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Impulsive Evolution Equations with Causal Operators

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Abstract: In this paper, we establish sufficient conditions for the existence of mild solutions for certain impulsive evolution differential equations with causal operators in separable Banach spaces. We rely on the existence of mild solutions for the strongly continuous semigroups theory, the measure of noncompactness and the Schauder fixed point theorem. We consider the impulsive integro-differential evolutions equation and impulsive reaction diffusion equations (which could include symmetric kernels) as applications to illustrate our main results.

Keywords: impulsive evolution equation; measure of noncompactness; existence result

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_+$ be the set of non-negative real numbers. Let E be a real Banach space endowed with the norm $\|\cdot\|$. We denote by $C([0,T],E)$ the Banach space of continuous functions from $[0,T]$ into E endowed with the norm $\|u(\cdot)\| = \sup\limits_{0\leq t\leq a} \|u(t)\|$. The space of all strongly measurable functions $u(\cdot) : [0,T] \to E$ such that

$$\|u(\cdot)\|_p := \left(\int_0^T \|u(t)\|^p\right)^{1/p} < \infty$$

for $1 \leq p < \infty$ and $\|u(\cdot)\|_\infty := ess\sup\limits_{t\in[0,T]} \|u(t)\| < \infty$, will be denoted by $L^p([0,T],E)$. This is a Banach space with respect to the norm $\|u(\cdot)\|_p$. Let $PC([0,T],E)$ be the set of all functions $u(\cdot) : [0,T] \to E$ such that $u(\cdot)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and the right limit $u(t_k^+)$ exists for $k = 1,2,...,N$. Then $PC([0,T],E)$ is a Banach space with respect to the norm $||u(\cdot)||_{PC} : = \sup\{||u(t)||; t\in[0,T]\}$. Moreover, we have that $PC([0,T],E) \subset L^p([0,T],E)$ and

$$\|u(\cdot)\|_1 \leq T^{1-1/p} \|u(\cdot)\|_p \leq T^{2-1/p} \|u(\cdot)\|_{PC}.$$

Let us denote by $\mathcal{F}_1([0,T],X)$ the space of all the functions from $[0,T]$ into X, and by $\mathcal{F}_2([0,T],Y)$ the space of all the functions from $[0,T]$ into Y. Then an operator $\mathfrak{C} : \mathcal{F}_1([0,T],X) \to \mathcal{F}_2([0,T],Y)$ is called a causal operator if for each $\tau \in (0,T)$ and for all $u(\cdot), v(\cdot) \in \mathcal{F}_1([0,T],X)$ such that $u(t) = v(t)$ for $t \in [0,\tau]$, we have that $(\mathfrak{C}u)(t) = (\mathfrak{C}v)(t)$ for $t \in [0,\tau]$, and $(\mathfrak{C}0)(t) = 0$ for all $t \in [0,T]$.

The aim of this paper is to establish existence results for mild solutions of the following impulsive evolution equation with the causal operator

$$\begin{cases} u'(t) = Au(t) + (\mathfrak{C}u)(t) \text{ for } t \in [0,T]\backslash\{t_1,...,t_N\} \\ u(t_k^+) = u(t_k^-) + (\mathfrak{I}_k u)(t_k^-) \\ u(0) = u_0, \end{cases} \tag{1}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t); t \geq 0\}$ and $\mathfrak{C} : PC([0,T],E) \to L^p([0,T],E)$ is a continuous causal operator; here $1 \leq p \leq \infty$, $N \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < ... < t_N < t_{N+1} = T$ and $\mathfrak{I}_k : PC([0,T],E) \to PC([0,T],E)$ is a continuous causal operator for each $k = 1, 2, ..., N$.

The theory of differential equations involving causal operators allows a unified treatment for general classes of differential equations, such as: Ordinary differential equations, differential equations with delay, integro-differential equations, Volterra integral equations and so on. The term causal operator (or Volterra abstract operator) was introduced by Tonelli [1], and the theory of these classes of operators was developed by Tychonoff [2]. The class of causal operators is quite large and it includes a number of operators that are used in mathematical modeling of some phenomena in engineering and physics. An important class of causal operators is the class of superposition operators or Nemytskij operators (see [3]) $\mathfrak{C} : L^p([0,T],E) \to L^p([0,T],E)$ defined by $(\mathfrak{C}u)(t) := F(t,u(t))$, where $F : [0,T] \times E \to E$ is a Caratheodory function. If $\sigma > 0$, then $\mathfrak{C} : C([-\sigma,T],E) \to L^p([0,T],E)$ defined by $(\mathfrak{C}u)(t) := F(t,u(t),u(t-\sigma))$ is another example of a causal operator. A more general example of causal operators is the operator $\mathfrak{C} : C([-\sigma,T],E) \to L^p([0,T],E)$ defined by

$$(\mathfrak{C}u)(t) := F\left(t, u(t), u(t-\sigma), \int_{t-\sigma}^{t} K(t,s,u(s))ds\right).$$

Several examples of causal operators and their applications can be found in the monograph [4]. Although it does not specifically study the theory of causal operators, several monographs, such as [5–9], address some aspects of differential equations involving causal operators. Detailed studies on differential equations with causal operators in finite dimensional spaces can be found in the monographs [4,10–12]. Applications of differential equations with causal operators in optimal control, adaptive control or hysteresis phenomena can be found in the papers [13–20]. Theoretical aspects regarding existence, stability or periodicity of solutions of differential equations with causal operators in finite or infinite dimensional spaces were presented in a series of works, such as: [21–39].

The study of evolution equations with causal operators was first presented in [40], where an existence result was obtained and some applications were given, but impulsive evolution differential equations with causal operators has not yet been studied. In this paper we study the class of impulsive evolution equations involving causal operators. In Section 2 we recall some results on C_0-semigroups of linear operators and some properties of the Hausdorff measure of noncompactness. In Section 3 we obtain the existence of mild solutions for a class of impulsive evolution equations with causal operators. Also, we show that a mild solution can be obtained as the limit of a sequence of successive approximations. In the last section we give some applications.

2. Preliminaries

We denote the space of all bounded linear operators acting on a Banach space E by $\mathcal{L}(E)$. We recall that a family $\{\mathcal{T}(t); t \geq 0\} \subset \mathcal{L}(E)$ is called a C_0-semigroup if the following three properties are satisfied:

(a) $\mathcal{T}(0) = I$, the identity operator on E;
(b) $\mathcal{T}(t)\mathcal{T}(s) = \mathcal{T}(t+s)$ for all $t, s \geq 0$;
(c) $\lim_{t \to 0^+} \mathcal{T}(t)u = u$ for all $u \in E$.

The infinitesimal generator of the C_0-semigroup $\{\mathcal{T}(t); t \geq 0\}$ is the operator $A : D(A) \subset E \to E$, defined by

$$D(A) = \{u \in E; \lim_{h \to 0^+} \frac{\mathcal{T}(h)u - u}{h} \text{ exists}\}$$

and

$$Au = \lim_{h \to 0^+} \frac{\mathcal{T}(h)u - u}{h}, \quad u \in D(A).$$

The generator is always a closed, densely defined operator. For further details on the theory of the C_0-semigroups see [41,42].

We denote by $\chi(B)$ the Hausdorff measure of non-compactness of a nonempty bounded set $B \subset E$, and it is defined by [43]:

$$\chi(B) = \inf\{\varepsilon > 0; B \text{ admits a finite cover by balls of radius} \leq \varepsilon\}.$$

We recall some properties of χ (see [43]). If A, B are bounded subsets of E, then

(χ1) $\chi(B) = 0$ if and only if $\overline{B}$ is compact;

(χ2) $\chi(B) = \chi(\overline{B}) = \chi(\overline{conv}(B))$;

(χ3) $\chi(\lambda B) = |\lambda| \chi(B)$ for every $\lambda \in \mathbb{R}$;

(χ4) $\chi(B) \leq \chi(C)$ if $B \subset C$;

(χ5) $\chi(\{x\} \cup B) = \chi(B)$ for every $x \in E$;

(χ6) $\chi(B + C) = \chi(B) + \chi(C)$.

(χ7) Generalized Cantor's intersection property : If $\{B_n\}_{n \geq 1}$ is a decreasing sequence of bounded closed nonempty subsets of E and $\lim_{n \to \infty} \chi(B_n) = 0$, then $\bigcap_{n=1}^{\infty} B_n$ is a nonempty and compact subset of E (see [44]).

Remark 1. *If $diam(B) = \sup\{||x - y||; x, y \in B\}$ is the diameter of the bounded set A, then we have that $\chi(B) \leq diam(B)$ and $\chi(B) \leq 2d$ if $\sup_{x \in B} ||x|| \leq d$.*

In the following, we denote by χ_c the Hausdorff measure of non-compactness in the space $C([0, T], E)$. Then it is well known that for every bounded set $B \subset C([0, T], E)$ we have

$$\chi(B(t)) \leq \chi_c(B),$$

for every $t \in [0, T]$, where $B(t) := \{u(t) : u \in B\}$. Moreover, for every bounded and equicontinuous set $B \subset C([0, T], E)$ we have (see [43])

$$\chi_c(B) = \sup_{0 \leq t \leq T} \chi(B(t)). \tag{2}$$

For each $k = 0, 1, 2, ..., N$ and $u(\cdot) \in PC([0, T], E)$ we set $J_k = (t_k, t_{k+1}]$, $\overline{J}_k = [t_k, t_{k+1}]$ and introduce the function $\widetilde{u}_k(\cdot) \in C(\overline{J}_k, E)$ defined by

$$\widetilde{u}_k(t) = \begin{cases} u(t), \text{ for } t \in J_k \\ u(t_k^+), \text{ for } t = t_k. \end{cases} \tag{3}$$

Also, for $B \subset PC([0,T],E)$ and $k = 0,1,2,...,N$, let us set $\widetilde{B}^k := \{\widetilde{u}_k(\cdot) \in C(\overline{J}_k, E); u(\cdot) \in B\}$. If we denote by χ_k the Hausdorff measure of noncompactness on $C(\overline{J}_k, E)$, then

$$\chi_{PC}(B) := \max_{0 \le k \le N} \chi_k \left(\widetilde{B}^k\right), B \subset PC([0,T],E)$$

defines the Hausdorff measure of noncompactness on $PC([0,T],E)$. Moreover, it is easy to see that

$$\chi_{PC}(B) = \sup_{t \in [0,T]} \chi\left(B\left(t\right)\right)$$

for every equicontinuous subset $B \subset PC([0,T],E)$.

Lemma 1 ([45], Lemma 2.1)**.** *A set $B \subset PC([0,T],E)$ is relatively compact in $PC([0,T],E)$ if and only if $\widetilde{B}^k$ is relatively compact in $C(\overline{J}_k, E)$ for every $k = 0,1,2,...,N$.*

Lemma 2 ([46], p. 125)**.** *If $B \subset E$ is a nonempty bounded set, then for every $\varepsilon > 0$ there exists a sequence $\{x_n\}_{n \ge 1}$ in E such that*

$$\chi(B) \le 2\chi\left(\{x_n; n \ge 1\}\right) + \varepsilon.$$

Lemma 3 ([47], Lemma 2.2)**.** *Let $\{u_n(\cdot); n \ge 1\}$ be a subset in $L^1([0,T],E)$ for which there exists $m(\cdot) \in L^1([0,T],\mathbb{R}_+)$ such that $\|u_n(t)\| \le m(t)$ for each $n \ge 1$ and for a.e. $t \in [0,T]$. Then the function $t \mapsto \chi(t) := \chi(\{u_n(t); n \ge 1\})$ is integrable on $[0,T]$ and, for each $t \in [0,T]$, we have*

$$\chi\left(\left\{\int_0^t u_n(s)ds; n \ge 1\right\}\right) \le \int_0^t \chi(s)ds.$$

3. Existence Result

Consider the following impulsive differential equation

$$\begin{cases} u'(t) = Au(t) + (\mathfrak{C}u)(t) \text{ for } t \in [0,T] \backslash \{t_1, ..., t_N\} \\ u(t_k^+) = u(t_k^-) + (\mathfrak{I}_k u)(t_k^-) \\ u(0) = u_0, \end{cases} \tag{4}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(t); t \ge 0\}$ and $\mathfrak{C} : PC([0,T],E) \to L^p([0,T],E)$ is a continuous causal operator; here $1 \le p \le \infty$, $N \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < ... < t_N < t_{N+1} = T$ and $\mathfrak{I}_k : PC([0,T],E) \to PC([0,T],E)$ is a continuous causal operator for each $k = 1,2,...,N$.

A function $u(\cdot) \in PC([0,T],E)$ is called a mild solution of (4) if it satisfies

$$u(t) = \mathcal{T}(t)u(0) + \int_0^t \mathcal{T}(t-s)(\mathfrak{C}u)(s)ds + \sum_{0<t_k<t} \mathcal{T}(t-t_k)\mathfrak{I}_k(u(t_k)), \ t \in [0,T].$$

Let us introduce the following conditions.

(H1) For each $k = 1,2,...,N$, $\mathfrak{I}_k : PC([0,T],E) \to PC([0,T],E)$ is continuous and a compact operator and there exists $c_k > 0$, with $M \sum\limits_{0<t_k<T} c_k < 1$, such that for each $u(\cdot) \in PC([0,T],E)$ we have

$$\|(\mathfrak{I}_k u)(t)\| \le c_k \|u(t)\| \ \text{ for every } t \in [0,T],$$

where $M := \sup\limits_{0 \le t \le T} \|\mathcal{T}(t)\|$.

(H2) (**a**) There exists a function $\xi(\cdot,\cdot) : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\xi(\cdot,\eta) \in L^1([0,T],\mathbb{R}_+)$ for every $\eta \in \mathbb{R}_+$, $\xi(t,\cdot)$ is continuous and increasing on $\mathbb{R}_+$ for a.e. $t \in [0,T]$ such that

$$\limsup_{\eta\to\infty} \frac{M}{\eta}\left(\|u_0\| + \int_0^T \xi(s,\eta)ds\right) < 1 - M \sum_{0<t_k<T} c_k \tag{5}$$

and

$$\|(\mathfrak{C}u)\,(t)\| \le \xi(t,||u(t)||), \text{ for a.e. } t \in [0,T], \tag{6}$$

for each $u(\cdot) \in PC([0,T],E)$.

(**b**) For each bounded subsets $B \subset PC([0,T],E)$ there exists $\gamma_B(\cdot) \in L^1([0,T],\mathbb{R}_+)$ such that

$$\int_0^T \gamma_B(t)dt < \frac{1}{2MT} \tag{7}$$

and

$$\chi((\mathfrak{C}B)(t)) \le \int_0^t \gamma_B(s)\chi(B(s))ds \text{ for } t \in [0,T], \tag{8}$$

where $(\mathfrak{C}B)(t) := \{(\mathfrak{C}u)(t) : u(\cdot) \in B\}$.

Theorem 1. *Let $\mathfrak{C} : PC([0,T],E) \to L^p([0,T],E)$ be a continuous causal operator such that conditions (H1) and (H2) hold. If $A : D(A) \subset E \to E$ is the generator of a C_0-semigroup $\{\mathcal{T}(t); t \ge 0\}$, then the evolution Equation (4) has at least one mild solution on $[0,T]$.*

Proof. First, we remark that there exists an $r > 0$ such that

$$M\,\|u_0\| + M\int_0^T \xi(s,r)ds + Mr \sum_{0<t_k<T} c_k < r. \tag{9}$$

Indeed, from (5) it follows that there exists $\eta_0 > 0$ such that

$$\frac{M}{\eta}\left(\|u_0\| + \int_0^T \xi(s,\eta)ds\right) < 1 - M \sum_{0<t_k<T} c_k,$$

for every $\eta > \eta_0$, so that

$$M\,\|u_0\| + M\int_0^T \xi(s,\eta)ds + M\eta \sum_{0<t_k<T} c_k < \eta,$$

for every $\eta > \eta_0$. Consequently, we can choose a $r > \eta_0$ such that (9) holds. Now, let

$$B_0 = \{u(\cdot) \in PC([0,T],E); \|u(\cdot)\|_{PC} \le r\}, \tag{10}$$

and define the operator $\mathfrak{F} : B_0 \to PC([0,T]),E$ by

$$(\mathfrak{F}u)(t) := \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)(\mathfrak{C}u)(s)ds + \sum_{0<t_k<t} \mathcal{T}(t-t_k)(\mathfrak{I}_k u)(t_k), \tag{11}$$

for $t \in [0,T]$. Since $\xi(t,\cdot)$ is increasing on $\mathbb{R}_+$ for a.e. $t \in [0,T]$ for every $u(\cdot) \in B_0$, using (5) we have

$$\|(\mathfrak{F}u)(t)\| \leq \|\mathcal{T}(t)u_0\| + \int_0^t \|\mathcal{T}(t-s)(\mathfrak{C}u)(s)\|\,ds + \sum_{0<t_k<t} \|\mathcal{T}(t-t_k)(\mathfrak{I}_k u)(t_k)\|$$

$$\leq M\,\|u_0\| + M\int_0^t \|(\mathfrak{C}u)(s)\|\,ds + M\sum_{0<t_k<t} c_k||u(t_k)||$$

$$\leq M\,\|u_0\| + M\int_0^t \xi(s,\|u(s)\|)ds + M\sum_{0<t_k<t} c_k||u(t_k)||$$

$$\leq M\,\|u_0\| + M\int_0^T \xi(s,\|u(s)\|)ds + M\sum_{0<t_k<T} c_k||u(t_k)||$$

$$\leq M\,\|u_0\| + M\int_0^T \xi(s,r)ds + Mr\sum_{0<t_k<T} c_k < r,$$

so that $\mathfrak{F}(B_0) \subset B_0$. We notice that $\|(\mathfrak{C}u)(t)\| \leq \psi(t)$ for a.e. on $[0,T]$, for every $u(\cdot) \in B_0$, where $\psi(\cdot) := \xi(\cdot,r) \in L^1([0,T],\mathbb{R}_+)$. Let $B_1 := \mathfrak{F}B_0$. Next, we will show that $\widetilde{B}_1^k$ is equicontinuous on $\overline{J}_k$ for every $k = 1,2,...,N$. For this, we shall write the operator $\mathfrak{F}$ as $(\mathfrak{F}u)(t) = (\mathfrak{F}_1 u)(t) + (\mathfrak{F}_2 u)(t)$, where

$$(\mathfrak{F}_1 u)(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-\tau)(\mathfrak{C}u)(\tau)d\tau,$$

$$(\mathfrak{F}_2 u)(t) = \sum_{0<t_k<t} \mathcal{T}(t-t_k)(\mathfrak{I}_k u)(t_k)$$

for $t \in [0,T]$.

First, we show that $G_1 := \mathfrak{F}_1 B_0$ is equicontinuous on $[0,T]$. Let $\varepsilon > 0$. Since $t \mapsto \mathcal{T}(t)u_0$ continuous on $[0,T]$ (see [42], Corollary 2.3), then there exists $\delta_1 = \delta_1(\varepsilon/5) > 0$ such that

$$\|\mathcal{T}(t+h)u_0 - \mathcal{T}(t)u_0\| \leq \frac{\varepsilon}{5} \tag{12}$$

for every $t \in [0,T]$ and $h \in \mathbb{R}$ with $|h| < \delta_1$ and $t+h \in [0,T]$. On $[0,T]$, the function $t \mapsto \int_0^t \psi(s)ds$ is uniformly continuous and thus there exists $\delta_2 = \delta_2(\varepsilon/5M) > 0$ such that

$$\left|\int_t^{t+h} \psi(\tau)d\tau\right| < \frac{\varepsilon}{5M} \tag{13}$$

for every $t \in [0,T]$ and $h \in \mathbb{R}$ with $|h| < \delta_2$ and $t+h \in [0,T]$. Then for $t = 0$ we have

$$\|(\mathfrak{F}_1 u)(h) - (\mathfrak{F}_1 u)(0)\| = \left\|\mathcal{T}(h)u_0 + \int_0^h [\mathcal{T}(h-\tau)(\mathfrak{C}u)(\tau)d\tau - u_0\right\|$$

$$\leq \|\mathcal{T}(h)u_0 - u_0\| + \int_0^h \|\mathcal{T}(h-\tau)(\mathfrak{C}u)(\tau)\|\,d\tau \leq \|\mathcal{T}(h)u_0 - u_0\| +$$

$$+M\int_0^h \leftarrow(\tau)d\tau < \tfrac{2\varepsilon}{5} < \varepsilon,$$

for each $u \in B_0$ and $h \in (0,T]$ with $h < \min\{\delta_1,\delta_2\}$. It follows that G_1 is equicontinuous at $t = 0$. Next, take $t \in (0,T]$ and let us choose $0 < \eta < \delta_2/2$ such that $t - \eta \in [0,T]$. For each $u \in B_0$ and $h \in \mathbb{R}$ such that $t+h \in [0,T]$ we have

$$\begin{aligned}&\|(\mathfrak{F}_1 u)(t+h)-(\mathfrak{F}_1 u)(t)\| \le \|(\mathfrak{F}_1 u)(t)-\mathcal{T}(\eta)[(\mathfrak{F}_1 u)(t-\eta)]\|\\ &+\|\mathcal{T}(\eta)[(\mathfrak{F}_1 u)(t-\eta)]-\mathcal{T}(\eta+h)[(\mathfrak{F}_1 u)(t-\eta)]\|\\ &+\|\mathcal{T}(\eta+h)[(\mathfrak{F}_1 u)(t-\eta)]-(\mathfrak{F}_1 u)(t+h)\|\,.\end{aligned} \tag{14}$$

Since

$$\begin{aligned}&\|(\mathfrak{F}_1 u)(t)-\mathcal{T}(\eta)[(\mathfrak{F}_1 u)(t-\eta)]\| = \Big\|\mathcal{T}(t)u_0+\textstyle\int_0^t \mathcal{T}(t-\tau)(\mathfrak{C}u)(\tau)d\tau-\\ &-\mathcal{T}(\eta)\Big[\mathcal{T}(t-\eta)u_0+\textstyle\int_0^{t-\eta}[\mathcal{T}(t-\eta-\tau)(\mathfrak{C}u)(\tau)d\tau\Big]\Big\|\\ &=\Big\|\textstyle\int_0^t \mathcal{T}(t-\tau)(\mathfrak{C}u)(\tau)d\tau-\int_0^{t-\eta}\mathcal{T}(t-\tau)(\mathfrak{C}u)(\tau)d\tau\Big\|\\ &=\Big\|\textstyle\int_{t-\eta}^t \mathcal{T}(t-\tau)(\mathfrak{C}u)(\tau)d\tau\Big\| \le M\int_{t-\eta}^t\psi(\tau)d\tau,\end{aligned}$$

it follows that

$$\|(\mathfrak{F}_1 u)(t)-\mathcal{T}(\eta)[(\mathfrak{F}_1 u)(t-\eta)]\| \le M\int_{t-\eta}^t \psi(\tau)d\tau<\frac{\varepsilon}{5} \tag{15}$$

for each $u\in B_0$. By similar reasoning, we obtain

$$\|\mathcal{T}(\eta+h)[(\mathfrak{F}_1 u)(t-\eta)]-(\mathfrak{F}_1 u)(t+h)\| \le M\int_{t-\eta}^{t+h}\psi(\tau)d\tau,$$

and so, by (13), we conclude that

$$\|\mathcal{T}(\eta+h)[(\mathfrak{F}_1 u)(t-\eta)]-(\mathfrak{F}_1 u)(t+h)\| \le M\int_{t-\eta}^{t+h}\psi(\tau)d\tau<\frac{\varepsilon}{5} \tag{16}$$

for each $u\in B_0$ and $h\in\mathbb{R}$ with $|h|<\eta$ and $t+h\in[0,T]$. Furthermore, we have

$$\begin{aligned}&\|\mathcal{T}(\eta)[(\mathfrak{F}_1 u)(t-\eta)]-\mathcal{T}(\eta+h)[(\mathfrak{F}_1 u)(t-\eta)]\| \le \|\mathcal{T}(t+h)u_0-\mathcal{T}(t)u_0\|\\ &+\textstyle\int_0^{t-\eta}\|\mathcal{T}(t+h-\tau)-\mathcal{T}(t-\tau)\|\,\psi(\tau)d\tau \le \|\mathcal{T}(t+h)u_0-\mathcal{T}(t)u_0\|+\\ &+2M\textstyle\int_0^{t-\eta}\psi(\tau)d\tau\le\frac{3\varepsilon}{5},\end{aligned} \tag{17}$$

that is,

$$\|\mathcal{T}(\eta)[(\mathfrak{F}_1 u)(t-\eta)]-\mathcal{T}(\eta+h)[(\mathfrak{F}_1 u)(t-\eta)]\| \le \frac{2\varepsilon}{5} \tag{18}$$

for each $u\in B_0$ and $h\in\mathbb{R}$ with $|h|<\min\{\eta,\delta_1,\delta_2\}$ and $t+h\in[0,T]$. Now, using (15), (16) and (18), from (14) it follows that

$$\|(\mathfrak{F}_1 u)(t+h)-(\mathfrak{F}_1 u)(t)\|<\varepsilon$$

for each $u\in B_0$ and $h\in\mathbb{R}$ with $|h|<\min\{\eta,\delta_1,\delta_2\}$ and $t+h\in[0,T]$. Thus G_1 is equicontinuous on $[0,T]$. From this it follows that $\widetilde{G}_1^k$ is equicontinuous on $\overline{J}_k$ for every $k=1,2,...,N$. Next, we show that, for a given $\nu\in\{1,2,...,N\}$, the set $\widetilde{G}_2^\nu$ is equicontinuous on $\overline{J}_\nu$, where $G_2:=\mathfrak{F}_2B_0$. Since $\mathfrak{I}_k$ is a compact operator, $\mathfrak{I}_kB_0$ is a relatively compact set in $PC([0,T],E)$ and so, by Lemma 1 $\mathfrak{I}_k\widetilde{B}_0$ is a relatively compact set in $C(\overline{J}_k,E)$ for each $k=1,2,...,N$. Using the Ascoli–Arzela theorem, from the compactness of $\mathfrak{I}_k\widetilde{B}_0$ in $C(\overline{J}_k,E)$, it follows that $(\mathfrak{I}_k\widetilde{B}_0)(t)$ is relatively compact in E for every $t\in\overline{J}_k$ and $k=1,2,...,N$. In particular, $(\mathfrak{I}_k\widetilde{B}_0)(t_k)$ is relatively compact for every $k=1,2,...,N$, and thus $K:=\bigcup\limits_{k=1}^{N}(\mathfrak{I}_k\widetilde{B}_0)(t_k)$ is relatively compact in E. Since $(t,x)\mapsto T(t)x$ is jointly continuous from $[0,\infty)\times K$ to E, it follows that there exists a $\delta>0$ such that

$$\|T(t-t_k)x - T(s-t_k)x\| < \varepsilon/N,\ x \in K$$

for every t_k, $k = 1, 2, ..., N$, $t, s \in \overline{J}_k$ with $|t-s| < \delta$. Next, for every $u(\cdot) \in B_0$, $t, s \in \overline{J}_\nu$ with $|t-s| < \delta$, we have

$$\|(\mathfrak{F}_2\widetilde{u}_\nu)(t) - (\mathfrak{F}_2\widetilde{u}_\nu)(s)\| = \|(\mathfrak{F}_2 u)(t) - (\mathfrak{F}_2 u)(s)\|$$

$$= \left\| \sum_{0<t_k<t} \mathcal{T}(t-t_k)(\mathfrak{I}_k u)(t_k) - \sum_{0<t_k<t} \mathcal{T}(s-t_k)(\mathfrak{I}_k u)(t_k) \right\|$$

$$\leq \sum_{k=1}^{\nu} \|T(t-t_k)(\mathfrak{I}_k u)(t_k) - \mathcal{T}(s-t_k)(\mathfrak{I}_k u)(t_k)\| < \varepsilon,$$

so that $\widetilde{G}_2^\nu$ is equicontinuous on $\overline{J}_\nu$. Since $\widetilde{B}_1^k = \widetilde{G}_1^k + \widetilde{G}_2^k$, $k = 1, 2, ..., N$, it follows that $\widetilde{B}_1^k$ is equicontinuous on $\overline{J}_k$ for every $k = 1, 2, ..., N$. Next, for each $n \geq 1$, we define $B_n = \overline{conv}(\mathfrak{F}B_{n-1})$. Then, for every $n \geq 1$, $B_n \subset PC([0,T], E)$ is a bounded, closed and convex set. Now, from $\mathfrak{F}B_0 \subset B_0$, it follows that

$$B_1 = \overline{conv}(\mathfrak{F}B_0) \subset \overline{conv}(B_0) = B_0.$$

If we suppose that $B_\nu \subset B_{\nu-1}$ for a given $\nu > 1$, then

$$B_{\nu+1} = \overline{conv}(\mathfrak{F}B_\nu) \subset \overline{conv}(\mathfrak{F}B_{\nu-1}) = B_\nu$$

and thus, by induction it follows that $B_n \subset B_{n-1}$ for every $n \geq 1$. Moreover, it is easy to see that $\widetilde{B}_n^k$ is equicontinuous on $\overline{J}_k$ for each $k = 1, 2, ..., N$ and for every $n \geq 1$. Now, we will show that $\chi_{PC}(B_n) \to 0$ as $n \to \infty$. From Lemma 2, it follows that there exists a sequence $\{v_m(\cdot)\}_{m\geq 1}$ in $\mathfrak{F}B_{n-1}$ such that

$$\chi_{PC}(B_n) = \chi_{PC}(\mathfrak{F}B_{n-1}) \leq 2\chi_{PC}(V) + \varepsilon,$$

where $V := \{v_m(\cdot); m \geq 1\}$. From the above inequality it follows that

$$\chi_{PC}(B_n) \leq 2 \max_{0\leq k\leq N} \chi_k(\overline{V}_n^k) + \varepsilon.$$

Since for each $k = 1, 2, ..., N$, the equicontinuity of $\overline{V}_n^k$ and Lemma 3 imply $\chi_k(\overline{V}_n^k) = \sup_{t\in J_k} \chi(V(t))$, we obtain

$$\begin{aligned} \chi_{PC}(B_n) &\leq 2 \max_{0\leq k\leq N} \left[\sup_{t\in J_k} \chi(V(t)) \right] + \varepsilon \leq 2 \sup_{t\in[0,T]} \chi(V(t)) + \varepsilon \\ &= 2 \sup_{t\in[0,T]} \chi(\{v_m(t); m \geq 1\}) + \varepsilon. \end{aligned}$$

Further, let $\{u_m(\cdot)\}_{m\geq 1}$ be a sequence in B_{n-1} such that $v_m(\cdot) = (\mathfrak{F}u_m)(\cdot)$, $m \geq 1$. If we put $V := \{u_m(\cdot); m \geq 1\}$ and $V(t) := \{u_m(t); m \geq 1\}$, $t \in [0,T]$, then from the previous inequality it follows that

$$\begin{aligned}\chi_{PC}(B_n) &\leq \varepsilon + 2\chi\left(\mathcal{T}(t-s)u_0 + \int_0^t \mathcal{T}(t-s)(\mathfrak{C}V)(s)ds\right.\\ &\quad \left. + \sum_{0<t_k<t-T/n} \mathcal{T}(t-t_k)(\mathfrak{I}_k V)(t_k)\right) \qquad (19)\\ &\leq \varepsilon + 2\chi\left(\int_0^t \mathcal{T}(t-s)(\mathfrak{C}V)(s)ds\right)\\ &\quad +2\chi\left(\sum_{0<t_k<t} \mathcal{T}(t-t_k)(\mathfrak{I}_k V)(t_k)\right). \qquad (20)\end{aligned}$$

Let $t \in [0,T]$ be fixed. Since

$$\|\mathcal{T}(t-s)\,(\mathfrak{C}u_m)\,(s)\| \leq M\psi(t) \text{ for a.e. } s \in [0,t],$$

for all $m \geq 1$, and

$$\begin{aligned}\chi(\{(\mathfrak{C}u_m)\,(s); m &\geq 1\}) \leq \int_0^s \gamma_V(\tau)\chi(\{u_m(\tau); m \geq 1\})d\tau\\ &\leq \int_0^s \gamma_V(\tau)\chi(B_{n-1}(\tau))d\tau \leq \chi_{PC}(B_{n-1})\int_0^s \gamma_V(\tau)d\tau\end{aligned}$$

for a.e. $s \in [0,t]$, by Lemma 3 it follows that

$$\begin{aligned}\chi\left(\int_0^t \mathcal{T}(t-s)(\mathfrak{C}V)(s)ds\right) &\leq \int_0^t \chi\left(\mathcal{T}(t-s)(\mathfrak{C}V)(s)\right)ds\\ &\leq 2M\int_0^t \chi\left((\mathfrak{C}V)(s)\right)ds\\ &\leq 2M\int_0^t\int_0^s \gamma_V(\tau)\chi\left((V(\tau)\right)d\tau ds\\ &\leq 2M\chi_{PC}(B_{n-1})\int_0^t\int_0^s \gamma_V(\tau)d\tau ds \qquad (21)\\ &= 2M\chi_{PC}(B_{n-1})\int_0^t\int_\tau^t \gamma_V(\tau)dsd\tau\\ &= 2M\chi_{PC}(B_{n-1})\int_0^t (t-\tau)\gamma_V(\tau)d\tau\\ &\leq 2MT\chi_{PC}(B_{n-1})\int_0^T \gamma_V(\tau)d\tau.\end{aligned}$$

Also, by the continuity of the operators $T(t)$ and by the compactness of the operators $\mathfrak{I}_k$, it follows that the set $\{\mathcal{T}(t-t_k)(\mathfrak{I}_k V)(t_k)\}$ is relatively compact for every $t \in [0,T]$. Therefore, we have that

$$\chi\left(\sum_{0<t_k<t} \mathcal{T}(t-t_k)(\mathfrak{I}_k V)(t_k)\right) \leq \sum_{0<t_k<t} \chi\left(\mathcal{T}(t-t_k)(\mathfrak{I}_k V)(t_k)\right) = 0. \qquad (22)$$

From (19), (21), and (22), we obtain

$$\chi_{PC}(B_n) \leq \varepsilon + \rho\chi_{PC}\left(B_{n-1}\right), \qquad (23)$$

where

$$\rho := 2MT\int_0^T \gamma_V(s)ds < 1.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\chi_{PC}(B_n) \leq \rho\chi_{PC}\left(B_{n-1}\right).$$

Also

$$\chi_{PC}(B_n) \leq \rho^{n-1}\chi_{PC}\left(B_1\right).$$

Since the last inequality is true for every $n \geq 1$ and $0 < \rho < 1$, passing to the limit as $n \to \infty$, we obtain $\lim\limits_{n\to\infty}\chi_{PC}(B_n) = 0$. Now, using the generalized Cantor's intersection property, it follows that the set $B := \bigcap\limits_{n=1}^{\infty} B_n$ is a nonempty and compact subset of $PC([0,T],E)$. Since every set B_n is a convex set, the set B is also a convex set. Next, we verify that $\mathfrak{F}B \subset B$. Indeed, for every $n \geq 1$, we have that $\mathfrak{F}B \subset \mathfrak{F}B_n \subset \overline{conv}\left(B_n\right) = B_{n+1}$, so that $\mathfrak{F}B \subset \bigcap\limits_{n=2}^{\infty} B_n$. Also, since $B_n \subset B_1$ for every $n \geq 1$, it follows that $\mathfrak{F}B \subset \bigcap\limits_{n=2}^{\infty} B_n \subset \bigcap\limits_{n=1}^{\infty} B_n \subset B$. Now, we show that $\mathfrak{F}$ is a continuous operator from B into itself. For this, let $u_n(\cdot) \to u(\cdot)$ in B. If $1 \leq p < \infty$ and $1/p + 1/q = 1$, then by Hölder's inequality we have

$$\begin{aligned}
&\|(\mathfrak{F}u_n)(t) - (\mathfrak{F}u)(t)\| \leq \\
\leq\ & \int_0^t [\|\mathcal{T}(t-s)[(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)]\|\, ds \\
& + \sum_{0<t_k<t} ||\mathcal{T}(t-t_k)(\mathfrak{I}_k u_n)(t_k) - (\mathfrak{I}_k u)(t_k)||] \\
\leq\ & \int_0^t \|\mathcal{T}(t-s)\|_{\mathcal{L}(E)} \|(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)\| ds \\
& + \sum_{0<t_k<t} \|\mathcal{T}(t-t_k)\|_{\mathcal{L}(E)}\, ||(\mathfrak{I}_k u_n)(t_k) - (\mathfrak{I}_k u)(t_k)||] \\
\leq\ & M \int_0^T \|(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)\| ds + \\
& M \sum_{0<t_k<T} ||(\mathfrak{I}_k u_n)(t_k) - (\mathfrak{I}_k u)(t_k)|| \\
\leq\ & MT^{1/q} \left(\int_0^T \|(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)\|^p ds \right)^{1/p} \\
& + M \sum_{0<t_k<T} ||(\mathfrak{I}_k u_n)(t_k) - (\mathfrak{I}_k u)(t_k)||,
\end{aligned}$$

and for $p = \infty$ we have

$$\begin{aligned}
& \|(\mathfrak{F}u_m)(t) - (\mathfrak{F}u)(t)\| \\
\leq\ & \int_0^t \|\mathcal{T}(t-s)\| \|(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)\| ds \\
& + \sum_{0<t_k<T} \|\mathcal{T}(t-t_k)\|\, ||(\mathfrak{I}_k u_n)(t_k^-) - (\mathfrak{I}_k u)(t_k^-)|| \\
\leq\ & MT \cdot ess \sup_{0\leq t\leq T} \|(\mathfrak{C}u_n)(t) - (\mathfrak{C}u)(t)\| \\
& + M \sum_{0<t_k<T} ||(\mathfrak{I}_k u_n)(t_k^-) - (\mathfrak{I}_k u)(t_k^-)||.
\end{aligned}$$

Using the continuity of the operators $\mathfrak{C}$ and $\mathfrak{I}_k$ it follows that for $1 \leq p \leq \infty$ we have that $\|(\mathfrak{F}u_n)(\cdot) - (\mathfrak{F}u)(\cdot)\|_{PC} \to 0$ as $n \to \infty$, so that $\mathfrak{F} : B \to B$ is a continuous operator. Since B is a nonempty compact convex set, and $\mathfrak{F} : B \to B$ is a continuous operator, by Schauder's fixed point theorem it follows that there exists at least one $u(\cdot) \in B$ such that

$$u(t) = (\mathfrak{F}u)(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)(\mathfrak{C}u)(s)ds + \sum_{0<t_k<t} \mathcal{T}(t-t_k)\left(\mathfrak{I}_k u\right)(t_k)$$

for all $t \in [0, T]$; that is, $u(\cdot) \in B$ is a mild solution for (4). □

Remark 2. *It is easy to see that the conclusion Theorem 1 remains true if (6) is replaced by*

$$\|(\mathfrak{C}u)\,(t)\| \le \xi(t, ||u(\cdot)||_{PC}), \text{for a.e. } t \in [0, T], \tag{24}$$

and for each $u(\cdot) \in PC([0, T], E)$. The conclusion of Theorem 1 remains true if (H2)(b) is replaced by:

(H2) ***(b′)*** *For each bounded subset $B \subset PC([0,T], E)$ there exists $\gamma_B(\cdot) \in L^1([0, T], \mathbb{R}_+)$ such that (7) holds and*

$$\chi(\mathcal{T}(t)(\mathfrak{C}B)(t)) \le \int_0^t \gamma_B(s)\chi(B(s))ds \text{ for } t \in [0, T].$$

If $\{\mathcal{T}(t), t \ge 0\}$ is a compact C_0-semigroup or $\mathfrak{C}$ is a compact operator, then $\chi(\mathcal{T}(t)(\mathfrak{C}B)(t)) = 0$ for $t \in [0, T]$ and for each bounded set $B \subset PC([0, T], E)$ (see [48, Remark 8.2.1]). Also, with a slight modification of the proof, the conclusion of Theorem 1 remains true if the condition (H2)(b) is replaced by:

(H2) ***(b′′)*** *For each bounded subset $B \subset PC([0,T], E)$ there exists $\gamma_B(\cdot) \in L^1([0, T], \mathbb{R}_+)$ such that (7) holds and*

$$\chi((\mathfrak{C}B)(t)) \le \gamma_B(t)\chi(B(t)) \text{ for a.e. } t \in [0, T].$$

Next, suppose that $f(\cdot, \cdot) : [0, T] \times E \to E$ is a function which satisfies the following condition:

(Hf) ***(a)*** *$f : [0, T] \times E \to E$ is a Carathéodory function; that is, $t \mapsto f(t, u)$ is strongly measurable for all $u \in E$, $u \mapsto f(t, u)$ is continuous for a.e. $t \in [0, T]$, and there exist $a > 0$ and $c(\cdot) \in L^P([0, T], \mathbb{R}_+)$ such that $aMT < 1 - M \sum_{0<t_k<T} c_k$ and*

$$\|f(t, u)\| \le c(t) + a\|u\|,\ t \in [0, T],\ u \in E,$$

where $p \ge 1$.

(b) *For each bounded set $B \subset E$ there exist $\gamma_B(\cdot) \in L^1([0, T], \mathbb{R}_+)$ such that (7) holds and*

$$\chi(f(t, B)) \le \gamma_B(t)\chi(B) \text{ a.e. on } [0, T]. \tag{25}$$

Then it is known (see [3, Theorem 3.1]) that the operator $\mathfrak{C}$, defined by $(\mathfrak{C}u)(t) := f(t, u(t))$, $t \in [0, T]$, is a continuous operator from $L^p([0, T], E)$ into $L^p([0, T], E)$, and thus it is continuous from $PC([0, T], E)$ into $L^p([0, T], E)$. Moreover, $\mathfrak{C}$ is a causal operator and it satisfies (H2)(a) with $\xi(t, \eta) := c(t) + a\eta$, $t \in [0, T]$, $\eta \in \mathbb{R}_+$. Also, it is easy to see that $\mathfrak{C}$ satisfies (H2)(b′′). We obtain the following result.

Corollary 1. *Assume that $f : [0, T] \times E \to E$ satisfies (H1) and (Hf). If $A : D(A) \subset E \to E$ is the generator of a C_0-semigroup $\{\mathcal{T}(t); t \ge 0\}$, then the impulsive evolution equation*

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)) \text{ for } t \in [0, T]\backslash\{t_1, ..., t_N\} \\ u(t_k^+) = u(t_k^-) + (\mathfrak{I}_k u)(t_k^-) \\ u(0) = u_0, \end{cases} \tag{26}$$

has at least one mild solution on $[0, T]$.

In the next result we show that, under the conditions (H1) and (H2), we can construct a sequence of successive approximations which converges to a mild solution of (4).

Theorem 2. *Assume that $\mathfrak{C} : PC([0,T],E) \to L^p([0,T],E)$ is a continuous causal operator such that the conditions (H1) and (H2) hold. If $A : D(A) \subset E \to E$ is the generator of a C_0-semigroup $\{\mathcal{T}(t); t \geq 0\}$, then there exists a sequence of functions $\{u_n(\cdot)\}_{n\geq 1}$ in $PC([0,T],E)$ such that $\|u_n(\cdot) - u(\cdot)\|_{PC} \to 0$ as $n \to \infty$, and $u(\cdot) : [0,T] \to E$ is a mild solution for (4).*

Proof. Let $r > 0$ be such that (9) holds, and let B_0 and $\mathfrak{F} : B_0 \to B_0$ be given by (10) and (11), respectively. We construct a sequence $\{u_n(\cdot)\}_{n\geq 1}$ of functions $u_n(\cdot) \in PC([0,T],E)$ as follows. Let $n \in \mathbb{N}$. For each $i \in \{1,2,...,n\}$, we define

$$u_n^1(t) = \mathcal{T}(t)u_0, \text{ for } t \in [0, T/n]$$

and

$$u_n^i(t) = \begin{cases} u_n^{i-1}(t), \text{ for } t \in [0,(i-1)T/n] \\ \mathcal{T}(t)u_0 + \int_0^{t-T/n} \mathcal{T}(t-s)(\mathfrak{C}u_n^{i-1})(s)ds \\ + \sum\limits_{0<t_k<t-\frac{T}{n}} \mathcal{T}(t-t_k)(\mathfrak{I}_k u_n^{i-1})(t_k), \text{ for } t \in ((i-1)T/n, iT/n] \end{cases}$$

for $i > 1$. Obviously, $\left\|u_n^1(t)\right\| \leq M\|u_0\| \leq r$ for $t \in [0, T/n]$. Let us suppose that $||u_n^i(t)|| \leq r$ for $t \in [0, iT/n]$ and $i \in \{1,2,...,\nu\}$ with $\nu \leq n-1$. Then we have

$$\begin{aligned} ||u_n^{i+1}(t)|| &\leq \|\mathcal{T}(t)u_0\| + M\int_0^{t-T/n} \left\|(\mathfrak{C}u_n^{i-1})(s)\right\| ds + \\ &\quad + \sum_{0<t_k<t-\frac{T}{n}} \left\|\mathcal{T}(t-t_k)(\mathfrak{I}_k u_n^{i-1})(t_k)\right\| \\ &\leq M\|u_0\| + \int_0^{t-T/n} \xi(s, \|u_n^{i-1}(s)\|)ds \\ &\quad + M\sum_{0<t_k<t-\frac{T}{n}} c_k\|u_n^{i-1}(t_k)\| \\ &\leq M\|u_0\| + \int_0^{t-T/n} \xi(s,r)ds + Mr\sum_{0<t_k<t-\frac{T}{n}} c_k \\ &< r \end{aligned}$$

for all $t \in [0,(i+1)T/n]$. Hence, by induction on i we have that $||u_n^i(t)|| < r$ for all $i \in \{1,2,...,n\}$ and $t \in [0, iT/n]$. In the following, to simplify the notation, we put $u_n(\cdot) = u_n^n(\cdot)$, $n \geq 1$. By the causality of $\mathfrak{C}$ and $\mathfrak{I}_k$, the sequence $\{u_n(\cdot)\}_{n\geq 1}$ can be written as

$$u_n(t) = \begin{cases} \mathcal{T}(t)u_0 \text{ if } t \in [0,T/n] \\ \mathcal{T}(t)u_0 + \int_0^{t-T/n} \mathcal{T}(t-s)(\mathfrak{C}u_n)(s)ds \\ + \sum_{0<t_k<t-T/n} \mathcal{T}(t-t_k)(\mathfrak{I}_k u_n)(t_k^-) \text{ if } t \in [T/n, T] \end{cases} \tag{27}$$

for every $n \geq 1$. Moreover, $u_n(\cdot) \in B_0$ for all $n \geq 1$. Next, if $0 \leq t \leq T/n$, then it is easy to see that

$$\|(\mathfrak{F}u_n)(t) - u_n(t)\| \leq \int_0^{T/n} \|\mathcal{T}(t-s)(\mathfrak{C}u_n)(s)\|ds \leq M\int_0^{T/n} \psi(s)ds.$$

If $T/n \le t \le T$, then we have

$$
\begin{aligned}
&\|(\mathfrak{F}u_n)(t) - u_n(t)\| \le \left\| \int_0^t \mathcal{T}(t-s)(\mathfrak{C}u_n)(s)ds - \int_0^{t-T/n} \mathcal{T}(t-s)(\mathfrak{C}u_n)(s)ds \right\| \\
&+ \left\| \sum_{0<t_k<t} \mathcal{T}(t-t_k)(\mathfrak{I}_k u_n)(t_k^-) - \sum_{0<t_k<t-T/n} \mathcal{T}(t-t_k)(\mathfrak{I}_k u_n)(t_k^-) \right\| \\
&\le \int_{t-T/n}^t \|\mathcal{T}(t-s)\|_{\mathcal{L}(E)} \|(\mathfrak{C}u_n)(s)\| \, ds + \sum_{t-T/n<t_k<t} \|\mathcal{T}(t-t_k)(\mathfrak{I}_k u_n)(t_k^-)\| \\
&\le M \int_{t-T/n}^t \|(\mathfrak{C}u_n)(s)\| ds + M \sum_{t-T/n<t_k<t} (\mathfrak{I}_k u_n)(t_k^-) \\
&\le M \int_{t-T/n}^t \psi(s) ds + M \sum_{t-T/n<t_k<t} (\mathfrak{I}_k u_n)(t_k^-).
\end{aligned}
$$

Therefore, we obtain that

$$
\|(\mathfrak{F}u_n)(\cdot) - u_n(\cdot)\|_{PC} \to 0 \text{ as } n \to \infty. \tag{28}
$$

Let $V = \{u_n(\cdot); n \ge 1\}$. Since

$$
\|u_n(\cdot)\|_{PC} \le \|u_n(\cdot) - (\mathfrak{F}u_n)(\cdot)\|_{PC} + \|(\mathfrak{F}u_n)(\cdot)\|_{PC},
$$

by (28) and the equicontinuity of $\mathfrak{F}(V)$ on $[0, T]$, it follows that V is also equicontinuous on $[0, T]$. Define $V(t) = \{u_n(t); n \ge 1\}$ for $t \in [0, T]$. Then by the property of the measure of non-compactness we obtain

$$
\begin{aligned}
&\chi(V(t)) \le \chi\left(\int_0^{t-T/n} \mathcal{T}(t-s)(\mathfrak{C}V)(s)ds + \sum_{0<t_k<t-T/n} \mathcal{T}(t-t_k)(\mathfrak{I}_k V)(t_k) \right) \\
&\le \chi\left(\int_0^t \mathcal{T}(t-s)(\mathfrak{C}V)(s)ds \right) + \chi\left(\int_{t-T/n}^t \mathcal{T}(t-s)(\mathfrak{C}V)(s)ds \right) \\
&+ \chi\left(\sum_{0<t_k<t-T/n} \mathcal{T}(t-t_k)(\mathfrak{I}_k V)(t_k) \right).
\end{aligned}
$$

Let $t \in [0, T]$ be fixed and let $\varepsilon > 0$. The we can find $n(\varepsilon) \ge 1$ such that $\int_{t-T/n}^t \psi(s)ds < \varepsilon/2M$ for $n \ge n(\varepsilon)$. Since

$$
\|\mathcal{T}(t-s)(\mathfrak{C}u_n)(s)\| \le M \|(\mathfrak{C}u_n)(s)\| \le M\psi(s)
$$

for a.e. $s \in [0, t]$ and $n \ge 1$, by Remark 1 we conclude that

$$
\begin{aligned}
&\chi\left(\int_{t-T/n}^t \mathcal{T}(t-s)(\mathfrak{C}V)(s)ds \right) \\
&= \chi\left(\left\{ \int_{t-T/n}^t \mathcal{T}(t-s)(\mathfrak{C}u_n)(s)ds; n \ge n(\varepsilon) \right\} \right) \\
&\le 2 \sup_{n \ge n(\varepsilon)} M \int_{t-T/n}^t \psi(s)ds < \varepsilon.
\end{aligned}
$$

Using the last inequality and the fact that $\chi\left(\sum_{0<t_k<t-T/n} \mathcal{T}(t-t_k)(\mathfrak{I}_k V)(t_k) \right) = 0$, we obtain that

$$
\chi(V(t)) \le \chi\left(\int_0^t \mathcal{T}(t-s)(\mathfrak{C}V)(s)ds \right).
$$

Since $V(t)$ is bounded, by Lemma 3 and (H2) it follows that

$$\begin{aligned}
\chi(V(t)) &\leq \int_0^t \chi\left(\mathcal{T}(t-s)(\mathfrak{C}V)(s)\right) ds \\
&\leq \int_0^t \int_0^s M\gamma_V(\tau)\chi\left(V(\tau)\right) d\tau ds \\
&= \int_0^t \int_\tau^t M\gamma_V(\tau)\chi\left(V(\tau)\right) ds d\tau \\
&= \int_0^t M(t-\tau)\gamma_V(\tau)\chi\left(V(\tau)\right) d\tau \\
&\leq \int_0^t MT\gamma_V(\tau)\chi\left(V(\tau)\right) d\tau.
\end{aligned}$$

Therefore, for each $t \in [0, T]$, we have

$$v(t) \leq \int_0^t MT\gamma_V(t)v(t)ds$$

where $v(t) := \chi(V(t), t \in [0, T]$. Then, by Gronwall's lemma, it follows that $v(t) = 0$ for every $t \in [0, T]$, so that $\chi(V(t)) = 0$ for every $t \in [0, T]$. Moreover, since $\chi_{PC}(V) = \sup_{0 \leq t \leq T} \chi(V(t))$, hence $\chi_{PC}(V) = 0$. Therefore, V is a relatively compact subset of B_0. Then, by the Arzela–Ascoli theorem, and extracting a subsequence if necessary, we may assume that the sequence $\{u_n(\cdot)\}_{n \geq 1}$ converges uniformly on $[0, T]$ to a continuous function $u(\cdot) \in B_0$. Since

$$\begin{aligned}
\|(\mathfrak{F}u)(\cdot) - u(\cdot)\|_{PC} \leq\ & \|(\mathfrak{F}u)(\cdot) - (\mathfrak{F}u_n)(\cdot)\|_{PC} + \|(\mathfrak{F}u_n)(\cdot) - u_n(\cdot)\|_{PC} \\
& + \|u_n(\cdot) - u(\cdot)\|_{PC},
\end{aligned}$$

by the continuity of $\mathfrak{F}$ and (28), we get $\|(\mathfrak{F}u)(\cdot) - u(\cdot)\|_{PC} = 0$. It follows that

$$u(t) = (\mathfrak{F}u)(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)(\mathfrak{C}u)(s)ds + \sum_{0<t_k<t} \mathcal{T}(t-t_k)\left(\mathfrak{I}_k u\right)(t_k)$$

for all $t \in [0, T]$; that is, $u(\cdot)$ is a mild solution of the causal evolution Equation (4). □

4. Applications

1. Consider the following impulsive integro-differential evolution equation

$$\begin{cases}
u'(t) = Au(t) + \int_0^t K(t,s)f(s,u(s))ds \text{ for a.e. } t \in [0, T], \\
u(t_k^+) = u(t_k^-) + \left(\mathfrak{I}_k u\right)(t_k^-), k = 1, 2, ..., N, \\
u(0) = u_0,
\end{cases} \tag{29}$$

where $f : [0, T] \times E \to E$ satisfies condition (Hf) and $K : [0, T] \times [0, T] \to \mathcal{L}(E)$ is strongly continuous. Put

$$(\mathfrak{C}u)(t) := \int_0^t K(t,s)f(s,u(s))ds,\ t \in [0, T]. \tag{30}$$

It is well known that $\mathfrak{C}$ defines a continuous operator from $L^p([0, T], E)$ into itself (see ([49], Proposition 9.5.2) or ([50], p. 160)). Then for each $u(\cdot) \in PC([0, T], E)$ we have

$$
\begin{aligned}
\|(\mathfrak{C}u)(t)\| &\leq \int_0^t \|K(s,\tau)\|_{\mathcal{L}(E)} \|f(\tau,u(\tau))\| \, d\tau \\
&\leq M_1 \int_0^t [c(\tau) + a\|u(\tau)\|] \, d\tau \\
&\leq \xi(t, \|u(\cdot)\|_{PC}) := M_1 \int_0^t c(\tau) d\tau + aM_1 T \|u(\cdot)\|_{PC},
\end{aligned}
$$

where $M_1 := \sup\{\|K(t,s)\|_{\mathcal{L}(E)}\,; t,s \in [0,T]\}$. Next, if B is a bounded set in $PC([0,T],E)$, then it is easy to show that $\mathfrak{C}B$ is bounded and equicontinuous on $[0,T]$ (as a subset of $PC([0,T],E)$). Thus, by (25) and Theorem 1.2.2 in [51], we have

$$
\begin{aligned}
\chi((\mathfrak{C}B)(t)) &= \chi\left(\int_0^t K(t,s) f(s,B(s)) ds\right) \leq \int_0^t \chi\left(K(t,s) f(s,B(s))\right) ds \\
&\leq \int_0^t M_1 \gamma_B(s) \chi\left(B(s)\right) ds, \, t \in [0,T].
\end{aligned}
$$

Consequently, the operator $\mathfrak{C}$, defined by (30), is a continuous causal operator from $PC([0,T],E)$ into $L^p([0,T],E)$ and it satisfies (24) and (8). Assume that hypotheses (H1), (Hf) hold, $\int_0^T M_1 \gamma_B(t) dt < \frac{1}{2MT}$ and $K : [0,T] \times [0,T] \to \mathcal{L}(E)$ is strongly continuous. Then, by Theorem 1, it follows that (29) has at least one mild solution in $PC([0,T],E)$ provided that (5) holds.

2. Consider the following impulsive reaction diffusion equation

$$
\begin{cases}
\frac{\partial z}{\partial t}(t,x) = \frac{\partial^2 z}{\partial x^2}(t,x) + \int_0^t k(t,s) f(s,z(s,x)) ds, x \in (0,\pi), t \in [0,T] \setminus \{t_1,..,t_N\} \\
z(t_k^+,x) - z(t_k^-,x) = \int_0^x z(t_k^-,y) g_k(x) dy, \quad x \in (0,\pi), k = 1,...,N \\
z(t,0) = z(t,\pi) = 0, \quad t \in [0,T], \\
z(0,x) = z_0(x), \quad x \in (0,\pi),
\end{cases}
\tag{31}
$$

where $0 = t_0 < t_1 < t_2 < ... < t_N < t_{N+1} = T$, $z(t_k^+,x) = \lim_{(h,x)\to(0^-,x)} z(t_k+h,x)$, $z(t_k^-,x) = \lim_{(h,x)\to(0^-,x)} z(t_k+h,x)$, $f(\cdot,\cdot) : [0,T] \times [0,\pi] \to \mathbb{R}$ and $k(\cdot,\cdot) : [0,T] \times [0,T] \to \mathbb{R}$ are given functions, $z_0(\cdot), g_k(\cdot) \in E := L^2[0,\pi]$ and $k = 1,...,N$. Also, we assume that $\|\pi_h g_k - g_k\|_{L^2[0,\pi-h]} \to 0$ as $h \to 0^+$ for $k = 1,...,N$, where $(\pi_h g_k)(t) = g_k(t+h)$. We can show that problem (31) is an abstract formulation of problem (29). For this, let

$$
\begin{aligned}
u(t)(x) \quad &: \quad = z(t,x), \, (t,x) \in [0,T] \times [0,\pi], \\
((\mathfrak{I}_k u)(t))(x) \quad &: \quad = \int_0^x u(t)(y) g_k(x) dy, \ (t,x) \in [0,T] \times [0,\pi], \ k = 1,2,...,N, \\
(\mathfrak{C}u)(t) \quad &: \quad = \int_0^t K(t,s) F(s,u(s)) ds, \, t \in [0,T],
\end{aligned}
$$

where $F(\cdot,\cdot) : [0,T] \times E \to E$ is given by $F(t,u(\cdot))(x) := f(t,z(\cdot,x))$ for $(t,x) \in [0,T] \times [0,\pi]$, and $K(\cdot,\cdot) : [0,T] \times [0,T] \to \mathcal{L}(E)$ is given by $(K(t,s)u(\cdot))(x) := k(t,s)u(\cdot)(x) = k(t,s)z(\cdot,x)$ for $t,s \in [0,T]$, $x \in [0,\pi]$. Next, let $t \geq 0$ and let us define $\mathcal{T}(t) : E \to E$ by

$$
(\mathcal{T}(t)u(\cdot))(x) := \sum_{n=0}^{\infty} a_n(u(\cdot)) e^{-n^2 t} \sin nx
$$

for each $u(\cdot) \in E$, where

$$
a_n(u(\cdot)) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u(\cdot)(x) \sin nx dx.
$$

Then it is known (see ([41], Problem 4.2 and Problem 7.8)) that $\{\mathcal{T}(t), t \geq 0\}$ is a compact C_0-semigroup and its infinitesimal generator $A : D(A) \subset E \to E$ is given by

$$(Au(\cdot))(x) = -\sum_{n=1}^{\infty} n^2 a_n(u(\cdot)) \sin nx, \; u(\cdot) \in D(A),$$

where $D(A)$ is the space of all functions $u(\cdot) \in E$ such that $u(\cdot), u'(\cdot)$ are absolutely continuous, $u''(\cdot) \in E$ and $u(\cdot)(0) = u(\cdot)(\pi) = 0$. Also, there exists $M > 1$ such that $\|\mathcal{T}(t)\|_{\mathcal{L}(E)} \leq M$ for $t \in [0, T]$. From the above it follows that (31) can be written in the abstract form (29). Now, assume that

(f) $f(\cdot,\cdot) : [0, T] \times [0, \pi] \to \mathbb{R}$ is a Carathéodory function; that is, $t \mapsto f(t, x)$ is measurable for all $x \in [0, \pi]$, $x \mapsto f(t, x)$ is continuous for a.e. $t \in [0, T]$, and there exist $a > 0$ and $c(\cdot) \in L^2([0, T], \mathbb{R}_+)$

$$\|f(t, x)\| \leq c(t) + a\|x\|, \; t \in [0, T], \; x \in [0, \pi].$$

(k) $k(\cdot,\cdot) : [0, T] \times [0, T] \to \mathbb{R}$ is continuous.

Then it is easy to check that $F(\cdot,\cdot)$ verifies the condition (Hf) and $K(\cdot,\cdot)$ is strongly continuous. Since $\{\mathcal{T}(t), t \geq 0\}$ is a compact C_0-semigroup, for any bounded set $B \subset PC([0,T], E)$, we have $\chi(\mathcal{T}(t)(\mathfrak{C}B)(t)) = 0$ for $t \in [0, T]$. It remains to show that $\mathfrak{I}_k$ is a compact operator for each $k = 1, ..., N$. For this, we must show that for any bounded set $B \subset PC([0,T], E)$, $\mathfrak{I}_k B$ is equicontinuos and $(\mathfrak{I}_k B)(t) \subset L^2[0, T]$ is relatively compact for every $t \in [0, T]$. For any $u(\cdot), v(\cdot) \in B$, using Hölder's inequality, we have

$$\begin{aligned}
\|(\mathfrak{I}_k u)(t) - (\mathfrak{I}_k v)(t)\|_{L^2[0,T]} &= \|\textstyle\int_0^x u(t)(y) g_k(x) dy - \int_0^x v(t)(y) g_k(x) dy\|_{L^2[0,T]} \\
&\leq \textstyle\int_0^x \|[u(t)(y) - v(t)(y)]\, g_k(x)\|_{L^2[0,T]}\, dy \\
&\leq \textstyle\int_0^\pi \left(\int_0^\pi |u(t)(y) - v(t)(y)|^2 |g_k(x)|^2 dx\right)^{1/2} dy \\
&= \left(\textstyle\int_0^\pi |u(t)(y) - v(t)(y)|\, dy\right) \|g_k\|_{L^2[0,T]} \\
&\leq \left(\textstyle\int_0^\pi dy\right)^{1/2} \left(\int_0^\pi |u(t)(y) - v(t)(y)|^2 dy\right)^{1/2} \|g_k\|_{L^2[0,T]} \\
&= \sqrt{\pi}\, \|g_k\|_{L^2[0,T]} \|u(t) - v(t)\|_{L^2[0,T]},
\end{aligned}$$

that is,

$$\|(\mathfrak{I}_k u)(t) - (\mathfrak{I}_k v)(t)\|_{L^2[0,T]} \leq c_k \|u(t) - v(t)\|_{L^2[0,T]}, \tag{32}$$

for every $u(\cdot), v(\cdot) \in B$, where $c_k = \sqrt{\pi}\, \|g_k\|_{L^2[0,T]}$. From the above inequality it follows that

$$\|\mathfrak{I}_k u - \mathfrak{I}_k v\|_{PC} \leq c_k \|u - v\|_{PC}$$

for every $u(\cdot), v(\cdot) \in B$, and so, $\mathfrak{I}_k B$ is equicontinuous. Using the compactness result from ([52], p. 74), $(\mathfrak{I}_k B)(t) \subset L^2[0, T]$ is relatively compact if and only if $(\mathfrak{I}_k B)(t)$ is bounded and

$$\|\pi_h(\mathfrak{I}_k u)(t) - (\mathfrak{I}_k u)(t)\|_{L^2[0,\pi-h]} \to 0 \text{ as } h \to 0^+ \tag{33}$$

for every $u(\cdot) \in B$. The boundedness of $(\mathfrak{I}_k B)(t)$ follows from (32) and the definition of the causal operator. More exactly, we have that

$$\|(\mathfrak{I}_k u)(t)\|_{L^2[0,T]} \leq c_k \|u(t)\|_{L^2[0,T]} \leq c_k \|u\|_{PC} < cr$$

for every $u(\cdot) \in B$, where $r := \sup_{u(\cdot) \in B} \|u\|_{PC}$ and $c := \sum_{k=1}^{N} c_k$.

Using the Hölder inequality it is not difficult to show that

$$\begin{aligned}
&\|\pi_h (\mathfrak{I}_k u)(t) - (\mathfrak{I}_k u)(t)\|_{L^2[0,\pi-h]} = \left(\int_0^{\pi-h} |(\mathfrak{I}_k u)(t)(x+h) - (\mathfrak{I}_k u)(t)(x)|^2 \, dx\right)^{1/2} \\
&= \left[\int_0^{\pi-h} \left(\int_0^{x+h} u(t)(y)\, g_k(x+h) dy - \int_0^{x} u(t)(y)\, g_k(x) dy\right)^2 dx\right]^{1/2} \\
&= \left[\int_0^{\pi-h} \left(\int_0^{x} u(t)(y)\,[g_k(x+h) - g_k(x)]\, dy + \int_x^{x+h} u(t)(y)\, g_k(x+h) dy\right)^2 dx\right]^{1/2} \\
&\le 2\sqrt{\pi}\, \|u\|_{PC} \, \|\pi_h g_k - g_k\|_{L^2[0,\pi-h]} + 2\left(\int_x^{x+h} u(t)(y)\, dy\right) \|g_k(\cdot + h)\|_{L^2[0,\pi-h]} .
\end{aligned}$$

Using the last estimation and the fact that $\|\pi_h g_k - g_k\|_{L^2[0,\pi-h]} \to 0$ as $h \to 0^+$, we obtain (33). Since, for each $k = 1, ..., N$, we showed that for any bounded set $B \subset PC([0,T], E)$, $\mathfrak{I}_k B$ is equicontinuous and $(\mathfrak{I}_k B)(t) \subset L^2[0,T]$ is relatively compact for every $t \in [0,T]$, it follows that $\mathfrak{I}_k$ is a compact operator for each $k = 1, ..., N$. Consequently, if $2Mc < 1$, then all the conditions of Theorem 1 are satisfied, so that (31) has a solution $z(\cdot,\cdot)$ on $[0,T] \times [0,\pi]$.

5. Conclusions

The theory of impulsive evolution differential equations with causal operators is an important one because it covers a large class of different types of impulsive evolution differential equations. The study of these evolution equations hopefully will be continued with impulsive evolution equations with nonlinear operators or impulsive evolution differential inclusions involving causal operators. Another direction of investigation is to study fractional differential equations with causal operators and their applications.

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