Pi_1(\varrho,q_1)\Pi_2(\rho,q_2)\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1),\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\,{}_0d_{q_1}\varrho_0 d_{q_2}\rho \\
&= \frac{1}{9\eta_1(\xi_2,\xi_1)\eta_2(\xi_3,\xi_4)}\left[\begin{array}{c} \Psi\left(\xi_1,\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)+\Psi\left(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right) \\ +4\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)+\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3\right) \\ +\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3+\eta_2\left(\xi_4,\xi_3\right)\right)\end{array}\right] \\
&+\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_3,\xi_4)}\left[\begin{array}{c}\Psi(\xi_1,\xi_3)+\Psi(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3)+\Psi(\xi_1,\xi_3+\eta_2\left(\xi_4,\xi_3\right)) \\ +\Psi(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3+\eta_2\left(\xi_4,\xi_3\right))\end{array}\right] \\
&-\frac{1-q_1}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_3,\xi_4)}\left[\begin{array}{c}\sum_{n=0}^{\infty}q_1^n\Psi\left(\xi_1+q_1^n\eta_1(\xi_2,\xi_1),\xi_3\right) \\ +4\sum_{n=0}^{\infty}q_1^n\Psi\left(\xi_1+q_1^n\eta_1(\xi_2,\xi_1),\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right) \\ +\sum_{n=0}^{\infty}q_1^n\Psi\left(\xi_1+q_1^n\eta_1(\xi_2,\xi_1),\xi_3+\eta_2(\xi_4,\xi_3)\right)\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1-q_2}{6\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\left[\begin{array}{c}\sum_{m=0}^{\infty}q_2^m\Psi\left(\xi_1,\xi_3+q_2^m\eta_2(\xi_4,\xi_3)\right) \\ +4\sum_{m=0}^{\infty}q_2^m\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3+q_2^m\eta_2(\xi_4,\xi_3)\right) \\ +\sum_{m=0}^{\infty}q_2^m\Psi\left(\xi_2,\xi_3+q_2^m\eta_2(\xi_4,\xi_3)\right)\end{array}\right] \\
&+\frac{(1-q_1)(1-q_2)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\Psi\left(\xi_1+q_1^n\eta_1(\xi_2,\xi_1),\xi_3+q_2^m\eta_2(\xi_4,\xi_3)\right)\right]. \\
&= \frac{1}{9\eta_1(\xi_2,\xi_1)\eta_2(\xi_3,\xi_4)}\left[\begin{array}{c} \Psi\left(\xi_1,\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)+\Psi\left(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right) \\ +4\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)+\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3\right) \\ +\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},\xi_3+\eta_2\left(\xi_4,\xi_3\right)\right)\end{array}\right] \qquad (7) \\
&+\frac{1}{36\eta_1(\xi_2,\xi_1)\eta_2(\xi_3,\xi_4)}\left[\begin{array}{c}\Psi(\xi_1,\xi_3)+\Psi(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3)+\Psi(\xi_1,\xi_3+\eta_2\left(\xi_4,\xi_3\right)) \\ +\Psi(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3+\eta_2\left(\xi_4,\xi_3\right))\end{array}\right] \\
&-\frac{1}{6\eta_1^2(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\xi_2,\xi_1)}\left[\Psi(x,\xi_3)+4\Psi\left(x,\frac{2\xi_3+\eta_2(\xi_4,\xi_3)}{2}\right)+\Psi(x,\xi_3)\right]{}_0d_{q_1}x \\
&-\frac{1}{6\eta_1(\xi_2,\xi_1)\eta_2^2(\xi_3,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\xi_4,\xi_3)}\left[\Psi(\xi_1,y)+4\Psi\left(\frac{2\xi_1+\eta_1(\xi_2,\xi_1)}{2},y\right)+\Psi(\xi_2,y)\right]{}_0d_{q_1}y \\
&+\frac{1}{\eta_1^2(\xi_2,\xi_1)\eta_2^2(\xi_3,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\xi_2,\xi_1)}\int_{\xi_3}^{\xi_3+\eta_2(\xi_4,\xi_3)}\Psi(x,y)_0d_{q_1}x_0d_{q_1}y.
\end{aligned}
$$

By multiplying both sides of (7) by $\eta_1(\xi_2,\xi_1)\eta_2(\xi_3,\xi_4)$, we get the desired result. □

4. Main Results

In order to provide compact demonstration, we are capable to determine the two-variables q_1q_2-integral inequalities of Simpson-type involving the class higher-order generalized strongly preinvex and quasi-preinvex functions.

Theorem 5. *Assume that* $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is a mixed partial* q_1q_2*-differentiable function on* Λ° *(the interior of* Λ*) with* $\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}$ *being continuous and integrable on* $[\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)] \subset \Lambda^{\circ}$ *for* $\eta_1(\xi_2, \xi_1), \eta_2(\xi_4, \xi_3) > 0$ *and* $q_1, q_2 \in (0, 1)$*. If* $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|$ *is a coordinated higher-order generalized strongly preinvex function, then one has following inequality:*

$$|\Gamma_\Psi(\xi_1, \xi_2, \xi_3, \xi_4; q_1, q_2)| \leq \mathcal{K}\left[\begin{array}{c} \left(\mathcal{A}_{q_1} + \mathcal{D}_{q_1}\right)\left(\mathcal{A}_{q_2} + \mathcal{D}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\left(\mathcal{A}_{q_1} + \mathcal{D}_{q_1}\right)\left(\mathcal{B}_{q_2} + \mathcal{E}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\left(\mathcal{B}_{q_1} + \mathcal{E}_{q_1}\right)\left(\mathcal{A}_{q_2} + \mathcal{D}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\left(\mathcal{B}_{q_1} + \mathcal{E}_{q_1}\right)\left(\mathcal{B}_{q_2} + \mathcal{E}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\mu_1\left(\mathcal{C}_{q_1} + \mathcal{F}_{q_1}\right)\mu_2\left(\mathcal{C}_{q_2} + \mathcal{F}_{q_2}\right)\eta_1^{\sigma}(\xi_2, \xi_1)\eta_2^{\sigma}(\xi_4, \xi_3) \end{array}\right],$$

where $\mathcal{A}_{q_k}$*,*$\mathcal{B}_{q_k}$*,* $\mathcal{C}_{q_k}$*,* $\mathcal{D}_{q_k}$*,* $\mathcal{E}_{q_k}$*, and* $\mathcal{F}_{q_k}$ *are given by*

$$\mathcal{A}_{q_k} = \begin{cases} \frac{1-4q_k^3}{24(1+q_k)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{1}{3}, 1 \leq k \leq 2, \\ \\ \frac{1+12q_k+12q_k^2+36q_k^3}{216(1+q_k)\left(1+q_k+q_k^2\right)}, \frac{1}{3} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

$$\mathcal{B}_{q_k} = \begin{cases} \frac{1-2q_k-2q_k^2}{24(1+q_k)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{1}{3}, 1 \leq k \leq 2, \\ \\ \frac{18q_k+18q_k^2-7}{216(1+q_k)\left(1+q_k+q_k^2\right)}, \frac{1}{3} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

$$\mathcal{C}_{q_k} = \begin{cases} \frac{1-2q_k-2q_k^3-4q_k^4}{48(1+q_k)\left(1+q_k^2\right)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{1}{3}, 1 \leq k \leq 2, \\ \\ \frac{108q_k^4+54q_k^3+12q_k^2+54q_k-17}{1296(1+q_k)\left(1+q_k^2\right)\left(1+q_k+q_k^2\right)}, \frac{1}{3} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

$$\mathcal{D}_{q_k} = \begin{cases} \frac{-5+8q+8q_k^2-8q_k^3}{24(1+q_k)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{5}{6}, 1 \leq k \leq 2, \\ \\ \frac{12q_k+12q_k^2+5}{216(1+q_k)\left(1+q_k+q_k^2\right)}, \frac{5}{6} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

$$\mathcal{E}_{q_k} = \begin{cases} \frac{5-2q_k-2q_k^2}{8(1+q_k)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{5}{6}, 1 \leq k \leq 2, \\ \\ \frac{18q_k+18q_k^2+25}{216(1+q_k)\left(1+q_k+q_k^2\right)}, \frac{5}{6} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

$$\mathcal{F}_{q_k} = \begin{cases} \frac{5-2q_k+28q_k^2-2q_k^3-12q_k^4}{48(1+q_k)\left(1+q_k^2\right)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{5}{6}, 1 \leq k \leq 2, \\ \\ \frac{108q_k^4-54q_k^3+96q_k^2-54q_k+115}{1296(1+q_k)\left(1+q_k^2\right)\left(1+q_k+q_k^2\right)}, \frac{5}{6} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

Proof. Utilizing Lemma 2 and the fact that $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,w)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|$ is a coordinated higher-order generalized strongly preinvex function, we have

$$\begin{aligned} &|\Gamma_\Psi(\xi_1,\xi_2,\xi_3,\xi_4;q_1,q_2)| \leq \mathcal{K} \\ &\times \int_0^1\int_0^1 |\Pi_1(\varrho,q_1)\Pi_2(w,q_2)| \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1),\xi_3+w\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| {}_0d_{q_1}\varrho_0 d_{q_2}w \\ &\leq \mathcal{K}\int_0^1 |\Pi_2(w,q_2)| \left\{ \int_0^1 |\Pi_1(\varrho,q_1)| \left[\begin{array}{c} (1-\varrho)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3+w\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\varrho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,(\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ -\mu_1\varrho(1-\varrho)\eta_1^\sigma(\xi_2,\xi_1) \end{array}\right] {}_0d_{q_1}\varrho \right\} {}_0d_{q_2}\rho. \end{aligned} \tag{8}$$

Computing the q_1-integral on the right-hand side of (8), we have

$$\begin{aligned} &\int_0^1 |\Pi_1(\varrho,q_1)| \left\{ \begin{array}{c} (1-\varrho)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\varrho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ -\mu_1\varrho(1-\varrho)\eta_1^\sigma(\xi_2,\xi_1) \end{array}\right\} {}_0d_{q_1}\varrho \\ &= \int_0^{\frac{1}{2}} \left|q_1\varrho - \frac{1}{6}\right| \left[\begin{array}{c} (1-\varrho)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\varrho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ -\mu_1\varrho(1-\varrho)\eta_1^\sigma(\xi_2,\xi_1) \end{array}\right] {}_0d_{q_1}\varrho \\ &+ \int_{\frac{1}{2}}^1 \left|q_1\varrho - \frac{5}{6}\right| \left[\begin{array}{c} (1-\varrho)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\varrho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ -\mu_1\varrho(1-\varrho)\eta_1^\sigma(\xi_2,\xi_1) \end{array}\right] {}_0d_{q_1}\varrho \end{aligned}$$

$$\begin{aligned} &= \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \left[\int_0^{\frac{1}{2}}(1-\varrho)\left|q_1\varrho-\frac{1}{6}\right| {}_0d_{q_1}\varrho + \int_{\frac{1}{2}}^1(1-\varrho)\left|q_1\varrho-\frac{5}{6}\right| {}_0d_{q_1}\varrho\right] \\ &+ \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \left[\int_0^{\frac{1}{2}}\varrho\left|q_1\varrho-\frac{1}{6}\right| {}_0d_{q_1}\varrho + \int_{\frac{1}{2}}^1\varrho\left|q_1\varrho-\frac{5}{6}\right| {}_0d_{q_1}\varrho\right] \\ &-\mu_1\eta_1^\sigma(\xi_2,\xi_1)\left[\int_0^{\frac{1}{2}}\varrho(1-\varrho)\left|q_1\varrho-\frac{1}{6}\right| {}_0d_{q_1}\varrho + \int_{\frac{1}{2}}^1\varrho(1-\varrho)\left|q_1\varrho-\frac{5}{6}\right| {}_0d_{q_1}\varrho\right]. \end{aligned}$$

In view of the Definitions 11 and 12, we get

$$\mathcal{A}_{q_k} = \int_0^{\frac{1}{2}}(1-\varrho)\left|q_k\varrho-\frac{1}{6}\right| {}_0d_{q_k}\varrho = \begin{cases} \frac{1-4q_k^3}{24(1+q_k)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{1}{3}, 1 \leq k \leq 2, \\ \\ \frac{1+12q_k+12q_k^2+36q_k^3}{216(1+q_k)\left(1+q_k+q_k^2\right)}, \frac{1}{3} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

$$\mathcal{B}_{q_k} = \int_0^{\frac{1}{2}} \varrho \left| q_k\varrho - \frac{1}{6} \right| {}_0d_{q_k}\varrho = \begin{cases} \frac{1-2q_k-2q_k^2}{24(1+q_k)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{1}{3}, 1 \le k \le 2, \\ \frac{18q_k+18q_k^2-7}{216(1+q_k)\left(1+q_k+q_k^2\right)}, \frac{1}{3} \le q_k < 1, 1 \le k \le 2, \end{cases}$$

$$\mathcal{C}_{q_k} = \int_0^{\frac{1}{2}} \varrho(1-\varrho) \left| q_k\varrho - \frac{1}{6} \right| {}_0d_{q_k}\varrho = \begin{cases} \frac{1-2q_k-2q_k^3-4q_k^4}{48(1+q_k)\left(1+q_k^2\right)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{1}{3}, 1 \le k \le 2, \\ \frac{108q_k^4+54q_k^3+12q_k^2+54q_k-17}{1296(1+q_k)\left(1+q_k^2\right)\left(1+q_k+q_k^2\right)}, \frac{1}{3} \le q_k < 1, 1 \le k \le 2, \end{cases}$$

$$\mathcal{D}_{q_k} = \int_{\frac{1}{2}}^1 (1-\varrho) \left| q_k\varrho - \frac{5}{6} \right| {}_0d_{q_k}\varrho = \begin{cases} \frac{-5+8q+8q_k^2-8q_k^3}{24(1+q_k)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{5}{6}, 1 \le k \le 2, \\ \frac{12q_k+12q_k^2+5}{216(1+q_k)\left(1+q_k+q_k^2\right)}, \frac{5}{6} \le q_k < 1, 1 \le k \le 2, \end{cases}$$

$$\mathcal{E}_{q_k} = \int_{\frac{1}{2}}^1 \varrho \left| q_k\varrho - \frac{5}{6} \right| {}_0d_{q_k}\varrho = \begin{cases} \frac{5-2q_k-2q_k^2}{8(1+q_k)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{5}{6}, 1 \le k \le 2, \\ \frac{18q_k+18q_k^2+25}{216(1+q_k)\left(1+q_k+q_k^2\right)}, \frac{5}{6} \le q_k < 1, 1 \le k \le 2, \end{cases}$$

$$\mathcal{F}_{q_k} = \int_{\frac{1}{2}}^1 \varrho(1-\varrho) \left| q_k\varrho - \frac{5}{6} \right| {}_0d_{q_k}\varrho = \begin{cases} \frac{5-2q_k+28q_k^2-2q_k^3-12q_k^4}{48(1+q_k)\left(1+q_k^2\right)\left(1+q_k+q_k^2\right)}, 0 < q_k < \frac{5}{6}, 1 \le k \le 2, \\ \frac{108q_k^4-54q_k^3+96q_k^2-54q_k+115}{1296(1+q_k)\left(1+q_k^2\right)\left(1+q_k+q_k^2\right)}, \frac{5}{6} \le q_k < 1, 1 \le k \le 2, \end{cases}$$

$$= \left(\mathcal{A}_{q_1} + \mathcal{D}_{q_1}\right) \left| \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1, \xi_3 + \rho\eta_2\left(\xi_4,\xi_3\right))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho} \right| + \left(\mathcal{B}_{q_1} + \mathcal{E}_{q_1}\right) \left| \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2, \xi_3 + \rho\eta_2\left(\xi_4,\xi_3\right))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho} \right|$$
$$-\mu_1\left(\mathcal{C}_{q_1} + \mathcal{F}_{q_1}\right)\eta_1^2(\xi_2, \xi_1).$$

Putting the above calculations into (8), we obtain

$$\le \mathcal{K} \int_0^1 \left|\Pi_2(\rho, q_2)\right| \left\{ \begin{array}{c} \left(\mathcal{A}_{q_1} + \mathcal{D}_{q_1}\right) \left| \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho} \right| \\ + \left(\mathcal{B}_{q_1} + \mathcal{E}_{q_1}\right) \left| \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho} \right| \\ -\mu_1\left(\mathcal{C}_{q_1} + \mathcal{F}_{q_1}\right)\eta_1^2(\xi_2, \xi_1) \end{array} \right\} {}_0d_{q_2}\rho. \tag{9}$$

Similarly, by computing the q_2-integral, by using Definitions 11 and 12 on the right-hand side of (9), we have

$$\leq \mathcal{K}\left[\begin{array}{c} \left(\mathcal{A}_{q_1}+\mathcal{D}_{q_1}\right)\left(\mathcal{A}_{q_2}+\mathcal{D}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\left(\mathcal{A}_{q_1}+\mathcal{D}_{q_1}\right)\left(\mathcal{B}_{q_2}+\mathcal{E}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\left(\mathcal{B}_{q_1}+\mathcal{E}_{q_1}\right)\left(\mathcal{A}_{q_2}+\mathcal{D}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\left(\mathcal{B}_{q_1}+\mathcal{E}_{q_1}\right)\left(\mathcal{B}_{q_2}+\mathcal{E}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| \\ +\mu_1\left(\mathcal{C}_{q_1}+\mathcal{F}_{q_1}\right)\mu_2\left(\mathcal{C}_{q_2}+\mathcal{F}_{q_2}\right)\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3) \end{array}\right].$$

Hence, we deduce the required result. □

Theorem 6. *Assume that $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a mixed partial q_1q_2-differentiable function on Λ° (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}$ being continuous and integrable on $[\xi_1, \xi_1+\eta_1(\xi_2,\xi_1)] \times [\xi_3, \xi_3+\eta_2(\xi_4,\xi_3)] \subset \Lambda^{\circ}$ for $\eta_1(\xi_2,\xi_1), \eta_2(\xi_4,\xi_3) > 0$ and $q_1, q_2 \in (0,1)$. If $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau}$ is a coordinated higher-order generalized strongly preinvex function where $\tau > 1$, then one has the following inequality:*

$$|\Gamma_{\Psi}(\xi_1,\xi_2,\xi_3,\xi_4;q_1,q_2)| \leq \mathcal{K}\left[\left(\mathcal{G}_{q_1}+\mathcal{H}_{q_1}\right)\left(\mathcal{G}_{q_2}+\mathcal{H}_{q_2}\right)\right]^{1-\frac{1}{\tau}}$$

$$\times\left[\begin{array}{c} \left(\mathcal{A}_{q_1}+\mathcal{D}_{q_1}\right)\left(\mathcal{A}_{q_2}+\mathcal{D}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau} \\ +\left(\mathcal{A}_{q_1}+\mathcal{D}_{q_1}\right)\left(\mathcal{B}_{q_2}+\mathcal{E}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau} \\ +\left(\mathcal{B}_{q_1}+\mathcal{E}_{q_1}\right)\left(\mathcal{A}_{q_2}+\mathcal{D}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau} \\ +\left(\mathcal{B}_{q_1}+\mathcal{E}_{q_1}\right)\left(\mathcal{B}_{q_2}+\mathcal{E}_{q_2}\right)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau} \\ +\mu_1\left(\mathcal{C}_{q_1}+\mathcal{F}_{q_1}\right)\mu_2\left(\mathcal{C}_{q_2}+\mathcal{F}_{q_2}\right)\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3) \end{array}\right]^{\frac{1}{\tau}},$$

where

$$\mathcal{G}_{q_k} = \begin{cases} \frac{1-2q_k}{12(1+q_k)}, 0 < q_k < \frac{1}{3}, 1 \leq k \leq 2, \\ \\ \frac{6q_k-1}{36(1+q_k)}, \frac{1}{3} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

$$\mathcal{H}_{q_k} = \begin{cases} \frac{5-4q_k}{12(1+q_k)}, 0 < q_k < \frac{5}{6}, 1 \leq k \leq 2, \\ \\ \frac{4q_k-5}{12(1+q_k)}, \frac{5}{6} \leq q_k < 1, 1 \leq k \leq 2, \end{cases}$$

and $\mathcal{A}_{q_k}$, $\mathcal{B}_{q_k}$, $\mathcal{C}_{q_k}$, $\mathcal{D}_{q_k}$, $\mathcal{E}_{q_k}$, and $\mathcal{F}_{q_k}$ are given by the same expressions as described in Theorem 5.

Proof. Utilizing Lemma 2, the Hölder inequality and the fact that $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau}$ is a coordinated higher-order generalized strongly preinvex function, we have

$$|\Gamma_{\Psi}(\xi_1,\xi_2,\xi_3,\xi_4;q_1,q_2)| \leq \mathcal{K}$$
$$\times \int_0^1\int_0^1 |\Pi_1(\varrho,q_1)\Pi_2(\rho,q_2)| \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1),\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right| {}_0d_{q_1}\varrho\, {}_0d_{q_2}\rho.$$

$$\leq \mathcal{K}\left(\int_0^1\int_0^1 |\Pi_1(\varrho,q_1)||\Pi_2(\rho,q_2)|_0d_{q_1}\varrho_0d_{q_2}\rho\right)^{1-\frac{1}{\tau}}$$

$$\times\left\{\int_0^1\int_0^1 |\Pi_1(\varrho,q_1)||\Pi_2(\rho,q_2)|\left[\begin{array}{c}(1-\varrho)(1-\rho)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau}\\ +(1-\varrho)\rho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau}\\ +(1-\rho)\varrho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau}\\ +\varrho\rho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^{\tau}\\ +\mu_1\varrho(1-\varrho)\mu_2\rho(1-\rho)\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3)\end{array}\right]\right\}^{\frac{1}{\tau}} {}_0d_{q_1}\varrho_0d_{q_2}\rho.$$

In view of Definitions 11 and 12, we get

$$\mathcal{G}_{q_k}=\int_0^{\frac{1}{2}}\left|q_k\varrho-\frac{1}{6}\right|{}_0d_{q_k}\varrho=\begin{cases}\frac{1-2q_k}{12(1+q_k)}, 0<q_k<\frac{1}{3}, 1\leq k\leq 2,\\[2ex] \frac{6q_k-1}{36(1+q_k)}, \frac{1}{3}\leq q_k<1, 1\leq k\leq 2,\end{cases}$$

$$\mathcal{H}_{q_k}=\int_{\frac{1}{2}}^{1}\left|q_k\varrho-\frac{5}{6}\right|{}_0d_{q_k}\varrho=\begin{cases}\frac{5-4q_k}{12(1+q_k)}, 0<q_k<\frac{5}{6}, 1\leq k\leq 2,\\[2ex] \frac{4q_k-5}{12(1+q_k)}, \frac{5}{6}\leq q_k<1, 1\leq k\leq 2,\end{cases}$$

and obtain the integral expressions of $\mathcal{A}_{q_k}$, $\mathcal{B}_{q_k}$, $\mathcal{C}_{q_k}$, $\mathcal{D}_{q_k}$, $\mathcal{E}_{q_k}$, and $\mathcal{F}_{q_k}$, which have the same formulas as those given in Theorem 5.

We observe that

$$\begin{aligned}
&\int_0^1\int_0^1 |\Pi_1(\varrho,q_1)||\Pi_2(\rho,q_2)|_0d_{q_1}\varrho_0d_{q_2}\rho\\
&=\left(\int_0^1 |\Pi_1(\varrho,q_1)|_0d_{q_1}\varrho\right)\left(\int_0^1 |\Pi_2(\rho,q_2)|_0d_{q_2}\rho\right)\\
&=\left(\mathcal{G}_{q_1}+\mathcal{H}_{q_1}\right)\left(\mathcal{G}_{q_2}+\mathcal{H}_{q_2}\right),\\
&\int_0^1\int_0^1 (1-\varrho)(1-\rho)|\Pi_1(\varrho,q_1)||\Pi_2(\rho,q_2)|_0d_{q_1}\varrho_0d_{q_2}\rho\\
&=\left(\int_0^1 (1-\varrho)|\Pi_1(\varrho,q_1)|_0d_{q_1}\varrho\right)\left(\int_0^1 (1-\rho)|\Pi_2(\rho,q_2)|_0d_{q_2}\rho\right)\\
&=\left(\mathcal{A}_{q_1}+\mathcal{D}_{q_1}\right)\left(\mathcal{A}_{q_2}+\mathcal{D}_{q_2}\right),\\
&\int_0^1\int_0^1 (1-\varrho)\rho|\Pi_1(\varrho,q_1)||\Pi_2(\rho,q_2)|_0d_{q_1}\varrho_0d_{q_2}\rho\\
&=\left(\int_0^1 (1-\varrho)|\Pi_1(\varrho,q_1)|_0d_{q_1}\varrho\right)\left(\int_0^1 \rho|\Pi_2(\rho,q_2)|_0d_{q_2}\rho\right)\\
&=\left(\mathcal{A}_{q_1}+\mathcal{D}_{q_1}\right)\left(\mathcal{B}_{q_2}+\mathcal{E}_{q_2}\right),\\
&\int_0^1\int_0^1 \varrho(1-\rho)|\Pi_1(\varrho,q_1)||\Pi_2(\rho,q_2)|_0d_{q_1}\varrho_0d_{q_2}\rho\\
&=\left(\int_0^1 \varrho|\Pi_1(\varrho,q_1)|_0d_{q_1}\varrho\right)\left(\int_0^1 (1-\rho)|\Pi_2(\rho,q_2)|_0d_{q_2}\rho\right)\\
&=\left(\mathcal{B}_{q_1}+\mathcal{E}_{q_1}\right)\left(\mathcal{A}_{q_2}+\mathcal{D}_{q_2}\right),
\end{aligned}$$

$$\int_0^1 \int_0^1 \varrho\rho |\Pi_1(\varrho, q_1)||\Pi_2(\rho, q_2)| {}_0d_{q_1}\varrho {}_0d_{q_2}\rho$$
$$= \left(\int_0^1 \varrho |\Pi_1(\varrho, q_1)| {}_0d_{q_1}\varrho \right) \left(\int_0^1 \rho |\Pi_2(\rho, q_2)| {}_0d_{q_2}\rho \right)$$
$$(\mathcal{B}_{q_2} + \mathcal{E}_{q2}) \left(\mathcal{A}_{q_2} + \mathcal{D}_{q_2}\right),$$
$$\int_0^1 \int_0^1 \varrho\rho(1-\varrho)(1-\rho) |\Pi_1(\varrho, q_1)||\Pi_2(\rho, q_2)| {}_0d_{q_1}\varrho {}_0d_{q_2}\rho$$
$$= \left(\int_0^1 \varrho(1-\varrho) |\Pi_1(\varrho, q_1)| {}_0d_{q_1}\varrho \right) \left(\int_0^1 \rho(1-\rho) |\Pi_2(\rho, q_2)| {}_0d_{q_2}\rho \right)$$
$$= \left(\mathcal{C}_{q_1} + \mathcal{F}_{q_1}\right) \left(\mathcal{C}_{q_2} + \mathcal{F}_{q_2}\right).$$

Utilizing the values of the above $q_1 q_2$-integrals, we get our required inquality. □

Theorem 7. *Assume that $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a mixed partial $q_1 q_2$-differentiable function on Λ^o (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}$ being continuous and integrable on $[\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)] \subset \Lambda^o$ for $\eta_1(\xi_2, \xi_1), \eta_2(\xi_4, \xi_3) > 0$ and $q_1, q_2 \in (0, 1)$. If $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau$ is a coordinated higher-order generalized strongly quasi-preinvex function where $\tau > 1$, then one has the following inequality*

$$|\Gamma_\Psi(\xi_1, \xi_2, \xi_3, \xi_4; q_1, q_2)| \leq \mathcal{K} \left(\left(\mathcal{G}_{q_1} + \mathcal{H}_{q_1}\right) \left(\mathcal{G}_{q_2} + \mathcal{H}_{q_2}\right) \right)^{1-\frac{1}{\tau}}$$
$$\times \left\{ \left[\begin{array}{c} \max \left(\begin{array}{c} \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau, \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau, \\ \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau, \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau \end{array} \right) \\ \times \left(\mathcal{G}_{q_1} + \mathcal{H}_{q_1}\right) \left(\mathcal{G}_{q_2} + \mathcal{H}_{q_2}\right) \\ +\mu_1 \left(\mathcal{C}_{q_1} + \mathcal{F}_{q_1}\right) \mu_2 \left(\mathcal{C}_{q_2} + \mathcal{F}_{q_2}\right) \eta_1^\sigma(\xi_2, \xi_1) \eta_2^\sigma(\xi_4, \xi_3) \end{array} \right] \right\}^{\frac{1}{\tau}} {}_0d_{q_1}\varrho {}_0d_{q_2}\rho,$$

where $\mathcal{C}_{q_k}$, $\mathcal{F}_{q_k}$ and $\mathcal{G}_{q_k}$, $\mathcal{H}_{q_k}$ are given by the same expressions as described in Theorems 5 and 6.

Proof. Utilizing Lemma 2, the Hölder inequality and the fact that $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau$ is a coordinated higher-order generalized strongly quasi-preinvex function, we have

$$|\Gamma_\Psi(\xi_1, \xi_2, \xi_3, \xi_4; q_1, q_2)| \leq \mathcal{K}$$
$$\times \int_0^1 \int_0^1 |\Pi_1(\varrho, q_1)\Pi_2(\rho, q_2)| \left| \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_1 + \varrho\eta_1(\xi_2,\xi_1), \xi_3 + \rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho} \right| {}_0d_{q_1}\varrho {}_0d_{q_2}\rho.$$

$$\leq \mathcal{K} \left(\int_0^1 \int_0^1 |\Pi_1(\varrho, q_1)||\Pi_2(\rho, q_2)| {}_0d_{q_1}\varrho {}_0d_{q_2}\rho \right)^{1-\frac{1}{\tau}}$$
$$\times \left\{ \int_0^1 \int_0^1 |\Pi_1(\varrho, q_1)||\Pi_2(\rho, q_2)| \left[\begin{array}{c} \max \left(\begin{array}{c} \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau, \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau, \\ \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau, \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}h(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{q_1}\varrho_{\xi_3}\partial_{q_2}\rho}\right|^\tau \end{array} \right) \\ +\mu_1\varrho(1-\varrho)\mu_2\rho(1-\rho)\eta_1^\sigma(\xi_2, \xi_1)\eta_2^\sigma(\xi_4, \xi_3) \end{array} \right] \right\}^{\frac{1}{\tau}} {}_0d_{q_1}\varrho {}_0d_{q_2}\rho,$$

and can obtain the integral expressions of $\mathcal{C}_{q_k}$, $\mathcal{F}_{q_k}$ and $\mathcal{G}_{q_k}$, $\mathcal{H}_{q_k}$, which have the same formulae as those given in Theorems 5 and 6. This completes our result. □

5. Conclusions

A new concept of higher-order generalized strongly preinvex functions with different kind of preinvexities is presented. Meanwhile, we establish an identity connected with two-variable q_1q_2-differentiable functions. Further, We derived several new consequences for the Simpson-type integral inequities by using the coordinated higher-order generalized strongly preinvex and quasi-preinvex function concerning quantum integrals. Here, we emphasize that all the derived outcomes in the present paper endured preserving for strongly preinvex functions, certainly, which can be seen by the unique values of $\sigma = 2$, $\mu = 0$ and $\eta = (\xi_2, \xi_1)$.The newly introduced numerical approximation can be used to solve problems in fluid mechanics and aerodynamics. We hope that the novel strategies of this paper will inspire researchers working in functional analysis (regarding uniform smoothness of norms in Banach space) [48], probability and statistics (by assessing the human behavior in mathematical psychology) [19]. This is a new path for future research.

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