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Parallel Tseng's Extragradient Methods for Solving Systems of Variational Inequalities on Hadamard Manifolds

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Abstract: The aim of this article is to study two efficient parallel algorithms for obtaining a solution to a system of monotone variational inequalities (SVI) on Hadamard manifolds. The parallel algorithms are inspired by Tseng's extragradient techniques with new step sizes, which are established without the knowledge of the Lipschitz constants of the operators and line-search. Under the monotonicity assumptions regarding the underlying vector fields, one proves that the sequences generated by the methods converge to a solution of the monotone SVI whenever it exists.

Keywords: system of variational inequalities; extragradient method; Hadamard manifolds; monotone vector fields

1. Introduction

Given an operator $A : H \to H$ and a convex and closed subset C in a real Hilbert space H, the well known variational inequality problem (VIP) indicates the one of finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in C. \tag{1}$$

It is well known that the variational inequality theory has been playing a big role in the study of signal processing, image reconstruction, mathematical programming, differential equations, and others; see, e.g., [1–5]. A large number of numerical methods has been designed for solving the VIPs and related optimization problems; see, e.g., [6–10]. With the help of an additional projection operator, Korpelevich [11] first introduced

$$\begin{cases} y_n = P_C(Id - \lambda A)x_n, \\ x_{n+1} = P_C(x_n - \lambda A y_n) \quad \forall n \geq 0, \end{cases} \tag{2}$$

where Id stands for the identity, λ is a real number in $(0, \frac{1}{L})$, where L is the Lipschitz module of A, and P_C stands for the nearest point projection operator onto subset C. Recently, the gradient (reduced) type iterative schemes are under the spotlight of engineers and mathematicians working in the communities of control theory and optimization. Based on the approach, a number of investigators have conducted various approaches on algorithms; see e.g., [12–18] and the references cited therein.

Let both A_1 and A_2 be single-valued self-operators on space H. Recently, Ceng et al. [19] considered and studied the following system problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - y^* + \mu_1 A_1 y^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \\ \langle y^* - x^* + \mu_2 A_2 x^*, x - y^* \rangle \geq 0 \quad \forall x \in C, \end{cases} \tag{3}$$

with two positive constants μ_1 and $\mu_2 > 0$, which is called a system of variational inequalities (SVI). In [19], system problem (3) was transformed into a fixed-point problem (FPP). Utilizing the equivalence relation between system problem (3) and the FPP of some operator, Ceng et al. [19] proposed and investigated a relaxed type method for solving system problem (3); see also [16,20–24] for recent investigations.

On the other hand, in 2003, Németh [25] introduced the VIP on Hadamard manifolds, that is, find $x^* \in C$ such that

$$\langle Ax^*, \exp_{x^*}^{-1} x\rangle \geq 0 \quad \forall x \in C, \tag{4}$$

where C is a nonempty, convex and closed set in Hadamard manifold $\mathcal{M}$, $A : \mathcal{M} \to T\mathcal{M}$ is a vector field, that is, $Ax \in T_x\mathcal{M}\ \forall x \in \mathcal{M}$, and $\exp^{-1}$ is the inverse of an exponential map. We denote by S the solution set of problem (4). Recently, some methods and techniques have been generalized from Euclidean spaces to Riemannian manifolds because the generalization has some important advantages; see, e.g., [26–28]. It is well known that the research progress on the problem (4) is limited by the nonlinearity of manifolds, and hence is slow. Moreover, the research on its algorithms is mainly focused on the proximal point algorithm and Korpelevich's method. Very recently, using Tseng's extragradient methods, Chen et al. [29] constructed two effective algorithms to solve the problem (4) on Hadamard manifolds. Moreover, their results gave a further answer to the open question put forth in Ferreira et al. [30].

Inspired by problems (3) and (4), this paper introduces and considers the SVI on Hadamard manifolds, that is, find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \exp_{y^*}^{-1} x^* + \mu_1 A_1 y^*, \exp_{x^*}^{-1} x\rangle \geq 0 \quad \forall x \in C, \\ \langle \exp_{x^*}^{-1} y^* + \mu_2 A_2 x^*, \exp_{y^*}^{-1} x\rangle \geq 0 \quad \forall x \in C, \end{cases} \tag{5}$$

with constants $\mu_1, \mu_2 > 0$, If $A_1 = A_2 = A$ and $x^* = y^*$, then SVI (5) reduces to VIP (4).

Inspired by the extragradient algorithms in Chen et al. [29], we propose and analyze two parallel effective algorithms for solving SVI (5), by virtue of Tseng's extragradient method. The first algorithm's step sizes are obtained by using line-search, and the second one only by using two previous iterates. In both algorithms, the Lipschitz constants are not required to be known. Moreover, our results improve and extend the corresponding results announced by some others, e.g., Ceng et al. [19] and Chen et al. [29].

The outline of the work is organized as follows. Some basic concepts, notations and important lemmas in Riemannian geometry are presented in Section 2. In Section 3, we present two algorithms based on the Tseng's extragradient method for SVI (5) on Hadamard manifolds and obtain the desired convergence theorems.

2. Preliminaries

This paper assumes that the Riemannian manifold $\mathcal{M}$ indicates a connected m-dimensional manifold endowed with a Riemannian metric. We use the same notations in [31]. For more details about these notations and relevant definitions, please consult relevant textbook on Riemannian geometry (see, e.g., [31]).

Definition 1. *(see [32]). Let $\mathcal{X}(\mathcal{M})$ be contain all univalued vector fields $V : \mathcal{M} \to T\mathcal{M}$ such that $V(x) \in T_x\mathcal{M}\ \forall x \in \mathcal{M}$ and the domain $\mathcal{D}(V)$ of V be defined by $\mathcal{D}(V) = \{x \in \mathcal{M} : V(x) \neq \emptyset\}$. Let $V \in \mathcal{X}(\mathcal{M})$. Then V is said to be pseudomonotone if, for any $x, y \in \mathcal{D}(V)$,*

$$\langle V(x), \exp_x^{-1} y\rangle \geq 0 \;\Rightarrow\; \langle V(y), \exp_y^{-1} x\rangle \leq 0.$$

Proposition 1. *(see [31]). (Comparison theorem for triangles). Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 (\mathrm{mod} 3)$, by $\gamma_i : [0, l_i] \to \mathcal{M}$ the geodesic joining p_i to p_{i+1}, and set $l_i = L(\gamma_i)$, and $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$. Then*

(i) $\alpha_1 + \alpha_2 + \alpha_3 \leq \pi$;
(ii) $l_i^2 + l_{i+1}^2 - 2 l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2$;

(iii) $l_{i+1} \cos \alpha_{i+2} + l_i \cos \alpha_i \geq l_{i+2}$.

In terms of the distance and the exponential map, the above inequality can be rewritten as

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2} \rangle \leq d^2(p_{i-1}, p_i),$$

since

$$\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}. \tag{6}$$

Lemma 1. *(see [33]). Let $x_0 \in \mathcal{M}$ and $\{x_n\} \subset \mathcal{M}$ with $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in \mathcal{M}$, we have $\exp^{-1}_{x_n} y \to \exp^{-1}_{x_0} y$ and $\exp^{-1}_{y} x_n \to \exp^{-1}_{y} x_0$;*
(ii) *If $v_n \in T_{x_n}\mathcal{M}$ and $v_n \to v_0$, then $v_0 \in T_{x_0}\mathcal{M}$;*
(iii) *Given $u_n, v_n \in T_{x_n}\mathcal{M}$ and $u_0, v_0 \in T_{x_0}\mathcal{M}$, if $u_n \to u_0$ and $v_n \to v_0$, then $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$;*
(iv) *For any $u \in T_{x_0}M$, the function $F : \mathcal{M} \to T\mathcal{M}$, defined by $F(x) = P_{x,x_0} u \ \forall x \in \mathcal{M}$ is continuous on $\mathcal{M}$.*

Lemma 2. *(see [34]). Given $p' \in \mathcal{M}$, there exists a unique projection $P_C(p')$. Furthermore, the following inequality holds:*

$$\langle \exp^{-1}_{P_C(p')} p', \exp^{-1}_{P_C(p')} p \rangle \leq 0 \quad \forall p \in C.$$

Proposition 2. *(see [32]). The following statements are equivalent:*

(i) *x^* is a solution of problem (4);*
(ii) *$x^* = P_C(\exp_{x^*}(-\beta_0 A x^*))$ for some $\beta_0 > 0$;*
(iii) *$x^* = P_C(\exp_{x^*}(-\beta A x^*))$ for all $\beta > 0$;*
(iv) *$r(x^*, \beta) = 0$, where $r(x^*, \beta) = \exp^{-1}_{x^*}[P_C(\exp_{x^*}(-\beta A x^*))]$.*

Lemma 3. *(see [35]). Let $\delta(p, q, r)$ be a geodesic triangle in $\mathcal{M}$ Hadamard manifold. Then, there exists $p', q', r' \in \mathbf{R}^2$ such that*

$$d(p, q) = \|p' - q'\|, \quad d(q, r) = \|q' - r'\| \quad \text{and} \quad d(r, p) = \|r' - p'\|.$$

Lemma 4. *(see [36]). Let $\delta(p, q, r)$ be a geodesic triangle in a Hadamard manifold $\mathcal{M}$ and $\delta(p', q', r')$ be its comparison triangle.*

(i) *If α, β, γ (resp., α', β', γ') be the angles of $\delta(p, q, r)$ (resp., $\delta(p', q', r')$) at the vertices p, q, r (resp., p', q', r'). Then, the following inequalities hold:*

$$\alpha' \geq \alpha, \quad \beta' \geq \beta \quad \text{and} \quad \gamma' \geq \gamma.$$

(ii) *If z is a point in the geodesic joining p to q and z' is its comparison point in the interval $[p', q']$ such that $d(z, p) = \|z' - p'\|$ and $d(z, q) = \|z' - q'\|$, then the following inequality holds:*

$$d(z, r) \leq \|z' - r'\|.$$

Definition 2. *A vector field f defined on a complete Riemannian manifold $\mathcal{M}$ is said to be Lipschitz continuous if there exists a constant $L(M) = L > 0$ such that*

$$d(f(x), f(x')) \leq L d(x, x') \quad \forall x, x' \in \mathcal{M}. \tag{7}$$

Besides this global concept, if for each $x_0 \in \mathcal{M}$, there exist $L(x_0) > 0$ and $\delta = \delta(x_0) > 0$ such that inequality (7) occurs, with $L = L(x_0)$, for all $x, x' \in B_\delta(x_0) := \{x \in \mathcal{M} : d(x_0, x) < \delta\}$, then f is said to be locally Lipschitz continuous.

Finally, utilizing a similar technique to that of transforming SVI (3) into the FPP in [19], we derive the following result.

Lemma 5. *A pair (x^*, y^*), with $x^*, y^* \in C$, is a solution of SVI (5) if and only if $x^* \in \mathrm{Fix}(G)$, i.e., $x^* = Gx^*$ where $\mathrm{Fix}(G)$ is the fixed-point set of the mapping $G := P_C(\exp_I(-\mu_1 A_1))P_C(\exp_I(-\mu_2 A_2))$ and $y^* = P_C(\exp_I(-\mu_2 A_2))x^*$.*

Proof. In terms of Lemma 2, we obtain that

$$\begin{cases} \langle \exp_{y^*}^{-1} x^* + \mu_1 A_1 y^*, \exp_{x^*}^{-1} x \rangle \geq 0 & \forall x \in C, \\ \langle \exp_{x^*}^{-1} y^* + \mu_2 A_2 x^*, \exp_{y^*}^{-1} x \rangle \geq 0 & \forall x \in C \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x^* = P_C(\exp_{y^*}(-\mu_1 A_1 y^*)), \\ y^* = P_C(\exp_{x^*}(-\mu_2 A_2 x^*)) \end{cases}$$

$$\Updownarrow$$

$$x^* = P_C(\exp_I(-\mu_1 A_1))y^* = P_C(\exp_I(-\mu_1 A_1))P_C(\exp_I(-\mu_2 A_2))x^*.$$

That is, $x^* \in \mathrm{Fix}(G)$. □

3. Main Results

In this section, inspired by the extragradient algorithms in Chen et al. [30], we propose the following Algorithms 1 and 2 for solving the system (5) of monotone variational inequalities on Hadamard manifolds, which are based on Tseng's extragradient method. The Algorithm 1 presents a simple and convenient way with the line-search for defining the step sizes. Meantime, in the Algorithm 3, the step sizes are computed by current information for the iterates, instead of requiring the knowledge of the Lipchitz constants of operators and additional projections. In particular, if we set $A_1 = A_2 = A$ in Algorithms 1 and 3, these algorithms are reduced to the following Algorithms 2 and 4, respectively, for solving the monotone VIP (4) on Hadamard manifolds. Let assume the following:

(H1) $\mathcal{S} \neq \emptyset$, where $\mathcal{S} \neq \emptyset$ is the set of solutions of SVI (5).

(H2) For $i = 1, 2$, $A_i : \mathcal{M} \to T\mathcal{M}$ is a vector field, that is, $A_i x \in T_x\mathcal{M}$ $\forall x \in \mathcal{M}$, and $\exp^{-1}$ is the inverse of exponential map.

(H3) For $i = 1, 2$, the mapping A_i is monotone, i.e.,

$$\langle A_i x - A_i y, \exp_y^{-1} x \rangle \geq 0 \quad \forall x, y \in \mathcal{M}.$$

(H4) For $i = 1, 2$, the mapping A_i is Lipschitz-continuous with constant $L_i > 0$, i.e., there exists $L_i > 0$ such that

$$d(A_i x, A_i y) \leq L_i d(x, y) \quad \forall x, y \in \mathcal{M}.$$

We first recall the concept of Fejér convergence and its related result.

Definition 3. *(see [37]). Let X be a complete metric space and $C \subset X$ be a nonempty set. A sequence $\{x_k\} \subset X$ is called Fejér convergent to C, if $d(x_{k+1}, y) \leq d(x_k, y)$ $\forall y \in C$, $k \geq 0$.*

Proposition 3. *(see [33]). Let X be a complete metric space and let $C \subset X$ be a nonempty set. Let $\{x_k\} \subset X$ be Fejér convergent to C and suppose that any cluster point of $\{x_k\}$ lies in C. Then $\{x_k\}$ converges to a point of C.*

3.1. Parallel Tseng's Extragradient Method with Line-Search

From the Lemma 5, we can obtain the following Algorithm 1. In particular, putting $A_1 = A_2 = A$ in the Algorithm 1, we can solve VIP (4).

Algorithm 1: Parallel Tseng's extragradient method with line-search.

Initialization: Given $\gamma_i > 0$, $l_i \in (0,1)$, $\lambda_i \in (0,1)$ for $i = 1,2$. Let $x_0 \in \mathcal{M}$.
Iterative Steps:
Step 1. Calculate

$$\begin{cases} \tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n} A_2 x_n)), \\ y_n = P_C(\exp_{z_n}(-\mu_{1,n} A_1 z_n)), \end{cases}$$

where $\mu_{i,n}$ is the largest $\mu_i \in \{\gamma_i, \gamma_i l_i, \gamma_i l_i^2, ...\}$ for $i = 1,2$, satisfying the Armijo-like search rule (ALSR)

$$\begin{cases} \mu_2 d(A_2 x_n, A_2 \tilde{z}_n) \leq \lambda_2 d(x_n, \tilde{z}_n), \\ \mu_1 d(A_1 z_n, A_1 y_n) \leq \lambda_1 d(z_n, y_n). \end{cases}$$

Step 2. Calculate

$$\begin{cases} z_n = \exp_{\tilde{z}_n}(\mu_{2,n}(A_2 x_n - A_2 \tilde{z}_n)), \\ x_{n+1} = \exp_{y_n}(\mu_{1,n}(A_1 z_n - A_1 y_n)). \end{cases}$$

$n \leftarrow n + 1$ and go to Step 1.

Algorithm 1 is well defined in the following lemma.

Lemma 6. *The Armijo-like search rule (ALSR) is well defined and* $\min\{\gamma_i, \frac{\lambda_i l_i}{L_i}\} \leq \mu_{i,n} \leq \gamma_i$ *for* $i = 1,2$.

Proof. Since A_i is L_i-Lipschitz continuous on $\mathcal{M}$ for $i = 1,2$, we have

$$d(A_i w_n, A_i P_C(\exp_{w_n}(-\mu_i A_i w_n))) \leq L_i d(w_n, P_C(\exp_{w_n}(-\mu_i A_i w_n))),$$

which is equivalent to

$$\frac{\lambda_i}{L_i} d(A_i w_n, A_i P_C(\exp_{w_n}(-\mu_i A_i w_n))) \leq \lambda_i d(w_n, P_C(\exp_{w_n}(-\mu_i A_i w_n))). \tag{8}$$

Thus, (ALSR) holds for all $\mu_i \leq \frac{\lambda_i}{L_i}$. So $\mu_{i,n}$ is well defined for $i = 1,2$.

Obviously, $\mu_{i,n} \leq \gamma_i$ for $i = 1,2$. If $\mu_{i,n} = \gamma_i$ then this lemma is valid; otherwise, if $\mu_{i,n} < \gamma_i$ by the search rule (ALSR), we know that $\frac{\mu_{i,n}}{l_i}$ must violate inequality (ALSR), i.e.,

$$d(A_i w_n, A_i P_C(\exp_{w_n}(-\frac{\mu_{i,n}}{l_i} A_i w_n))) > \frac{\lambda_i}{\frac{\mu_{i,n}}{l_i}} d(w_n, P_C(\exp_{w_n}(-\frac{\mu_{i,n}}{l_i} A_i w_n)),$$

Again from L_i-Lipschitz continuity of A_i on $\mathcal{M}$, we obtain $\mu_{i,n} > \frac{\lambda_i l_i}{L_i}$. □

Corollary 1. *The Armijo-like search rule (ALSR) with* $A_1 = A_2 = A$ *is well defined and* $\min\{\gamma_i, \frac{\lambda_i l_i}{L}\} \leq \mu_{i,n} \leq \gamma_i$ *for* $i = 1,2$.

Now, we analyze the convergence of Algorithm 1.

Lemma 7. *Let $\{x_n\}$ and $\{z_n\}$ be the iterative sequences constructed via Algorithm 1. Then both $\{x_n\}$ and $\{z_n\}$ are bounded iterative sequences, provided for all $(p,q) \in S$ and $n \geq 0$,*

$$\begin{cases} (1-\lambda_2^2)d^2(\tilde{z}_n, x_n) + (1-\lambda_1^2)d^2(y_n, z_n) \\ \quad + 2\mu_{2,n}\langle A_2 p, \exp_p^{-1}\tilde{z}_n\rangle + 2\mu_{1,n}\langle A_1 p, \exp_p^{-1} y_n\rangle \geq 0, \\ (1-\lambda_2^2)d^2(\tilde{z}_{n+1}, x_{n+1}) + (1-\lambda_1^2)d^2(y_n, z_n) \\ \quad + 2\mu_{2,n+1}\langle A_2 q, \exp_q^{-1}\tilde{z}_{n+1}\rangle + 2\mu_{1,n}\langle A_1 q, \exp_q^{-1} y_n\rangle \geq 0. \end{cases}$$

Proof. Take a fixed $(p,q) \in C \times C$ arbitrarily. Then, noticing

$$\begin{cases} \tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n} A_2 x_n)), \\ y_n = P_C(\exp_{z_n}(-\mu_{1,n} A_1 z_n)), \end{cases}$$

we deduce from Lemma 2 that

$$\begin{cases} \langle \exp_{x_n}^{-1}\tilde{z}_n + \mu_{2,n} A_2 x_n, \exp_p^{-1}\tilde{z}_n\rangle \leq 0, \\ \langle \exp_{z_n}^{-1} y_n + \mu_{1,n} A_1 z_n, \exp_q^{-1} y_n\rangle \leq 0, \end{cases}$$

and hence

$$\begin{cases} \langle \exp_{x_n}^{-1}\tilde{z}_n, \exp_p^{-1}\tilde{z}_n\rangle \leq -\mu_{2,n}\langle A_2 x_n, \exp_p^{-1}\tilde{z}_n\rangle, \\ \langle \exp_{z_n}^{-1} y_n, \exp_q^{-1} y_n\rangle \leq -\mu_{1,n}\langle A_1 z_n, \exp_q^{-1} y_n\rangle. \end{cases} \tag{9}$$

Also, from the monotonicity of A_2 on $\mathcal{M}$ it follows that

$$\begin{aligned} &\langle A_2\tilde{z}_n - A_2 x_n, \exp_p^{-1}\tilde{z}_n\rangle \\ &= \langle A_2\tilde{z}_n - A_2 p, \exp_p^{-1}\tilde{z}_n\rangle + \langle A_2 p - A_2 x_n, \exp_p^{-1}\tilde{z}_n\rangle \\ &\geq \langle A_2 p - A_2 x_n, \exp_p^{-1}\tilde{z}_n\rangle = \langle A_2 p, \exp_p^{-1}\tilde{z}_n\rangle - \langle A_2 x_n, \exp_p^{-1}\tilde{z}_n\rangle. \end{aligned} \tag{10}$$

We now fix $n \geq 0$. Consider the geodesic triangle $\delta(x_n, \tilde{z}_n, p)$ and its comparison triangle $\Delta(x_n', \tilde{z}_n', p')$. Then by Lemma 3, we have $d(x_n, p) = d(x_n', p')$, $d(\tilde{z}_n, p) = d(\tilde{z}_n', p')$, and $d(x_n, \tilde{z}_n) = d(x_n', \tilde{z}_n')$. Recall that $z_n = \exp_{\tilde{z}_n}(\mu_{2,n}(A_2 x_n - A_2\tilde{z}_n))$. The comparison point of z_n' is $\tilde{z}_n' + \mu_{2,n}(A_2 x_n' - A_2\tilde{z}_n')$. By Lemma 4, we have

$$\begin{aligned} d^2(z_n, p) &\leq d^2(z_n', p') = \|p'\tilde{z}_n' + \mu_{2,n}(A_2 x_n' - A_2\tilde{z}_n')\|^2 \\ &= \|p' - \tilde{z}_n'\|^2 + \mu_{2,n}^2\|A_2 x_n' - A_2\tilde{z}_n'\|^2 + 2\mu_{2,n}\langle A_2 x_n' - A_2\tilde{z}_n', \tilde{z}_n' - p'\rangle \\ &= \|\tilde{z}_n' - x_n'\|^2 + \|p' - x_n'\|^2 + 2\langle \tilde{z}_n' - x_n', x_n' - p'\rangle \\ &\quad + \mu_{2,n}^2\|A_2 x_n' - A_2\tilde{z}_n'\|^2 + 2\mu_{2,n}\langle A_2 x_n' - A_2\tilde{z}_n', \tilde{z}_n' - p'\rangle \\ &= \|p' - x_n'\|^2 + \|\tilde{z}_n' - x_n'\|^2 - 2\langle \tilde{z}_n' - x_n', \tilde{z}_n' - x_n'\rangle + 2\langle \tilde{z}_n' - x_n', \tilde{z}_n' - p'\rangle \\ &\quad + \mu_{2,n}^2\|A_2\tilde{z}_n' - A_2 x_n'\|^2 + 2\mu_{2,n}\langle A_2 x_n' - A_2\tilde{z}_n', \tilde{z}_n' - p'\rangle \\ &= \|p' - x_n'\|^2 - \|\tilde{z}_n' - x_n'\|^2 + \mu_{2,n}^2\|A_2\tilde{z}_n' - A_2 x_n'\|^2 \\ &\quad + \langle 2\tilde{z}_n' - 2x_n' + 2\mu_{2,n}A_2 x_n' - 2\mu_{2,n}A_2\tilde{z}_n', \tilde{z}_n' - p'\rangle \\ &\leq d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + \mu_{2,n}^2\|A_2\tilde{z}_n' - A_2 x_n'\|^2 \\ &\quad + \langle 2\tilde{z}_n' - 2x_n' + 2\mu_{2,n}A_2 x_n' - 2\mu_{2,n}A_2\tilde{z}_n', \tilde{z}_n' - p'\rangle. \end{aligned} \tag{11}$$

In the geodesic triangle $\Delta(A_2 x_n, A_2\tilde{z}_n, z_n)$ and its comparison triangle $\Delta(A_2 x_n', A_2\tilde{z}_n', z_n')$. Using Lemma 3 again, we have $d(A_2 x_n', A_2\tilde{z}_n') = d(A_2 x_n, A_2\tilde{z}_n)$. From (11), we obtain

$$\begin{aligned} d^2(z_n, p) &\leq d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + \mu_{2,n}^2 d^2(A_2\tilde{z}_n, A_2 x_n) \\ &\quad + \langle 2\tilde{z}_n' - 2x_n' + 2\mu_{2,n}A_2 x_n' - 2\mu_{2,n}A_2\tilde{z}_n', \tilde{z}_n' - p'\rangle. \end{aligned} \tag{12}$$

Consider the geodesic triangle $\Delta(a,b,c)$ and its comparison triangle $\Delta(a',b',c')$. Then set $a = 2\exp_{x_n}^{-1} y_n - 2\mu_{2,n}(A_2\tilde{z}_n - A_2 x_n)$ and $b = \exp_p^{-1}\tilde{z}_n$, (resp., $a' = 2\tilde{z}_n' - 2x_n' + 2\mu_{2,n}A_2 x_n') -$

$2\mu_{2,n}A_2\tilde{z}'_n$ and $b' = \tilde{z}'_n - p'$). Let β and β' denote the angles at c and c', respectively. Then by Lemma 4 (i), we have $\beta' \geq \beta$ and so $\cos\beta' \leq \cos\beta$. Then by Proposition 1 and Lemma 3, we have

$$\langle a', b'\rangle = \|a'\|\|b'\|\cos\beta' \leq \|a'\|\|b'\|\cos\beta = \|a\|\|b\|\cos\beta = \langle a, b\rangle.$$

It is easy to see that

$$\begin{aligned} &\langle 2\tilde{z}'_n - 2x'_n + 2\mu_{2,n}A_2x'_n - 2\mu_{2,n}A_2\tilde{z}'_n, \tilde{z}'_n - p'\rangle \\ &\leq \langle 2\exp^{-1}_{x_n}\tilde{z}_n - 2\mu_{2,n}(A_2\tilde{z}_n - A_2x_n), \exp^{-1}_p\tilde{z}_n\rangle. \end{aligned} \tag{13}$$

Due to (12) and (13), it follows that

$$\begin{aligned} d^2(z_n, p) &\leq d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + \mu^2_{2,n}d^2(A_2\tilde{z}_n, A_2x_n) \\ &\quad + \langle 2\tilde{z}'_n - 2x'_n + 2\mu_{2,n}A_2x'_n - 2\mu_{2,n}A_2\tilde{z}'_n, \tilde{z}'_n - p'\rangle \\ &\leq d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + \mu^2_{2,n}d^2(A_2\tilde{z}_n, A_2x_n) \\ &\quad + \langle 2\exp^{-1}_{x_n}\tilde{z}_n - 2\mu_{2,n}(A_2\tilde{z}_n - A_2x_n), \exp^{-1}_p\tilde{z}_n\rangle \\ &= d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + \mu^2_{2,n}d^2(A_2\tilde{z}_n, A_2x_n) \\ &\quad - 2\mu_{2,n}\langle A_2\tilde{z}_n - A_2x_n, \exp^{-1}_p\tilde{z}_n\rangle + 2\langle\exp^{-1}_{x_n}\tilde{z}_n, \exp^{-1}_p\tilde{z}_n\rangle. \end{aligned} \tag{14}$$

From (9), (10) and (14), we have

$$\begin{aligned} d^2(z_n, p) &\leq d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + \mu^2_{2,n}d^2(A_2\tilde{z}_n, A_2x_n) \\ &\quad + 2\mu_{2,n}(\langle A_2x_n, \exp^{-1}_p\tilde{z}_n\rangle - \langle A_2p, \exp^{-1}_p\tilde{z}_n\rangle) - 2\mu_{2,n}\langle A_2x_n, \exp^{-1}_p\tilde{z}_n\rangle \\ &= d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + \mu^2_{2,n}d^2(A_2\tilde{z}_n, A_2x_n) - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p\tilde{z}_n\rangle. \end{aligned} \tag{15}$$

According to (ALSR) and (15), we obtain

$$\begin{aligned} d^2(z_n, p) &\leq d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + \lambda_2^2d^2(\tilde{z}_n, x_n) - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p\tilde{z}_n\rangle \\ &= d^2(x_n, p) - (1-\lambda_2^2)d^2(\tilde{z}_n, x_n) - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p\tilde{z}_n\rangle. \end{aligned} \tag{16}$$

In a similar way,

$$d^2(x_{n+1}, q) \leq d^2(z_n, q) - (1-\lambda_1^2)d^2(y_n, z_n) - 2\mu_{1,n}\langle A_1q, \exp^{-1}_q y_n\rangle. \tag{17}$$

Next, we restrict $(p, q) \in \mathcal{S}$. Then, substituting (16) for (17) with $q := p$ sends us to

$$\begin{aligned} d^2(x_{n+1}, p) &\leq d^2(x_n, p) - (1-\lambda_2^2)d^2(\tilde{z}_n, x_n) - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p\tilde{z}_n\rangle \\ &\quad - (1-\lambda_1^2)d^2(y_n, z_n) - 2\mu_{1,n}\langle A_1p, \exp^{-1}_p y_n\rangle \\ &= d^2(x_n, p) - (1-\lambda_2^2)d^2(\tilde{z}_n, x_n) - (1-\lambda_1^2)d^2(y_n, z_n) \\ &\quad - 2\mu_{2,n}\langle A_2p, \exp^{-1}_p\tilde{z}_n\rangle - 2\mu_{1,n}\langle A_1p, \exp^{-1}_p y_n\rangle. \end{aligned}$$

This together with the hypotheses, implies that $d(x_{n+1}, p) \leq d(x_n, p)$. So the sequence $\{x_n\}$ is bounded. In the same way, substituting (17) for (16) with $n := n+1$ and $p := q$ implies

$$\begin{aligned} d^2(z_{n+1}, q) &\leq d^2(z_n, q) - (1-\lambda_1^2)d^2(y_n, z_n) - 2\mu_{1,n}\langle A_1q, \exp^{-1}_q y_n\rangle \\ &\quad - (1-\lambda_2^2)d^2(\tilde{z}_{n+1}, x_{n+1}) - 2\mu_{2,n+1}\langle A_2q, \exp^{-1}_q\tilde{z}_{n+1}\rangle \\ &= d^2(z_n, q) - (1-\lambda_2^2)d^2(\tilde{z}_{n+1}, x_{n+1}) - (1-\lambda_1^2)d^2(y_n, z_n) \\ &\quad - 2\mu_{2,n+1}\langle A_2q, \exp^{-1}_q\tilde{z}_{n+1}\rangle - 2\mu_{1,n}\langle A_1q, \exp^{-1}_q y_n\rangle. \end{aligned}$$

This together with the hypotheses, implies that $d(z_{n+1}, q) \leq d(z_n, q)$. So the sequence $\{z_n\}$ is bounded. $\square$

Corollary 2. *Let $\{x_n\}$ and $\{z_n\}$ be the iterative sequences constructed via Algorithm 1 with $A_1 = A_2 = A$. Then both $\{x_n\}$ and $\{z_n\}$ are bounded.*

Proof. Take a fixed $p \in S$ arbitrarily. Noticing $A_1 = A_2 = A$, we obtain from (16) and (17)

$$\begin{cases} d^2(x_{n+1}, p) \le d^2(x_n, p) - (1-\lambda_2^2)d^2(\tilde{z}_n, x_n) - (1-\lambda_1^2)d^2(y_n, z_n), \\ d^2(z_{n+1}, p) \le d^2(z_n, p) - (1-\lambda_2^2)d^2(\tilde{z}_{n+1}, x_{n+1}) - (1-\lambda_1^2)d^2(y_n, z_n). \end{cases}$$

i.e., $d(x_n, p) \ge d(x_{n+1}, p)$ and $d(z_n, p) \ge d(z_{n+1}, p)$. So the sequences $\{x_n\}$ and $\{z_n\}$ are bounded. Also, noticing $\lambda_1, \lambda_2 \in (0,1)$ in Algorithm 2, we have

$$\begin{cases} (1-\lambda_2^2)d^2(\tilde{z}_n, x_n) + (1-\lambda_1^2)d^2(y_n, z_n) \le d^2(x_n, p) - d^2(x_{n+1}, p), \\ (1-\lambda_2^2)d^2(\tilde{z}_{n+1}, x_{n+1}) + (1-\lambda_1^2)d^2(y_n, z_n) \le d^2(z_n, p) - d^2(z_{n+1}, p), \end{cases}$$

and so $\lim_{n\to\infty} d(\tilde{z}_n, x_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$. So the sequences $\{\tilde{z}_n\}$ and $\{y_n\}$ are bounded. Note that

$$\begin{cases} z_n = \exp_{\tilde{z}_n}(\mu_{2,n}(Ax_n - A\tilde{z}_n)), \\ x_{n+1} = \exp_{y_n}(\mu_{1,n}(Az_n - Ay_n)). \end{cases}$$

Since A_i is L_i-Lipschitz continuous for $i = 1, 2$, we conclude that $\lim_{n\to\infty} d(z_n, \tilde{z}_n) = 0$ and $\lim_{n\to\infty} d(x_{n+1}, y_n) = 0$. □

Algorithm 2: Parallel Tseng's extragradient method with line-search.

Initialization: Given $\gamma_i > 0$, $l_i \in (0,1)$, $\lambda_i \in (0,1)$ for $i = 1, 2$. Let $x_0 \in \mathcal{M}$.

Iterative Steps:

Step 1. Calculate

$$\begin{cases} \tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n}Ax_n)), \\ y_n = P_C(\exp_{z_n}(-\mu_{1,n}Az_n)), \end{cases}$$

where $\mu_{i,n}$ is chosen to be the largest $\mu_i \in \{\gamma_i, \gamma_i l_i, \gamma_i l_i^2, ...\}$ for $i = 1, 2$, satisfying

$$\begin{cases} \mu_2 d(Ax_n, A\tilde{z}_n) \le \lambda_2 d(x_n, \tilde{z}_n), \\ \mu_1 d(Az_n, Ay_n) \le \lambda_1 d(z_n, y_n). \end{cases}$$

Step 2. Calculate

$$\begin{cases} z_n = \exp_{\tilde{z}_n}(\mu_{2,n}(Ax_n - A\tilde{z}_n)), \\ x_{n+1} = \exp_{y_n}(\mu_{1,n}(Az_n - Ay_n)). \end{cases}$$

$n \leftarrow n+1$ and go to Step 1.

Theorem 1. *Let $\{x_n\}$ and $\{z_n\}$ be the iterative sequences constructed via Algorithm 1. Assume that the hypotheses in Lemma 7 hold. Then $\{(x_n, z_n)\}$ is convergent to a solution of SVI (5) provided $d(x_n, y_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$ as $n \to \infty$.*

Proof. First of all, by Lemma 7, we know that $\{x_n\}$ and $\{z_n\}$ are bounded, and

$$d(x_{n+1}, p) \le d(x_n, p) \quad \text{and} \quad d(z_{n+1}, q) \le d(z_n, q) \quad \forall (p, q) \in \mathcal{S}, n \ge 0.$$

Utilizing the assumption that $d(x_n, y_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$ as $n \to \infty$, we obtain that $\{y_n\}$ and $\{\tilde{z}_n\}$ are bounded. We define the sets S_1, S_2 as follows:

$$S_1 = \{p \in C : \exists q \in C \text{ such that } (p, q) \in \mathcal{S}\} \ \text{ and } \ S_2 = \{q \in C : \exists p \in C \text{ such that } (p, q) \in \mathcal{S}\}.$$

From Definition 2, we know that $\{x_n\}$ and $\{z_n\}$ are Fejér convergent to S_1 and S_2, respectively. Let $\bar{p}$ be an accumulation point of $\{x_n\}$. Then $\exists\ \{x_{n_k}\} \subset \{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \bar{p}$. Since $\{z_n\}, \{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ all are bounded, we may assume, without loss of generality, that $z_{n_k} \to \bar{q}$, $\mu_{1,n_k} \to \bar{\mu}_1$ and $\mu_{2,n_k} \to \bar{\mu}_2$ as $k \to \infty$. Since $d(x_n, y_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$ as $n \to \infty$, we deduce that $y_{n_k} \to \bar{p}$ and $\tilde{z}_{n_k} \to \bar{q}$. Note that

$$\begin{cases} \tilde{z}_{n_k} = P_C(\exp_{x_{n_k}}(-\mu_{2,n_k} A_2 x_{n_k})), \\ y_{n_k} = P_C(\exp_{z_{n_k}}(-\mu_{1,n_k} A_1 z_{n_k})). \end{cases}$$

Hence by Lemma 2, we get

$$\begin{aligned} 0 &\leq \langle \exp^{-1}_{x_{n_k}} \tilde{z}_{n_k} + \mu_{2,n_k} A_2 x_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x \rangle \\ &= \langle \exp^{-1}_{x_{n_k}} \tilde{z}_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x \rangle + \mu_{2,n_k} \langle A_2 x_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x \rangle \\ &= \langle \exp^{-1}_{x_{n_k}} \tilde{z}_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x \rangle + \mu_{2,n_k} \langle A_2 x_{n_k}, \exp^{-1}_{\tilde{z}_{n_k}} x_{n_k} \rangle + \mu_{2,n_k} \langle A_2 x_{n_k}, \exp^{-1}_{x_{n_k}} x \rangle, \end{aligned} \tag{18}$$

and

$$\begin{aligned} 0 &\leq \langle \exp^{-1}_{z_{n_k}} y_{n_k} + \mu_{1,n_k} A_1 z_{n_k}, \exp^{-1}_{y_{n_k}} x \rangle \\ &= \langle \exp^{-1}_{z_{n_k}} y_{n_k}, \exp^{-1}_{y_{n_k}} x \rangle + \mu_{1,n_k} \langle A_1 z_{n_k}, \exp^{-1}_{y_{n_k}} x \rangle \\ &= \langle \exp^{-1}_{z_{n_k}} y_{n_k}, \exp^{-1}_{y_{n_k}} x \rangle + \mu_{1,n_k} \langle A_1 z_{n_k}, \exp^{-1}_{y_{n_k}} z_{n_k} \rangle + \mu_{1,n_k} \langle A_1 z_{n_k}, \exp^{-1}_{z_{n_k}} x \rangle. \end{aligned} \tag{19}$$

By Lemma 6 we have $\mu_{i,n} > \frac{\lambda_i l_i}{L_i}$ for $i = 1, 2$. Passing to the limit, and combining (6) with (18) and (19), respectively, we get

$$\begin{cases} 0 \leq \langle \exp^{-1}_{\bar{p}} \bar{q}, \exp^{-1}_{\bar{q}} x \rangle + \bar{\mu}_2 \langle A_2 \bar{p}, \exp^{-1}_{\bar{q}} \bar{p} \rangle + \bar{\mu}_2 \langle A_2 \bar{p}, \exp^{-1}_{\bar{p}} x \rangle, \\ 0 \leq \langle \exp^{-1}_{\bar{q}} \bar{p}, \exp^{-1}_{\bar{p}} x \rangle + \bar{\mu}_1 \langle A_1 \bar{q}, \exp^{-1}_{\bar{p}} \bar{q} \rangle + \bar{\mu}_1 \langle A_1 \bar{q}, \exp^{-1}_{\bar{q}} x \rangle. \end{cases}$$

Consequently,

$$\begin{cases} \langle \exp^{-1}_{\bar{q}} \bar{p} + \bar{\mu}_1 \langle A_1 \bar{q}, \exp^{-1}_{\bar{p}} x \rangle \geq 0 \quad \forall x \in C, \\ \langle \exp^{-1}_{\bar{p}} \bar{q} + \bar{\mu}_2 \langle A_2 \bar{p}, \exp^{-1}_{\bar{q}} x \rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{20}$$

This means that $(\bar{p}, \bar{q}) \in \mathcal{S}$, and hance $\bar{p} \in S_1$. So it follows from Proposition 3 that $x_n \to \bar{p}$ as $n \to \infty$.

On the other hand, suppose that $\hat{q}$ is an accumulation point of $\{z_n\}$. Then $\exists\ \{z_{m_k}\} \subset \{z_n\}$ such that $\lim_{k\to\infty} z_{m_k} = \hat{q}$. Since $\{x_n\}, \{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ all are bounded, we may assume, without loss of generality, that $x_{m_k} \to \hat{p}$, $\mu_{1,m_k} \to \hat{\mu}_1$ and $\mu_{2,m_k} \to \hat{\mu}_2$ as $k \to \infty$. Since $d(x_n, y_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$ as $n \to \infty$, we deduce that $y_{m_k} \to \hat{p}$ and $\tilde{z}_{m_k} \to \hat{q}$. Note that

$$\begin{cases} \tilde{z}_{m_k} = P_C(\exp_{x_{m_k}}(-\mu_{2,m_k} A_2 x_{m_k})), \\ y_{m_k} = P_C(\exp_{z_{m_k}}(-\mu_{1,m_k} A_1 z_{m_k})). \end{cases}$$

Similar ideas to (20) give

$$\begin{cases} \langle \exp^{-1}_{\hat{q}} \hat{p} + \hat{\mu}_1 \langle A_1 \hat{q}, \exp^{-1}_{\hat{p}} x \rangle \geq 0 \quad \forall x \in C, \\ \langle \exp^{-1}_{\hat{p}} \hat{q} + \hat{\mu}_2 \langle A_2 \hat{p}, \exp^{-1}_{\hat{q}} x \rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{21}$$

This means that $(\hat{p}, \hat{q}) \in \mathcal{S}$, and hance $\hat{q} \in S_2$. By Proposition 3, we get $z_n \to \hat{q}$ as $n \to \infty$. Therefore, in terms of the uniqueness of the limit, we have $\{(x_n, z_n)\}$ converges to $(\hat{p}, \hat{q}) \in \mathcal{S}$ to the SVI (5). This completes the proof. □

Theorem 2. *Let $\{x_n\}$ and $\{z_n\}$ be the iterative sequences constructed via Algorithm 2. Then $\{x_n\}$ and $\{z_n\}$ both are convergent to a solution of VIP (4).*

Proof. By Corollary 2 and Definition 2, we know that $\{x_n\}$ and $\{z_n\}$ both are Fejér convergent to the same S. Let $\bar{p}$ be an accumulation point of $\{x_n\}$. Then $\exists \{x_{n_k}\} \subset \{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \bar{p}$. Hence, from $\lim_{k\to\infty} d(\tilde{z}_{n_k}, x_{n_k}) = 0$ we get $\lim_{k\to\infty} \tilde{z}_{n_k} = \bar{p}$. Since $\{\mu_{2,n}\}$ is bounded, we may assume, without loss of generality, that $\lim_{k\to\infty} \mu_{2,n_k} = \bar{\mu}_2$. So it follows from $\tilde{z}_{n_k} = P_C(\exp_{x_{n_k}}(-\mu_{2,n_k} A x_{n_k}))$ that $\bar{p} = P_C(\exp_{\bar{p}}(-\bar{\mu}_2 A \bar{p}))$. In terms of Proposition 2, we get $\bar{p} \in S$. Thus, by Proposition 3 we infer that $x_n \to \bar{p}$ as $n \to \infty$. In a similar way, we can show that $z_n \to \bar{q}$ as $n \to \infty$ for some $\bar{q} \in S$. Since $d(\tilde{z}_n, x_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$, we derive the desired result. $\square$

3.2. Parallel Tseng's Extragradient Method

To solve problem (5), we give the following Algorithm 3, that is, a parallel Tseng's extragradient algorithm. The step sizes in this algorithm are obtained by simple updating, rather than using the line-search, which results in a lower computational cost.

Algorithm 3: Parallel Tseng's extragradient method.

Initialization. Given $\mu_{i,0} > 0$, $\lambda_i \in (0,1)$ for $i = 1,2$, and $x_0 \in \mathcal{M}$ an arbitrary starting point.
Iterative Steps:
Step 1. Compute

$$\begin{cases} \tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n} A_2 x_n)), \\ y_n = P_C(\exp_{z_n}(-\mu_{1,n} A_1 z_n)). \end{cases}$$

Step 2. Compute

$$\begin{cases} z_n = \exp_{\tilde{z}_n} \mu_{2,n}(A_2 x_n - A_2 \tilde{z}_n), \\ x_{n+1} = \exp_{y_n} \mu_{1,n}(A_1 z_n - A_1 y_n), \end{cases}$$

and

$$\begin{cases} \mu_{2,n+1} = \begin{cases} \min\{\frac{\lambda_2 d(x_n,\tilde{z}_n)}{d(A_2 x_n, A_2 \tilde{z}_n)}, \mu_{2,n}\}, & \text{if } d(A_2 x_n, A_2 \tilde{z}_n) \neq 0, \\ \mu_{2,n}, & \text{otherwise}, \end{cases} \\ \mu_{1,n+1} = \begin{cases} \min\{\frac{\lambda_1 d(z_n,y_n)}{d(A_1 z_n, A_1 y_n)}, \mu_{1,n}\}, & \text{if } d(A_1 z_n, A_1 y_n) \neq 0, \\ \mu_{1,n}, & \text{otherwise}. \end{cases} \end{cases}$$

$n \leftarrow n+1$ and go to Step 1.

In particular, putting $A_1 = A_2 = A$ in Algorithm 3, we obtain the following Algorithm 4, that is, a parallel Tseng's extragradient method for solving VIP (4).

Algorithm 4: Parallel Tseng's extragradient method.

Initialization. Given $\mu_{i,0} > 0$, $\lambda_i \in (0,1)$ for $i = 1,2$, and $x_0 \in \mathcal{M}$ an arbitrary starting point.
Iterative Steps:
Step 1. Compute

$$\begin{cases} \tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n} A x_n)), \\ y_n = P_C(\exp_{z_n}(-\mu_{1,n} A z_n)). \end{cases}$$

Step 2. Compute

$$\begin{cases} z_n = \exp_{\tilde{z}_n} \mu_{2,n}(A x_n - A \tilde{z}_n), \\ x_{n+1} = \exp_{y_n} \mu_{1,n}(A z_n - A y_n), \end{cases}$$

and

$$\begin{cases} \mu_{2,n+1} = \begin{cases} \min\{\frac{\lambda_2 d(x_n,\tilde{z}_n)}{d(A x_n, A \tilde{z}_n)}, \mu_{2,n}\}, & \text{if } d(A x_n, A \tilde{z}_n) \neq 0, \\ \mu_{2,n}, & \text{otherwise}, \end{cases} \\ \mu_{1,n+1} = \begin{cases} \min\{\frac{\lambda_1 d(z_n,y_n)}{d(A z_n, A y_n)}, \mu_{1,n}\}, & \text{if } d(A z_n, A y_n) \neq 0, \\ \mu_{1,n}, & \text{otherwise}. \end{cases} \end{cases}$$

$n \leftarrow n+1$ and go to Step 1.

Lemma 8. *For $i = 1, 2$, the sequence $\{\mu_{i,n}\}$ constructed via Algorithm 3 is monotonically decreasing with lower bound* $\min\{\frac{\lambda_i}{L_i}, \mu_{i,0}\}$.

Proof. Obviously, the sequence $\{\mu_{i,n}\}$ is monotonically decreasing for $i = 1, 2$. Note that A_i is a Lipschitz continuous mapping with constant $L_i > 0$ for $i = 1, 2$. Then, in the case of $d(A_2x_n, A_2\tilde{z}_n) \neq 0$, we have

$$\frac{\lambda_2 d(x_n, \tilde{z}_n)}{d(A_2x_n, A_2\tilde{z}_n)} \geq \frac{\lambda_2 d(x_n, \tilde{z}_n)}{L_2 d(x_n, \tilde{z}_n)} = \frac{\lambda_2}{L_2}. \tag{22}$$

Thus, the sequence $\{\mu_{2,n}\}$ has the lower bound $\min\{\frac{\lambda_2}{L_2}, \mu_{2,0}\}$. In a similar way, we can show that $\{\mu_{1,n}\}$ has the lower bound $\min\{\frac{\lambda_1}{L_1}, \mu_{1,0}\}$. □

Corollary 3. *For $i = 1, 2$, the sequence $\{\mu_{i,n}\}$ constructed via Algorithm 4 is monotonically decreasing with lower bound* $\min\{\frac{\lambda_i}{L}, \mu_{i,0}\}$.

Lemma 9. *Let $\{x_n\}$ and $\{z_n\}$ be the iterative sequences constructed via Algorithm 3. Then both $\{x_n\}$ and $\{z_n\}$ are bounded, provided for all $(p, q) \in \mathcal{S}$ and $n \geq 0$,*

$$\begin{cases} (1 - \mu_{2,n}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+1}^2}) d^2(\tilde{z}_n, x_n) + (1 - \mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}^2}) d^2(y_n, z_n) \\ \quad + 2\mu_{2,n} \langle A_2 p, \exp_p^{-1} \tilde{z}_n \rangle + 2\mu_{1,n} \langle A_1 p, \exp_p^{-1} y_n \rangle \geq 0, \\ (1 - \mu_{2,n+1}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+2}^2}) d^2(\tilde{z}_{n+1}, x_{n+1}) + (1 - \mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}^2}) d^2(y_n, z_n) \\ \quad + 2\mu_{2,n+1} \langle A_2 q, \exp_q^{-1} \tilde{z}_{n+1} \rangle + 2\mu_{1,n} \langle A_1 q, \exp_q^{-1} y_n \rangle \geq 0. \end{cases}$$

Proof. Similar to the proof of Lemma 7, we get

$$\begin{aligned} d^2(z_n, p) \quad &\leq d^2(x_n, p) + d^2(\tilde{z}_n, x_n) + \mu_{2,n}^2 d^2(A_2\tilde{z}_n, A_2x_n) \\ &\quad - 2\mu_{2,n} \langle A_2\tilde{z}_n - A_2x_n, \exp_p^{-1} \tilde{z}_n \rangle + 2\langle \exp_{x_n}^{-1} \tilde{z}_n, \exp_p^{-1} x_n \rangle. \end{aligned} \tag{23}$$

Then from (23), we have

$$\begin{aligned} d^2(z_n, p) \quad &\leq d^2(x_n, p) + d^2(\tilde{z}_n, x_n) - 2\langle \exp_{x_n}^{-1} \tilde{z}_n, \exp_{x_n}^{-1} \tilde{z}_n \rangle + 2\langle \exp_{x_n}^{-1} \tilde{z}_n, \exp_p^{-1} \tilde{z}_n \rangle \\ &\quad + \mu_{2,n}^2 d^2(A_2\tilde{z}_n, A_2x_n) - 2\mu_{2,n} \langle A_2\tilde{z}_n - A_2x_n, \exp_p^{-1} \tilde{z}_n \rangle \\ &= d^2(x_n, p) - d^2(\tilde{z}_n, x_n) + 2\langle \exp_{x_n}^{-1} \tilde{z}_n, \exp_p^{-1} \tilde{z}_n \rangle \\ &\quad + \mu_{2,n}^2 d^2(A_2\tilde{z}_n, A_2x_n) - 2\mu_{2,n} \langle A_2\tilde{z}_n - A_2x_n, \exp_p^{-1} \tilde{z}_n \rangle. \end{aligned} \tag{24}$$

Since $\tilde{z}_n = P_C(\exp_{x_n}(-\mu_{2,n} A_2 x_n))$, by Lemma 2, we have

$$\langle \exp_{x_n}^{-1} \tilde{z}_n + \mu_{2,n} A_2 x_n, \exp_p^{-1} \tilde{z}_n \rangle \leq 0,$$

that is,

$$\langle \exp_{x_n}^{-1} \tilde{z}_n, \exp_p^{-1} \tilde{z}_n \rangle \leq -\mu_{2,n} \langle A_2 x_n, \exp_p^{-1} \tilde{z}_n \rangle. \tag{25}$$

Combining (24), (25) and (22) yields

$$\begin{aligned} d^2(z_n, p) \quad &\leq d^2(x_n, p) - d^2(\tilde{z}_n, x_n) - 2\mu_{2,n} \langle A_2 x_n, \exp_p^{-1} \tilde{z}_n \rangle \\ &\quad + \mu_{2,n}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+1}^2} d^2(\tilde{z}_n, x_n) - 2\mu_{2,n} \langle A_2\tilde{z}_n - A_2x_n, \exp_p^{-1} \tilde{z}_n \rangle. \end{aligned} \tag{26}$$

Also, from the monotonicity of A_2 on $\mathcal{M}$ it follows that

$$\begin{aligned}&\langle A_2\tilde{z}_n - A_2x_n, \exp_p^{-1}\tilde{z}_n\rangle\\ &= \langle A_2\tilde{z}_n - A_2p, \exp_p^{-1}\tilde{z}_n\rangle + \langle A_2p - A_2x_n, \exp_p^{-1}\tilde{z}_n\rangle\\ &\geq \langle A_2p - A_2x_n, \exp_p^{-1}\tilde{z}_n\rangle = \langle A_2p, \exp_p^{-1}\tilde{z}_n\rangle - \langle A_2x_n, \exp_p^{-1}\tilde{z}_n\rangle.\end{aligned} \tag{27}$$

From (26) and (27), we obtain

$$\begin{aligned}d^2(z_n,p) &\leq d^2(x_n,p) - d^2(\tilde{z}_n,x_n) - 2\mu_{2,n}\langle A_2x_n, \exp_p^{-1}\tilde{z}_n\rangle\\ &+ \mu_{2,n}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+1}^2} d^2(\tilde{z}_n,x_n) - 2\mu_{2,n}(\langle A_2p, \exp_p^{-1}\tilde{z}_n\rangle - \langle A_2x_n, \exp_p^{-1}\tilde{z}_n\rangle)\\ &= d^2(x_n,p) - (1-\mu_{2,n}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+1}^2})d^2(\tilde{z}_n,x_n) - 2\mu_{2,n}\langle A_2p, \exp_p^{-1}\tilde{z}_n\rangle.\end{aligned} \tag{28}$$

In a similar way, we get

$$d^2(x_{n+1},q) \leq d^2(z_n,q) - (1-\mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}^2})d^2(y_n,z_n) - 2\mu_{1,n}\langle A_1q, \exp_q^{-1}y_n\rangle. \tag{29}$$

Next, we restrict $(p,q) \in \mathcal{S}$. Then, substituting (28) for (29) with $q := p$, we have

$$\begin{aligned}d^2(x_{n+1},p) \quad &\leq d^2(x_n,p) - (1-\mu_{2,n}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+1}^2})d^2(\tilde{z}_n,x_n) - 2\mu_{2,n}\langle A_2p, \exp_p^{-1}\tilde{z}_n\rangle\\ &\quad - (1-\mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}})d^2(y_n,z_n) - 2\mu_{1,n}\langle A_1p, \exp_p^{-1}y_n\rangle\\ &= d^2(x_n,p) - (1-\mu_{2,n}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+1}^2})d^2(\tilde{z}_n,x_n) - (1-\mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}^2})d^2(y_n,z_n)\\ &\quad - 2\mu_{2,n}\langle A_2p, \exp_p^{-1}\tilde{z}_n\rangle - 2\mu_{1,n}\langle A_1p, \exp_p^{-1}y_n\rangle.\end{aligned}$$

This together with the hypotheses, implies that $d(x_{n+1},p) \leq d(x_n,p)$. So the sequence $\{x_n\}$ is bounded. In the same way, substituting (29) for (28) with $n := n+1$ and $p := q$, we have

$$\begin{aligned}d^2(z_{n+1},q) \quad &\leq d^2(z_n,q) - (1-\mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}})d^2(y_n,z_n) - 2\mu_{1,n}\langle A_1q, \exp_q^{-1}y_n\rangle\\ &\quad - (1-\mu_{2,n+1}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+2}^2})d^2(\tilde{z}_{n+1},x_{n+1}) - 2\mu_{2,n+1}\langle A_2q, \exp_q^{-1}\tilde{z}_{n+1}\rangle\\ &= d^2(z_n,q) - (1-\mu_{2,n+1}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+2}^2})d^2(\tilde{z}_{n+1},x_{n+1}) - (1-\mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}})d^2(y_n,z_n)\\ &\quad - 2\mu_{2,n+1}\langle A_2q, \exp_q^{-1}\tilde{z}_{n+1}\rangle - 2\mu_{1,n}\langle A_1q, \exp_q^{-1}y_n\rangle.\end{aligned}$$

This together with the hypotheses, implies that $d(z_{n+1},q) \leq d(z_n,q)$. So the sequence $\{z_n\}$ is bounded. □

Corollary 4. *Let $\{x_n\}$ and $\{z_n\}$ be constructed via Algorithm 4. Then $\{x_n\}$ and $\{z_n\}$ are bounded.*

Proof. Take a fixed $p \in S$ arbitrarily. Noticing $A_1 = A_2 = A$, we deduce from (28) and (29) that

$$\begin{cases} d^2(x_{n+1},p) \leq d^2(x_n,p) - (1-\mu_{2,n}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+1}^2})d^2(\tilde{z}_n,x_n) - (1-\mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}^2})d^2(y_n,z_n),\\ d^2(z_{n+1},p) \leq d^2(z_n,p) - (1-\mu_{2,n+1}^2 \cdot \frac{\lambda_2^2}{\mu_{2,n+2}^2})d^2(\tilde{z}_{n+1},x_{n+1}) - (1-\mu_{1,n}^2 \cdot \frac{\lambda_1^2}{\mu_{1,n+1}})d^2(y_n,z_n).\end{cases}$$

Since $\lim_{n\to\infty}(1-\mu_{i,n}^2 \cdot \frac{\lambda_i^2}{\mu_{i,n+1}^2}) = 1-\lambda_i^2 > 0$ for $i = 1,2$, we know that there exists $n_0 \geq 0$ such that $1-\mu_{i,n}^2 \cdot \frac{\lambda_i^2}{\mu_{i,n+1}^2} > 0\ \forall n \geq n_0$ for $i = 1,2$. This implies that $d(x_n,p) \geq d(x_{n+1},p)$ and $d(z_n,p) \geq$

$d(z_{n+1}, p)$. So the sequences $\{x_n\}$ and $\{z_n\}$ are bounded. It can be seen that $\lim_{n\to\infty} d(\tilde{z}_n, x_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$. Hence $\{\tilde{z}_n\}$ and $\{y_n\}$ are bounded. Note that

$$\begin{cases} z_n = \exp_{\tilde{z}_n}(\mu_{2,n}(Ax_n - A\tilde{z}_n)), \\ x_{n+1} = \exp_{y_n}(\mu_{1,n}(Az_n - Ay_n)). \end{cases}$$

Since A_i is L_i-Lipschitz continuous for $i = 1, 2$, we conclude that $\lim_{n\to\infty} d(z_n, \tilde{z}_n) = 0$ and $\lim_{n\to\infty} d(x_{n+1}, y_n) = 0$. □

Theorem 3. *Let $\{x_n\}$ and $\{z_n\}$ be constructed via Algorithm 3. Assume that the hypotheses in Lemma 9 hold. Then $\{(x_n, z_n)\}$ is convergent to a solution of SVI (5) provided $d(x_n, y_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$ as $n \to \infty$.*

Proof. First of all, by Lemma 8, the limit of $\{\mu_{i,n}\}$ exists for $i = 1, 2$. We denote $\mu_i = \lim_{n\to\infty} \mu_{i,n}$, then $\mu_i > 0$ for $i = 1, 2$. By Lemma 9 $\{x_n\}$ and $\{z_n\}$ are bounded, and

$$d(x_n, p) \geq d(x_{n+1}, p) \quad \text{and} \quad d(z_n, q) \geq d(z_{n+1}, q) \quad \forall (p, q) \in \mathcal{S}, n \geq 0.$$

By assumption that $d(x_n, y_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$ as $n \to \infty$, we obtain that $\{y_n\}$ and $\{\tilde{z}_n\}$ are bounded. Define the sets S_1, S_2 as follows:

$$S_1 = \{p \in C : \exists q \in C \text{ such that } (p, q) \in \mathcal{S}\} \text{ and } S_2 = \{q \in C : \exists p \in C \text{ such that } (p, q) \in \mathcal{S}\}.$$

From Definition 2, we know that $\{x_n\}$ and $\{z_n\}$ are Fejér convergent to S_1 and S_2, respectively. Then $\exists\ \{x_{n_k}\} \subset \{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \bar{p}$. Since $\{z_n\}$ is bounded, we may assume, without loss of generality, that $z_{n_k} \to \bar{q}$. Meantime, it is clear that $\mu_{1,n_k} \to \mu_1$ and $\mu_{2,n_k} \to \mu_2$ as $k \to \infty$. Since $d(x_n, y_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$ as $n \to \infty$, we deduce that $y_{n_k} \to \bar{p}$ and $\tilde{z}_{n_k} \to \bar{q}$. Note that

$$\begin{cases} \tilde{z}_{n_k} = P_C(\exp_{x_{n_k}}(-\mu_{2,n_k} A_2 x_{n_k})), \\ y_{n_k} = P_C(\exp_{z_{n_k}}(-\mu_{1,n_k} A_1 z_{n_k})). \end{cases}$$

Letting $k \to \infty$ and using Lemma 1 and Lipschitz continuity of $A_i, i = 1, 2$, we obtain

$$\begin{cases} \bar{q} = P_C(\exp_{\bar{p}}(-\mu_2 A_2 \bar{p})), \\ \bar{p} = P_C(\exp_{\bar{q}}(-\mu_1 A_1 \bar{q})). \end{cases}$$

Thus, $\bar{p} = P_C(\exp_I(-\mu_1 A_1))P_C(\exp_I(-\mu_2 A_2))\bar{p}$. By Lemma 5 we get $(\bar{p}, \bar{q}) \in \mathcal{S}$, and hence $\bar{p} \in S_1$. By Proposition 3, $x_n \to \bar{p}$ as $n \to \infty$.

On the other hand, suppose that $\hat{q}$ is an accumulation point of $\{z_n\}$. Then $\lim_{k\to\infty} z_{m_k} = \hat{q}$, where $\{z_{m_k}\} \subset \{z_n\}$ is some subsequence. Since $\{x_n\}$ is a bounded iterative sequence, we may suppose $x_{m_k} \to \hat{p}$. Meantime, it is clear that $\mu_{1,m_k} \to \mu_1$ and $\mu_{2,m_k} \to \mu_2$ as $k \to \infty$. Since $d(x_n, y_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$ as $n \to \infty$, we deduce that $y_{m_k} \to \hat{p}$ and $\tilde{z}_{m_k} \to \hat{q}$. Note that

$$\begin{cases} \tilde{z}_{m_k} = P_C(\exp_{x_{m_k}}(-\mu_{2,m_k} A_2 x_{m_k})), \\ y_{m_k} = P_C(\exp_{z_{m_k}}(-\mu_{1,m_k} A_1 z_{m_k})). \end{cases}$$

Letting $k \to \infty$ and using Lemma 1 and Lipschitz continuity of $A_i, i = 1, 2$ reaches

$$\begin{cases} \hat{q} = P_C(\exp_{\hat{p}}(-\mu_2 A_2 \hat{p})), \\ \hat{p} = P_C(\exp_{\hat{q}}(-\mu_1 A_1 \hat{q})). \end{cases}$$

Thus, $\hat{p} = P_C(\exp_I(-\mu_1 A_1))P_C(\exp_I(-\mu_2 A_2))\hat{p}$. By Lemma 5 we get $(\hat{p}, \hat{q}) \in \mathcal{S}$, and hance $\hat{q} \in S_2$. By Proposition 3, $z_n \to \hat{q}$ as $n \to \infty$. Therefore, in terms of the uniqueness of the limit, we infer that $(x_n, z_n) \to (\hat{p}, \hat{q}) \in \mathcal{S}$ to the SVI (5). □

Theorem 4. *Assume that $\{x_n\}$ and $\{z_n\}$ are constructed via Algorithm 4. Then $\{x_n\}$ and $\{z_n\}$ both are convergent to a solution of VIP (4).*

Proof. By Corollary 4 and Definition 2, one knows that $\{x_n\}$ and $\{z_n\}$ both are Fejér convergent to the same S. Let $\bar{p}$ be an accumulation point of $\{x_n\}$. Thus, $\exists\ \{x_{n_k}\} \subset \{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \bar{p}$. Hence, from $\lim_{k\to\infty} d(\tilde{z}_{n_k}, x_{n_k}) = 0$ we get $\lim_{k\to\infty} \tilde{z}_{n_k} = \bar{p}$. Since $\lim_{n\to\infty} \mu_{2,n} = \mu_2$, we have $\lim_{k\to\infty} \mu_{2,n_k} = \mu_2$. So it follows from $\tilde{z}_{n_k} = P_C(\exp_{x_{n_k}}(-\mu_{2,n_k} A x_{n_k}))$ that $\bar{p} = P_C(\exp_{\bar{p}}(-\mu_2 A\bar{p}))$. In terms of Proposition 2, we get $\bar{p} \in S$. Thus, by Proposition 3 we infer that $x_n \to \bar{p}$ as $n \to \infty$. Similarly, one can obtain $z_n \to \bar{q}$ as $n \to \infty$ for some $\bar{q} \in S$. Since $d(\tilde{z}_n, x_n) \to 0$ and $d(z_n, \tilde{z}_n) \to 0$, we derive the desired result. □

4. Concluding Remark

In this paper, we focused to systems of variational inequalities on Hadamard manifolds and present two algorithms to deal with it under the monotonicity assumption on the underlying vector fields. We considered two strategies for obtaining step sizes. The second has many advantages; simple structure, low computational cost and no requiring extra projection. To design more effective methods for the problem (5) on Hadamard manifolds, we will consider the geometric structure of manifolds and some numerical implementations in the future. Since every complete and connected Riemannian manifold is a geodesic metric space (see, e.g., [38]), our results can be obtained in geodesic spaces.

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