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Analysis on Complete Set of Fock States with Explicit Wavefunctions for the Covariant Harmonic Oscillator Problem

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Received: 5 November 2019; Accepted: 17 December 2019; Published: 23 December 2019

Abstract: The earlier treatments of the Lorentz covariant harmonic oscillator have brought to light various difficulties, such as reconciling Lorentz symmetry with the full Fock space, and divergence issues with their functional representations. We present here a full solution avoiding those problems. The complete set of Fock states is obtained, together with the corresponding explicit wavefunctions and their inner product integrals free from any divergence problem and with Lorentz symmetry fully maintained without additional constraints imposed. By a simple choice of the pseudo-unitary representation of the underlying symmetry group, motivated from the perspective of the Minkowski spacetime as a representation for the Lorentz group, we obtain the natural non-unitary Fock space picture commonly considered, although not formulated and presented in the careful details given here. From a direct derivation of the appropriate basis state wavefunctions of the finite-dimensional irreducible representations of the Lorentz symmetry, the relation between the latter and the Fock state wavefunctions is also explicitly shown. Moreover, the full picture, including the states with a non-positive norm, may give a consistent physics picture as a version of Lorentz covariant quantum mechanics. The probability interpretation for the usual von Neumann measurements is not a problem, as all wavefunctions restricted to a definite value for the 'time' variable are just like those of the usual time independent quantum mechanics. A further understanding from a perspective of the dynamics from the symplectic geometry of the phase space is shortly discussed.

Keywords: Covariant Harmonic Oscillator; Pseudo-unitary Representation; Lorentz Symmetry

1. Introduction

The importance of the harmonic oscillator problem in quantum mechanics can hardly be overstated. It is then easy to appreciate that the problem, as formulated with the classical Minkowski spacetime instead of the Newtonian one as the starting point, received a lot of attention, since physicists started to think about 'relativistic quantum mechanics' [1]. Here, we ar talking about the latter relaxed from the usual textbook usage of the term. In fact, one may think about the quantum theory that is found in the textbooks as having become a kind of a standard only because of the failure to obtain a nice covariant formulation physicists would like to have. The covariant harmonic oscillator problem cannot avoid such a setting though. The obvious theoretical principle one would want to impose is covariance under the Lorentz symmetry $SO(1,3)$. Perhaps we should make it clear that it is not our intention to fully address the general issue of the 'relativistic' generalization of the standard quantum harmonic oscillator problem here. Nor do we want to discuss different formulations of 'relativistic quantum mechanics'. Such tasks are certainly much beyond our scope here. For that matter, there have been various different approaches including ones in favor of going outside the framework of Lorentz or Poincaré symmetry [2–6], which are also of interest. For our case, it suffices to say that the big practical success of high energy physics under the framework of quantum field theory certainly suggests that the Lorentz covariant problem we focus on here deserves serious study. Furthermore, there certainly has been no lack of efforts in that direction from the beginning. The topic has been revisited more recently in Reference [7]. The noncompact nature of the Lorentz group leads to some quite nontrivial issues, as illustrated therein. We present here a full analysis on the natural non-unitary Fock space formulation which gives the complete set of Fock states with explicit wavefunction solutions that meet the best expectations one could have for the Lorentz covariance feature. Moreover, there is no divergence in any of the wavefunctions or integrals for their inner products. This can only be achieved by giving up on the complete unitarity. We explain why replacing it with a pseudo-unitarity, reflecting the Minkowski instead of the Euclidean nature of the classical spacetime, is not only reasonable but desirable (see more in Ref. [8]).

The consideration of unitarity as a necessary requirement for quantum mechanics is tied to the Born probability interpretation. There is, however, a very sensible way to look at quantum mechanics without the latter [9,10]. Furthermore, the simple bottom line here is that even in the setting of quantum mechanics with the Copenhagen interpretation, the Born probability picture should not be strictly required to be extended to a spacetime description. For a particle, maintaining the total probability of finding it somewhere in the space, at a particular moment of its existence, to be unity is one thing, asking for the same in the spacetime is quite another. A von Neumann measurement of an observable for a spacetime wavefunction without specifying, or at least restricting, the time does not seem to be anything we can do anyway.

To lay the background for comparison, we summarize here the key features of the usual unitary formulations below. Readers may consult Reference [7] for details.

A naive formulation of a version of the quantum harmonic oscillator problem on the otherwise classical Minkowski spacetime can be seen as a solution to the eigenvalue equation

$$\frac{1}{2\hbar}\left(\hat{X}_\mu \hat{X}^\mu + \hat{P}_\mu \hat{P}^\mu\right)\psi_\lambda(x^\mu) = \lambda\psi_\lambda(x^\mu)\;, \tag{1}$$

with, in a direct analog to a three dimensional harmonic oscillator problem, the position, $\hat{X}_\mu$, and the momentum, $\hat{P}_\mu$, operators represented by x_μ and $-i\hbar\frac{\partial}{\partial x^\mu}$, respectively, satisfying the commutation relation $[\hat{X}_\mu, \hat{P}_\nu] = i\hbar\eta_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$ is the Minkowski metric.

The operator on the left hand side of Equation (1) can be written in terms of the Lorentz covariant ladder operators

$$\hat{a}_\mu = \hat{X}_\mu + i\hat{P}_\mu\;, \qquad \hat{a}^\dagger_\mu = \hat{X}_\mu - i\hat{P}_\mu\;; \qquad \left[\hat{a}_\mu, \hat{a}^\dagger_\nu\right] = 2\hbar\eta_{\mu\nu}\;, \tag{2}$$

while the eigenfunctions $\psi_\lambda(x^\mu)$ correspond to the eigenstates of a (shifted) number operator. The nth-level states are obtained by applying n raising operators on a ground state, denoted $|0\rangle$, annihilated by all the lowering operators. There is a freedom in the choice of operators to be taken as raising/lowering, giving rise to different Fock spaces. For a ground-state wavefunction $\langle x^\mu|0\rangle \sim e^{\mp\frac{x_\mu x^\mu}{2\hbar}}$, annihilated by $\hat{a}_\mu$ ($\hat{a}^\dagger_\mu$) for upper (lower) sign, to avoid divergence, one has to constrain the x^μ vectors to the spacelike (timelike) domain. There is still a very tricky normalization problem. In fact, because of the infinite range of the boost parameter, the squared-integral norm still diverges and has to be handled somehow tactically (e.g., by redefining the norm so as to factor out that infinite volume element). It is not clear at all there can be a mathematically consistent definition of the norm that leaves an interesting enough set of Fock states normalizable. Moreover, an abstract algebraic analysis yields many states with a negative norm since, for the spacelike Fock space, $\langle 0|\hat{a}_0\hat{a}^\dagger_0|0\rangle = -1$. Similarly, $\langle 0|\hat{a}^\dagger_i\hat{a}_i|0\rangle = -1$ for the timelike case. With $\hat{X}_\mu$ and $\hat{P}_\mu$ defined Hermitian, all Lorentz transformations are unitary, which means they preserve an inner product of positive definite norms. That is in direct conflict with the notion that the four $\hat{a}^{\dagger}_{\mu}$, hence four $\hat{a}^{\dagger}_{\mu}\left|0\right\rangle$ states (or four $\hat{a}_{\mu}\left|0\right\rangle$ states), should transform as a Minkowski four-vector. It is then not surprising at all that Bars [7] concluded that unitarity and covariance together leave only Lorentz invariant states as admissible, which is however really saying that Lorentz symmetry is completely trivialized, hence essentially not there. The paper does give some results and discussion about the nonunitary Fock states though, only far from the explicit full results we present below.

One different way to obtain a unitary positively-normed space of states is to take as the ground state the one annihilated by $\hat{a}^{\dagger}_{0}$ and $\hat{a}_{i}$ $(i = 1, 2, 3)$. All Focks states obtained from it have positive norms and states at each level n and form an infinite dimensional irreducible unitary representation of the Lorentz group. However, the ground state is not a Lorentz invariant [7]. Lorentz symmetry should hence be considered spontaneously broken, in contrast to our objectives.

The structure of the paper is the following. Section 2 is dedicated to the pseudo-unitary representation for the Lorentz covariant harmonic oscillator problem. In Section 2.1 it is motivated from a parallel with the Minkowski spacetime representation of the Lorentz symmetry. Section 2.2 contains the explicit operator formulation and Fock states wavefunctions, whose transformation properties under Lorentz boosts are illustrated in Section 2.3. In Section 2.4 we present the Lorentz invariant pseudo-unitary inner product on the Hilbert space spanned by Fock states, in an algebraic as well as integral form, and elaborate further on the Lorentz structure of the Hilbert space. Section 3 gives a direct derivation of the functional form of the basis states of finite dimensional irreducible representations of the Lorentz symmetry in relation to the problem and their explicit connection to the results in Section 2. We address issues related to interpretations of the results in Section 4, before concluding in Section 5.

2. A Pseudo-Unitary Representation

2.1. Motivation

The Minkowski spacetime is a pseudo-unitary irreducible representation of the Lorentz symmetry. Its associated invariant is an indefinite vector norm of signature $(1, 3)$. Each transformation acts on a four-vector as a (real) $SU(1, 3)$ matrix. It is this pseudo-unitary representation that reduces properly back to the reducible $1 + 3$ dimensional representation of the Newtonian space and time. Such non-unitarity we see as the defining signature of spacetime physics. The full $SO(3)$ invariant Fock space of the harmonic oscillator serves well as a solid picture of the single particle phase space under quantum mechanics, especially under the serious treatment of rigged Hilbert space formulation [11,12], giving full justice to the Hermitian nature of the position and momentum operators. A similar treatment of the $SO(1, 3)$ version could play an equivalent role in the proper formulation of Lorentz covariant quantum mechanics [8]. One way or another, the essence of going from Newtonian physics to 'relativistic' physics should be like a direct consequence of embedding the Newtonian space and time into the Minkowski spacetime. It is then very desirable to have the $SO(3)$ invariant Fock space, for the 'three dimensional' quantum harmonic oscillator problem, sits inside a full $SO(1, 3)$ invariant Fock space in a manner directly analogous to how the Newtonian space sits inside the Minkowski spacetime.

We want to have a complete set of Fock states with sensible norms as solutions to the problem, keeping the Lorentz symmetry while maintaining that there are four $n = 1$ states transforming as a Minkowski four-vector. As an irreducible representation of the Lorentz group, the latter corresponds to a non-unitary one. However, it is the same non-unitarity of the Minkowski spacetime as a representation space. That is the natural framework to see the problem as a Lorentz covariant version of the rotational covariant picture of the 'three dimensional' theory. The indefinite Minkowski norm is what is preserved by the Lorentz transformations. We seek its natural extension in the form of pseudo-unitary norm for the Fock space, upon the restriction to the subspace of the four $n = 1$ states.

2.2. Operator Representations and Fock States with Hermite Polynomials

We start at the level of symmetry or algebraic structure at the abstract level. The symbols X_μ, P_μ, a_μ, $\bar{a}_\mu$, …etc. are to be seen as abstract algebraic quantities, for which we seek a representation as operators on a Hilbert space. The relevant Lie algebra is that of $H_R(1,3)$, given as

$$\begin{aligned}
&[J_{\mu\nu}, J_{\rho\sigma}] = i\hbar \left(\eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma}\right) \ , \\
&[J_{\mu\nu}, X_\rho] = i\hbar \left(\eta_{\mu\rho} X_\nu - \eta_{\nu\rho} X_\mu\right) \ , \\
&[J_{\mu\nu}, P_\rho] = i\hbar \left(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu\right) \ , \\
&[X_\mu, P_\nu] = i\hbar \eta_{\mu\nu} I \ ,
\end{aligned} \tag{3}$$

for which we focus on representations of the Lorentz symmetry with vanishing spins, i.e., its six generators $J_{\mu\nu}(\equiv -J_{\nu\mu})$ can be taken as $J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$. The unitary representation as a direct extension of the $H_R(3)$ case, with only X_i and P_i, for standard quantum mechanics is straightforward [13,14]. Yet, at least when applied to the harmonic oscillator problem, the Fock state wavefunctions have undesirable behavior and divergence unless restricted to spacelike or timelike domains, under which there may be other mathematical issues for the full theory. Additionally, the integral inner products in either case contain a divergent volume factor that has to be artificially dropped for them to make sense. These have been well analyzed in Reference [13], with their undesirable Lorentz transformation properties also well addressed in Reference [7], as summarized above. The pseudo-unitary representation is obtained as

$$\begin{aligned}
X_i &\to \hat{X}_i & \qquad P_i &\to \hat{P}_i \\
X_0 &\to i\hat{X}_4 & \qquad P_0 &\to i\hat{P}_4 \ ,
\end{aligned} \tag{4}$$

where, as operators on a space of functions of real variables x^a $(a = 1,2,3,4)$,

$$\hat{X}_a = x_a, \qquad \hat{P}_a = -i\hbar \frac{\partial}{\partial x^a} \equiv -i\hbar \partial_a \ . \tag{5}$$

We have $[\hat{X}_a, \hat{P}_b] = i\hbar\delta_{ab}$, with δ_{ab} being the Kronecker delta symbol. Note that X_0 and P_0, and hence $J_{0\nu}$, are represented by anti-Hermitian operators, therefore we have a non-unitary representation of the group $H_R(1,3)$ or its subgroup $SO(1,3)$. For the complex combinations a_μ and $\bar{a}_\mu$, we have then

$$\begin{aligned}
\hat{a}_0 &= i\left(\hat{X}_4 + i\hat{P}_4\right) = i\hat{a}_4 \\
\hat{\bar{a}}_0 &= i\left(\hat{X}_4 - i\hat{P}_4\right) = i\hat{a}_4^\dagger \ ,
\end{aligned} \tag{6}$$

while $\hat{a}_a = \hat{X}_a + i\hat{P}_a$, $\hat{a}_a^\dagger = \hat{X}_a - i\hat{P}_a$, satisfying $[\hat{a}_\mu, \hat{\bar{a}}_\nu] = 2\hbar\eta_{\mu\nu}$ and $[\hat{a}_a, \hat{a}_b^\dagger] = 2\hbar\delta_{ab}$. The (total) number operator can be written as

$$\hat{N} = \frac{1}{2\hbar}\eta^{\mu\nu}\hat{\bar{a}}_\mu\hat{a}_\nu = \frac{1}{2\hbar}\delta^{ab}\hat{a}_a^\dagger\hat{a}_b \ , \tag{7}$$

and decomposed into a sum of the Hermitian number operators $\hat{N} = \hat{N}_0 + \hat{N}_1 + \hat{N}_2 + \hat{N}_3$, where $\hat{N}_i = \frac{1}{2\hbar}\hat{a}_i^\dagger\hat{a}_i$ and $\hat{N}_0 = -\frac{1}{2\hbar}\hat{\bar{a}}_0\hat{a}_0 = \frac{1}{2\hbar}\hat{a}_4^\dagger\hat{a}_4 = \hat{N}_4$, easily seen from (6). We have $[\hat{N}_a, \hat{a}_a] = -\hat{a}_a$ and $[\hat{N}_a, \hat{a}_a^\dagger] = \hat{a}_a^\dagger$. The normalized Fock states are eigenstates of $\hat{N}_a$ operators,

$$\hat{N}_a \left|n_1, n_2, n_3; n_4\right\rangle = n_a \left|n_1, n_2, n_3; n_4\right\rangle \ . \tag{8}$$

Solving (8) in x^a coordinates, in which

$$\begin{aligned}\hat{N}_a &= \frac{1}{2\hbar}\left(\hat{X}_a^2 + \hat{P}_a^2\right) - \frac{1}{2} \\ &= \frac{1}{2\hbar}\left(x_a^2 - \hbar^2\partial_a^2\right) - \frac{1}{2}\,,\end{aligned} \tag{9}$$

we obtain the eigenstate wavefunctions as

$$\langle x^a|n_1,n_2,n_3;n_4\rangle = \frac{1}{\pi\hbar}e^{-\frac{x_a x^a}{2\hbar}}\,\tilde{H}_{n_1}\!\left(\frac{x^1}{\sqrt{\hbar}}\right)\tilde{H}_{n_2}\!\left(\frac{x^2}{\sqrt{\hbar}}\right)\tilde{H}_{n_3}\!\left(\frac{x^3}{\sqrt{\hbar}}\right)\tilde{H}_{n_4}\!\left(\frac{x^4}{\sqrt{\hbar}}\right)\,, \tag{10}$$

where $\tilde{H}_{n_a} = [2^{n_a} n_a!]^{-\frac{1}{2}} H_{n_a}$, $H_{n_a}\!\left(\frac{x^a}{\sqrt{\hbar}}\right)$ are the standard Hermite polynomials. Hence, we have an explicit solution for a complete set of the Fock states wavefunctions without any problem of the other formulations.

In terms of $\hat{X}_a$ and $\hat{P}_a$, the above is just like a quantum version of the harmonic oscillator in the four Euclidean classical dimensions. The Hilbert space spanned by the eigenstate wavefunctions looks completely conventional with an inner product giving a positive definite norm for the eigenstates in a usual manner. However, we only have to introduce the notation $\hat{X}_0 = i\hat{X}_4$ and $\hat{P}_0 = i\hat{P}_4$ to see that $(\hat{N}+2) = \frac{1}{2\hbar}\eta^{\mu\nu}\left(\hat{X}_\mu\hat{X}_\nu + \hat{P}_\mu\hat{P}_\nu\right)$ corresponds exactly to the naively expected Hamiltonian of the covariant harmonic oscillator in Equation (1). It is interesting to note that identifying $\hat{X}_0$ simply as $-\hat{X}_4$ (and $\hat{X}^0$ as $\hat{X}^4$), and the same for $\hat{P}_0$, works too though the Hermitian $\hat{X}_0$ and $\hat{P}_0$ then differs from the representations of X_0 and P_0 with an i factor. The non-unitary nature of the representation and a sensible notion of a pseudo-unitary inner product on the Hilbert space can be seen by looking into the eigenstates and their transformation properties under the Lorentz symmetry, which we turn to next.

2.3. Transformation Properties under the Lorentz Boosts

The Lorentz-algebra generators $J_{\mu\nu}$ are represented by the operators $\hat{J}_{\mu\nu} = \hat{X}_\mu\hat{P}_\nu - \hat{X}_\nu\hat{P}_\mu$, where $\hat{J}_{ij}$ form a usual, unitarily represented $SO(3)$ subalgebra of spatial rotations, while

$$\hat{J}_{0i} = \hat{X}_0\hat{P}_i - \hat{X}_i\hat{P}_0 = i\left(\hat{X}_4\hat{P}_i - \hat{X}_i\hat{P}_4\right) \tag{11}$$

are the anti-Hermitian boost operators. To examine the nature of the obtained states under the Lorentz transformation, we act with the boost generator in the, arbitrarily chosen, x^3 direction on the eigenstate (10). Using the properties of Hermite polynomials we get

$$\hat{J}_{03}\,|n_1,n_2,n_3;n_4\rangle = \hbar\left(n_3\sqrt{\frac{n_3}{n_4+1}}\,|n_1,n_2,n_3{-}1;n_4{+}1\rangle - n_4\sqrt{\frac{n_4}{n_3+1}}\,|n_1,n_2,n_3{+}1;n_4{-}1\rangle\right)\,. \tag{12}$$

We look into $n=1$ level, as those four states should correspond to the components of a four vector. From (12) we can obtain $\hat{J}_{03}$ as a matrix

$$\hat{J}_{03} = \hbar\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\,. \tag{13}$$

Exponentiating, we get

$$e^{i\frac{\alpha}{\hbar}\hat{J}_{03}} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cosh\alpha&-i\sinh\alpha\\0&0&i\sinh\alpha&\cosh\alpha\end{pmatrix} \equiv \hat{\Lambda}_{03}\,, \tag{14}$$

the corresponding finite boost by the real parameter α. Alternatively, we can see the transformation as a rotation in x^3-x^4 plane by a purely imaginary angle $i\alpha$. We find the action of $\hat{\Lambda}_{03}$ on arbitrary function $f(x^a)$ as

$$\left(\pi(\hat{\Lambda}_{03})f\right)(x^a) = f\left(\hat{\Lambda}_{03}^{-1}x\right) = f\left(x^1, x^2, \cosh\alpha\, x^3 + i\sinh\alpha\, x^4, -i\sinh\alpha\, x^3 + \cosh\alpha\, x^4\right) . \tag{15}$$

In particular,

$$\begin{aligned}\pi(\hat{\Lambda}_{03})\,\langle x^a|0,0,1;0\rangle &= \frac{1}{\pi\hbar}e^{-\frac{x_a x^a}{2\hbar}}\tilde{H}_1\!\left(\frac{\cosh\alpha\, x^3 + i\sinh\alpha\, x^4}{\sqrt{\hbar}}\right) \\ &= \cosh\alpha\,\langle x^a|0,0,1;0\rangle + i\sinh\alpha\,\langle x^a|0,0,0;1\rangle \ ,\end{aligned} \tag{16}$$

$$\begin{aligned}\pi(\hat{\Lambda}_{03})\,\langle x^a|0,0,0;1\rangle &= \frac{1}{\pi\hbar}e^{-\frac{x_a x^a}{2\hbar}}\tilde{H}_1\!\left(\frac{-i\sinh\alpha\, x^3 + \cosh\alpha\, x^4}{\sqrt{\hbar}}\right) \\ &= -i\sinh\alpha\,\langle x^a|0,0,1;0\rangle + \cosh\alpha\,\langle x^a|0,0,0;1\rangle \ ,\end{aligned} \tag{17}$$

while $\langle x^a|1,0,0;0\rangle$ and $\langle x^a|0,1,0;0\rangle$ are invariant as $f(x_a x^a)$ is obviously invariant under any Lorentz transformation.

Seen differently, we can introduce $|n\rangle_0 \equiv |n_0; n_1, n_2, n_3\rangle = (i)^{n_4}\,|n_1, n_2, n_3; n_4\rangle$, with $n_0 = n_4$, to show that in the basis formed by four $|n=1\rangle_0$ states, $\hat{\Lambda}_{03}$ takes the usual form

$$\begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} , \tag{18}$$

preserving a Minkowski norm on the real span of the four $|n=1\rangle_0$ vectors. The states hence transform as components of a Minkowski four-vector. In fact, that real span can actually be seen as a model of the Minkowski spacetime with the $SO(3)$ invariant subspace spanned by the single $n_4 = 1$ state and the complementary subspace spanned by the three $n_4 = 0$ states, modeling the Newtonian time and space, respectively.

The Minkowski norm, or the extension of it to the complex span of the $|n = 1\rangle_0$ vectors, and further to the whole Hilbert space spanned by all the Fock states, is definitely not unitary. We seek exactly an inner product, or rather an invariant bilinear functional [15], different from the standard $\langle\phi|\phi'\rangle$ corresponding to the L^2-norm for the wavefunctions, one that is invariant under any Lorentz transformation.

2.4. The Pseudo-Unitary Inner Product or Invariant Bilinear Functional

Fock states wavefunctions, given in Equation (10), are orthonormal according to $\int \langle n_1, n_2, n_3; n_4|x^a\rangle\,\langle x^a|m_1, m_2, m_3; m_4\rangle\; d^4x = \delta_{nm}$, as the inner product is usually defined on a unitary Hilbert space. Label n here is to be understood as $(n_1, n_2, n_3; n_4)$, and similar for m. Therefore, we have $\langle n|m\rangle = \delta_{nm}$. Since the Lorentz transformations, boosts in particular, are not represented by unitary operators, such an inner product cannot be preserved in general, as can easily be seen from the results above, e.g., we have

$$\left\langle \hat{\Lambda}_{03}\,(0,0,1;0)\,|\hat{\Lambda}_{03}\,(0,0,1;0)\right\rangle = \cosh^2\alpha + \sinh^2\alpha \neq 1 \, . \tag{19}$$

Instead, we define another inner product given through the Fock state basis as

$$\langle\!\langle n|m\rangle\!\rangle = (-1)^{n_4}\,\langle n_1, n_2, n_3; n_4|m_1, m_2, m_3; m_4\rangle = (-1)^{n_4}\delta_{nm} \ , \tag{20}$$

and extend it to the full vector space assuming sesqulinearity. It gives an indefinite norm, which is the natural extension of the Minkowski norm on the subspace of the real span of the four $|n=1\rangle_0$ vectors, and is invariant under the Lorentz transformations. In particular, for the boost $\hat{\Lambda}_{03}$ we have

$$\left\langle\!\!\left\langle \hat{\Lambda}_{03}(0,0,1;0)|\hat{\Lambda}_{03}(0,0,1;0)\right\rangle\!\!\right\rangle = \langle\langle 0,0,1;0|0,0,1;0\rangle\rangle = 1\ , \tag{21}$$

$$\left\langle\!\!\left\langle \hat{\Lambda}_{03}(0,0,0;1)|\hat{\Lambda}_{03}(0,0,0;1)\right\rangle\!\!\right\rangle = \langle\langle 0,0,0;1|0,0,0;1\rangle\rangle = -1\ . \tag{22}$$

Moreover, a state vector that is proportional to the sum or difference of the two states here above have zero norm under the inner product. We have, in general, spacelike, timelike, and lightlike state vectors with positive, negative, and vanishing pseudo-unitary norms, respectively. All vectors have finite norm and are all normalizable to the norm values of $1, 0$, and -1, though the notion of normalization is an empty one for the lightlike states, obviously. It is important to note that normalizations with respect to $\langle\cdot|\cdot\rangle$ and $\langle\langle\cdot|\cdot\rangle\rangle$ are in general not the same. All the basis Fock states are, however, normalized with respect to both, and none of them is lightlike.

Splitting the pseudo-unitary inner product notation $\langle\langle\phi|\phi'\rangle\rangle$, one should consider the ket $|\phi'\rangle\rangle$ as simply another notation for $|\phi'\rangle$, while the bra $\langle\langle\phi|$ as a linear functional is in general different from $\langle\phi|$. We have explicitly $\langle\langle n| = (-1)^{n_4}\langle n|$ which defines all $\langle\langle\phi|$ implicitly. We have then for the inner product

$$\begin{aligned}
\langle\langle\phi|\phi'\rangle\rangle \quad &= \sum_n \langle\langle\phi|n\rangle\langle n|\phi'\rangle\rangle = \sum_n \langle\langle\phi|n\rangle\rangle\,\langle n|\phi'\rangle \\
&= \sum_n \overline{\langle\langle n|\phi\rangle\rangle}\,\langle n|\phi'\rangle = \int d^4x \sum_n \langle\phi|n\rangle\,(-1)^{n_4}\,\langle n|x^a\rangle\,\langle x^a|\phi'\rangle \\
&= \int d^4x \sum_n \langle\phi|n\rangle \left\langle n|x^i,-x^4\right\rangle \langle x^a|\phi'\rangle \\
&= \int d^4x \left\langle\phi|x^i,-x^4\right\rangle \langle x^a|\phi'\rangle\ ,
\end{aligned} \tag{23}$$

where we have used the fact that the wavefunctions $\langle x^a|n\rangle$, given explicitly in Equation (10), are odd and even in x^4 for odd and even n_4, respectively. This gives a nice integral representation of it in terms of the wavefunctions (In some sense, it may be more proper to write things in terms of an alternative formulation of the wavefunctions as $\langle\langle x^a|\phi\rangle\rangle = \sum_n (-1)^{n_4}\overline{\langle\langle n|x^a\rangle\rangle}\langle\langle n|\phi\rangle\rangle = \sum_n (-1)^{n_4}\langle x^a|n\rangle\langle n|\phi\rangle$. The latter is however a lot more clumsy to work with. Moreover, having two wavefunction representations of the states here only causes potential confusion.). Note that on the Hilbert space for a non-unitary representation of a noncompact group, there may not exist an invariant inner product. Certainly not a positive definite one. The wavefunctions of the states may not be squared integrable either. The appropriate structure to look for is an invariant bilinear functional [15]. Our $\langle\langle\phi|\phi'\rangle\rangle$ inner product is exactly a gadget of that kind.

There is a simple way to write the mathematical relation between the two inner products that gives also an easy way to see the Lorentz invariance of the pseudo-unitary one. It is given in terms of a parity operator $\mathcal{P}_4$, which sends x^4 to $-x^4$, as

$$\langle\langle\phi|\phi'\rangle\rangle = \langle\phi|\mathcal{P}_4|\phi'\rangle\ . \tag{24}$$

We can actually take this as the definition. The $(-1)^{n_4}$ factor in our definition of the inner product in term of the Fock state basis above is exactly the $\mathcal{P}_4$ eigenvalue of $|n\rangle$. With it, we have nicely

$$\langle\langle\phi|\phi'\rangle\rangle = \int d^4x\,\langle\phi|\mathcal{P}_4|x^a\rangle\,\langle x^a|\phi'\rangle = \int d^4x \left\langle\phi|x^i,-x^4\right\rangle \langle x^a|\phi'\rangle\ . \tag{25}$$

A good way to appreciate the Lorentz structure of the Hilbert space spanned by the Fock states is the following. We first look at the parallel for the case of the 'three dimensional' quantum harmonic oscillator. The three $n = 1$ states transform under $SO(3)$ as components of an Euclidean three-vector. The $n = 0$ state is invariant. The two constant n-level subspaces are vector spaces for the three dimensional defining representation and the trivial representations of $SO(3)$. For the $n = 2$ level, it corresponds exactly to the symmetric part of the product of two $n = 1$ representations, i.e., transforming as the Euclidean symmetric two tensor and the invariant ($n = 0$). The standard $n = 2$ wavefunctions clearly show that, for an explicit check. One goes on to the higher n-tensors for the higher n levels. As also similarly discussed in Reference [7], actually for the general Minkowski case, at the n level, the full set of Fock states is a symmetric tensor of $SU(1,3)$ which reduces to irreducible representations of $SO(1,3)$ corresponding to the rank of the traceless tensors in the decomposition. The rank numbers are n, $n-2$, $\ldots$, (0 or 1). The pattern is essentially the same for any '$l+m$ dimensional' harmonic oscillator with the Fock states at level n obtained by the action of n creation operators on the $n = 0$ state, with the $l+m$ independent creation operators transforming as a $(l+m)$-vector. The structure is not sensitive to the actual background signature (l,m) the latter has. Such representations, of $SU(l,m)$ or $SO(l,m)$, are unitary only for the Euclidean case. For 'three dimensional' case, the rank of each of those traceless (Cartesian) tensors is exactly the j value. Back to our $1+3$ Lorentzian case, the finite dimensions of those traceless irreducible tensors are given by the square of rank plus one, with the full result explicitly illustrated in the next section. The way the Fock states for the 'three dimensional' states sit inside our Fock states at each n-level can also be easily traced from the perspective of the Cartesian tensors.

The nature of the n-level states as components of the symmetric n-tensors can also be directly seen by looking at the wavefunctions given in terms of products of the Hermite polynomials with the common invariant factor $e^{-\frac{x_a x^a}{2\hbar}}$, which is essentially the $n = 0$ state wavefunction. It is then easy to appreciate the right pseudo-unitary inner product as given by Equation (20) or Equation (24). The norm as an invariant should better be expressed as $\int d^4x \, \langle\phi|x_a\rangle\langle x^a|\phi\rangle$ so that the upper indices in the wavefunction $\langle x^a|\phi\rangle$ can be contracted with the lower indices in the otherwise conjugate function $\langle\phi|x_a\rangle$. For an Euclidean case $x_a = x^a$, as for the unitary inner product. To get to the pseudo-unitary inner product which goes along with the Minkowski nature of the tensors, it is then obvious that we only need to turn the x_4 variable appearing in $\langle\phi|x_a\rangle$, which are the tensors with lower indices, into $-x^4$ or $\eta_{00}x^4$. The extra i factor involved in the exact state for the $n = 1$ level as the component of a Minkowski four-vector, as discussed right above and in relation to Equation (18), does not matter, due to the sequlinearity of the inner product. The invariant factor $e^{-\frac{x_a x^a}{2\hbar}}$, of course, does not change, though it is to be interpreted as $e^{-\frac{1}{2\hbar}\Sigma(x^a)^2}$ and $e^{-\frac{1}{2\hbar}(-x_4)^2-\frac{1}{2\hbar}\Sigma(x_i)^2}$, accordingly.

3. Fock States as Representations of Lorentz Symmetry

Our final task is to relate our Fock states to the basis states of the irreducible representations of the Lorentz symmetry explicitly derived. Since the number operator $\hat{N}$ commutes with Lorentz transformations, the collection of Fock states at a fixed level n spans a Lorentz representation, generally reducible into a sum of finite-dimensional irreducible ones. Noncompactness of the Lorentz group implies the non-unitarity of the latter. They can be labeled [16] (see also Ref. [15]) by two independent numbers (j_o, c), the integer or half-integer j_o, corresponding to the spin, and the complex number c, characterizing the spin independent Casimir invariant. Since our problem at hand is spinless, we simply drop the vanishing j_o. A convenient labeling of basis states is given by $|j_o, c; j, m\rangle$, hence $|c; j, m\rangle$, which transform as the familiar angular momentum states $|j, m\rangle$ under the $SO(3)$ subgroup. For finite dimensional representations, which we are interested in, c is a natural number and $j = 0, 1, \ldots, c-1$. $c = 1$ is a trivial representation. All the others are nonunitary. The smallest nontrivial one, $c = 2$, is then a sum of $j = 0$ and $j = 1$ representations of $SO(3)$, which is the complex extension of the one for the Minkowski spacetime. The dimensions of such irreducible representations are simply given by c^2.

To find the explicit formulation of Fock states in terms of the irreducible Lorentz representations we first solve the relevant differential equations for the latter, obtaining the corresponding functions in a coordinate form, and show the way to obtain any Fock state as their linear combinations. The result is completely in accordance with discussions in the last part of the previous section.

The basis functions for irreducible Lorentz representation can be found as solutions of the eigenvalue equations of the nonzero Lorentz algebra Casimir operator $\hat{C}$, and the standard spherical harmonic in three dimensions. Additionally, we impose condition on solutions to be eigenfunctions of the number operator $\hat{N}$ and denote such functions by $\psi^n_{c;jm} \equiv \langle x^a|n;c;j,m\rangle$. We have the following system of equations

$$\left\{\hat{N}, \hat{C}, \hat{J}^2, \hat{J}_{12}\right\} \psi^n_{c;jm} = \left\{n, \hbar^2(c^2-1), \hbar^2 j(j+1), \hbar m\right\} \psi^n_{c;jm} \,, \tag{26}$$

where $\hat{C} = \frac{1}{2}\hat{J}_{\mu\nu}\hat{J}^{\mu\nu}$ is here represented by

$$\hat{C} = -\hbar^2(x^a x_a)(\partial^a\partial_a) + 2\hbar^2 x^a\partial_a + \hbar^2(x^a\partial_a)^2 \,. \tag{27}$$

We make the following general coordinatization

$$\begin{aligned} x^1 &= r\sqrt{1-u^2}\cos\phi\sin\theta\,, & x^2 &= r\sqrt{1-u^2}\sin\phi\sin\theta\,, \\ x^3 &= r\sqrt{1-u^2}\cos\theta\,, & x^4 &= ru\,, \end{aligned} \tag{28}$$

where $x_a x^a = r^2$, hence u has the range $[-1,1]$. A measure according to which the solutions have to be normalized is

$$dx = r^3\sqrt{1-u^2}\sin\theta\, dr\, du\, d\theta\, d\phi \,. \tag{29}$$

We have $\psi^n_{c;jm} \propto Y_{jm}(\theta,\phi)$, and assume factorization $\psi^n_{c;jm} \propto R(r)U(u)Y_{jm}(\theta,\phi)$. We are left with two equations; one is $\hat{C}\psi^n_{c;jm} = \hbar^2(c^2-1)\psi^n_{c;jm}$, and the other one is a number operator equation which can be casted simply as

$$-\hbar^2\partial^a\partial_a\psi^n_{c;jm} = \left(2\hbar(\hat{N}+2) - x^a x_a\right)\psi^n_{c;jm} = \left(2\hbar(n+2) - r^2\right)\psi^n_{c;jm} \,. \tag{30}$$

Expressed in terms of the new coordinates and combined, we get

$$\left(1-u^2\right)\frac{d^2U(u)}{du^2} - 3u\frac{dU(u)}{du} + \left(c^2 - 1 - \frac{j(j+1)}{(1-u^2)}\right)U(u) = 0\,, \tag{31}$$

$$\frac{d^2R(r)}{dr^2} + \frac{3}{r}\frac{dR(r)}{dr} + \left(\frac{2\hbar(2+n) - r^2}{\hbar^2} + \frac{1-c^2}{r^2}\right)R(r) = 0\,. \tag{32}$$

With the condition that the obtained functions are orthogonal, solutions are related to Legendre functions and Laguerre polynomials,

$$U(u) = (1-u^2)^{-1/4}\left(a_1\, P^{j+\frac{1}{2}}_{c-\frac{1}{2}}(u) + a_2\, Q^{j+\frac{1}{2}}_{c-\frac{1}{2}}(u)\right)\,, \tag{33}$$

$$R(r) \propto e^{-\frac{r^2}{2\hbar}}\left(\frac{r^2}{\hbar}\right)^{\frac{c-1}{2}} L^c_k\left(\frac{r^2}{\hbar}\right)\,, \tag{34}$$

where

$$k = \frac{n+1-c}{2} = 0, 1, 2, \ldots\,, \tag{35}$$

i.e., normalizable solutions of (32) exist when $c = n+1, n-1, n-3\ldots$ with $k = 0, 1, 2, \ldots$ Furthermore, to set the coefficients a_1 and a_2 in $U(u)$, given in (33), we compare solutions $\psi^n_{c;jm}$ with the previously obtained Fock states wavefunctions (10), upon applying the coordinate transformation (28), and deduce

$a_1 = 0$. Using the relation between Legendre functions (see e.g., [17]), $Q^{j+\frac{1}{2}}_{c-\frac{1}{2}} \propto P^{-j-\frac{1}{2}}_{c-\frac{1}{2}}$, we finally obtain normalized solutions ($\int \psi^* \psi \, d^4x = 1$)

$$\psi^n_{c;jm} = R_{nc}(r) U_{cj}(u) Y_{jm}(\theta, \phi) \,, \tag{36}$$

where

$$\begin{aligned} Y_{jm}(\theta,\phi) =& \sqrt{\frac{(2j+1)}{4\pi}\frac{(j-m)!}{(m-m)!}} e^{im\phi} P^m_j(\cos\theta) \,, \\ R_{nc}(r) =& e^{-\frac{r^2}{2\hbar}} \sqrt{\frac{2k!}{(k+c)!}} \left(\frac{r^2}{\hbar}\right)^{\frac{c-1}{2}} L^c_k\left(\frac{r^2}{\hbar}\right), \qquad k = \frac{n+1-c}{2} \,, \\ U_{cj}(u) =& \sqrt{\frac{c(c+j)!}{(c-j-1)!}} (1-u^2)^{-1/4} P^{-j-\frac{1}{2}}_{c-\frac{1}{2}}(u) \,. \end{aligned} \tag{37}$$

Particular linear combinations of such functions form the Fock state wavefunctions, i.e.

$$\langle r, u, \theta, \phi | n_1, n_2, n_3; n_4 \rangle = \sum_{c=n+1,n-1,\ldots} R_{nc}(r) \sum_{j=0}^{c-1} U_{cj}(u) \sum_{m=-j}^{j} Y_{jm}(\theta,\phi) \, A^n_{cjm} \,. \tag{38}$$

We can find the coefficients A^n_{cjm} by comparison with $\langle x^a | n_1, n_2, n_3; n_4 \rangle$, given in Equation (10). Explicit results for some of the lower n states are given in the Appendix A.

The above results match exactly to what we discuss above in terms of the Cartesian tensors. The n-level is a sum of irreducible representations of $c = n+1, n-1, \ldots$, which are exactly the traceless irreducible tensors of rank $c-1$. Note that the c value uniquely specifies the irreducible representation to which a basis state belongs. The n value does not otherwise matter. The $|n;1;0,0\rangle$ states for example, all transform exactly in the same way as the $|0;1;0,0\rangle$ state, with wavefunctions all of the form $f(x_a x^a) = f(r)$. Note that the Laguerre polynomial L^c_k for our k values is simply an order k polynomial, hence R_{nc} an order n polynomial in r times $e^{-\frac{r^2}{2\hbar}}$. The pseudo-unitary inner product among the $|n;c;j,m\rangle$ states can be written as

$$\langle\langle n';c';j',m' | n;c;j,m \rangle\rangle = (-1)^{n-j} \langle n';c';j',m' | n;c;j,m \rangle = (-1)^{n-j} \delta_{nn'} \delta_{cc'} \delta_{jj'} \delta_{mm'} \,. \tag{39}$$

That is easy to appreciate as $\mathcal{P}_4$ or simply $\mathcal{P}\mathcal{P}^{-1}_{(3)}$, where $\mathcal{P}$ is the full parity operator sending all x^a to $-x^a$ and $\mathcal{P}_{(3)}$ the corresponding one for the '3D' problem sending x^i to $-x^i$ for which $|n;c;j,m\rangle$ is an eigenstate with eigenvalues $(-1)^n$ and $(-1)^j$, respectively.

4. Discussions on Issues of Interpretations

Issues on the practical interpretation of the results are tricky. In lack of a solid practical setting that has been identified as to be depicted by a theory of the covariant harmonic oscillator, it is not quite possible to put the theory to the test directly. Representing observables by non-Hermitian operators surely does not fit into the conventional interpretations of quantum theories, specifically in regard to the probability postulates. However, as stated above, it is not clear at all that the usual probability notion should be a part of a theory of wavefunctions over the 'spacetime' variables. The formulation here has the position operators $\hat{X}_i$ and momentum operators $\hat{P}_i$, and hence any observable corresponding to the function of those six basics observables, represented Hermitianly, in fact, in exactly the same standard way. Naively, that should include all physical observables, which would say that our formulation has no difficulty at all when applied to look at any of the physical observables at any specific value of the variable describing 'time'. The bottom line, again, is that the pseudo-unitary theory is fully unitary when the 'time' variable for the wavefunctions is restricted to a fixed value. Hence, the usual probability interpretation in connection to von Neumann measurements performed at a definite time is not a problem. The latter seems to be good enough for a theory of Lorentz covariant quantum mechanics interpreted along the usual perspective.

Taking up issues related to the nonhermitain time $\hat{X}_0 = ix_4$ and energy $\hat{P}_0 = \hbar\partial_{x^4}$ operators, some discussions about the notion of time in physical theories in Rovelli's book on quantum gravity [18] is very relevant, from which we would like to extract a few quotes here. For example, the author noted that time in Newtonian mechanics is really an "unobservable physical quantity", that it is enough for a theory to predict "correlations between physical variables" but not necessarily values of the observables at any particular time. Rovelli also observed that in general relativity, "the coordinate time is not an observable", while "dynamics cannot be expressed as evolution in τ" (the proper time), and that "a fundamental concept of time may be absent in quantum gravity". In view of all that, we can better consider the physics picture of our time operator $\hat{X}_0$. The first thing to note is that it should really be thought of more like the coordinate time. Taking quantum observables as noncommutative coordinates has been established as fully valid, for example in Reference [19], where it is shown how the six operators $\hat{X}_i$ and $\hat{P}_i$ can be seen as coordinates of a noncommutative symplectic geometry, which can alternatively be described as a commutative/real manifold of the projective Hilbert space, in the explicit language of a coordinate transformation map. Our formulation of the pseudo-unitary Lorentz covariant harmonic oscillator can be expected to fit well into the Lorentz covariant generalization of that [8]. Note that without the notion of a noncommutative value for an observable [20], the quite intuitive picture of the quantum phase space cannot be made logically sound. It is also relevant to note that there has been a very substantial number of studies on a plausible time operator in quantum mechanics, though mostly not in a Lorentz covariant theory, since the old days. A common conclusion from those studies is exactly that a Hermitian time operator is not compatible with the theory. On the other hand, from the more mathematical perspective, the observable algebra is essentially agreed to be taken as a C^*-algebra, which corresponds to including all complex linear combinations of the physical/Hermitian operators. After all, a complex linear combination is in no sense any less 'observable' than a real one.

At first sight, having an energy operator $\hat{P}_0$ to be nonhermitian posits a serious problem. However, it is a mathematically unavoidable consequence of having nonhermitian $\hat{X}_0$. Upon a more careful thinking, it is not at all clear that the $\hat{P}_0$ operator here has to be the physical energy as in the usual quantum mechanics or classical special relativity. In fact, it is easy to see that the $H_R(1,3)$ symmetry at the classical limit is still a symmetry bigger than the Poincaré symmetry. The corresponding theory is certainly a theory more general than Einstein special relativity, more like the so-called 'parameterized relativistic' theory (see for example Ref. [21]). In fact, that kind of theory has essentially a translational symmetry in the energy variable [22], which can be seen as the usual notion of the physically indeterminate zero reference point of energy measurements, in line with the notion of the energy or the energy operator as a coordinate variable.

Another important point of view related to the idea of position and momentum operators as, actually canonical, coordinate variables for the quantum theory as symplectic dynamics is the symplectic geometric picture for the basic quantum mechanics (see for example Refs. [19,23] and references therein) with the infinite real dimensional projective Hilbert space as the phase space. The Hamiltonian mechanics presents the dynamical theory well, at least when the measurement problem is not included. While the Copenhagen school framework with the probability picture gives a scheme to describe von Neumann measurements, it is hardly a dynamical/theoretical description. The decoherence theory [24], with the statistical results from an open system perspective, we consider quite successful in that direction. In principle, there is no fundamental difficulty in formulating the latter equally successfully in the symplectic geometric approach. The key point here is that a physical state, as a point in the corresponding symplectic manifold, is completely unambiguous. At least in principle, the state can be determined and the 'full values' of all observables, as known functions of the state [19,20], completely fixed accordingly. Such a 'full value' can be described as the noncommutative number [20] which contains full information about the observable beyond the complete statistics of repeated von Neumann measurements. None of all that requires the observables to be Hermitian. In fact, in the noncommutative geometric picture the geometry is the dual object of the observable algebra, which is basically the representation of the group C^*-algebra matching with the quantum theory as the representation of the basic/relativity symmetry of $H_R(3)$ [25]. We plan on going with the studies of a pseudo-unitary Lorentz covariant quantum theory along this line, with the latter generalized to the $H_R(1,3)$ symmetry, results and lessons from which would help us fully understand the physics of the covariant harmonic oscillator solutions given here.

It may be of interest to note further that from the symplectic point of view, in terms of commutative or noncommutative coordinates, dynamics is a specific case of one-parameter Hamiltonian flows [25]. The generator of the latter generally does not need to be inside the basic symmetry algebra. Even in the case of Galilean symmetry, the only case of the physical Hamiltonian being included is the case of a free particle. Moreover, for the Lorentz covariant formulations, the evolution parameter should probably not be taken as the proper time. A parameter that corresponds to a proper time divided by the particle mass at the Einstein limit [22], as first introduced by Feynman back in 1950 [26], works better. With respect to the evolution parameter, for the properly generalized Lorentz covariant Schrödinger equation, we are, here, only solving that which corresponds to the 'time-independent', i.e., evolution parameter-independent covariant Schrödinger equation.

We can only sketch here how the interpretational issues may be approached on the basis of the known alternative perspectives. Beyond that, more studies within a full setting of the Lorentz covariant quantum theory, still to be explicitly formulated, have to be performed to help lighten up the physics picture.

5. Conclusions

A basic picture of the Fock states for the pseudo-unitary representation we present here is more or less known. Because of the lack of interest from the conventional unitary quantum theory line of thinking or otherwise, a detailed analysis with comprehensive, consistent, explicit wavefunctions, the inner product, and the full matching to the irreducible representations of the $SO(1,3)$ has not yet been available. We present here such a study.

The covariant harmonic oscillator problem in a general setting of $SO(l,m)$ symmetry may serve as an important background for formulating the corresponding quantum theory. It is all about an irreducible representation of the $H_R(l,m)$ symmetry. In fact, the authors approach the problem with the formulation of such a covariant quantum theory in mind. We see the un-conventional approach in the direction of a pseudo-unitary representation as a sensible one to explore, and the only reasonable approach from a certain kind of background perspective. Better appreciation of the physics picture of the theoretical framework could be obtained with a full dynamical formulation of such a quantum theory and, furthermore, by analyzing its application to various physical systems, especially the experimentally accessible cases, like the motion of an electron under an electromagnetic field.

Looking carefully into the other theoretical applications of the Lorentz covariant harmonic oscillator would, of course, also be useful, although the question of a solid practical setting for the experimental applicability of such theories may not be very well established. All that takes more effort, to which we hope to be able to contribute. Our results here are given to provide a firm mathematical background for these kind of studies.

Author Contributions: Conceptualization, O.K.; formal analysis, S.B. and O.K.; writing—original draft preparation, S.B. and O.K.; writing—review and editing, S.B. and O.K.

Funding: This research was funded by MOST of Taiwan grant number 107-2119-M-008-011.

Acknowledgments: We thank H.K. Ting for discussions. O.K. is partially supported by research grant number 107-2119-M-008-011 of the MOST of Taiwan.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. List of Explicit Relations between Some of the Fock States and the $|n;c;j,m\rangle$ Basis States of $SO(1,3)$ Irreducible Representations

$$
\begin{aligned}
|n_1,n_2,n_3;n_4\rangle &= \sum A^{n}_{cjm}\,|n;c;j,m\rangle \\
|0,0,0;0\rangle &= |0;1;0,0\rangle \\
|0,0,0;1\rangle &= |1;2;0,0\rangle \\
|0,0,1;0\rangle &= |1;2;1,0\rangle \\
|0,1,0;0\rangle &= \frac{i}{\sqrt{2}}\left(|1;2;1,1\rangle + |1;2;1,-1\rangle\right) \\
|1,0,0;0\rangle &= \frac{1}{\sqrt{2}}\left(|1;2;1,-1\rangle - |1;2;1,1\rangle\right) \\
|0,0,0;2\rangle &= -\frac{1}{2}\,|2;1;0,0\rangle + \frac{\sqrt{3}}{2}\,|2;3;0,0\rangle \\
|0,0,2;0\rangle &= -\frac{1}{2}\,|2;1;0,0\rangle - \frac{1}{2\sqrt{3}}\,|2;3;0,0\rangle + \sqrt{\frac{2}{3}}\,|2;3;2,0\rangle \\
|0,2,0;0\rangle &= -\frac{1}{2}\,|2;1;0,0\rangle - \frac{1}{2\sqrt{3}}\,|2;3;0,0\rangle - \frac{1}{\sqrt{6}}\,|2;3;2,0\rangle - \frac{1}{2}\left(|2;3;2,2\rangle + |2;3;2,-2\rangle\right) \\
|2,0,0;0\rangle &= -\frac{1}{2}\,|2;1;0,0\rangle - \frac{1}{2\sqrt{3}}\,|2;3;0,0\rangle - \frac{1}{\sqrt{6}}\,|2;3;2,0\rangle + \frac{1}{2}\left(|2;3;2,2\rangle + |2;3;2,-2\rangle\right) \\
|0,0,1;1\rangle &= |2;3;1,0\rangle \\
|1,1,0;0\rangle &= \frac{i}{\sqrt{2}}\left(|2;3;2,-2\rangle - |2;3;2,2\rangle\right) \\
|1,0,1;0\rangle &= \frac{1}{\sqrt{2}}\left(|2;3;2,-1\rangle - |2;3;2,1\rangle\right) \\
|1,0,0;1\rangle &= \frac{1}{\sqrt{2}}\left(|2;3;1,-1\rangle - |2;3;1,1\rangle\right) \\
|0,1,1;0\rangle &= \frac{i}{\sqrt{2}}\left(|2;3;2,-1\rangle + |2;3;2,1\rangle\right) \\
|0,1,0;1\rangle &= \frac{i}{\sqrt{2}}\left(|2;3;1,-1\rangle + |2;3;1,1\rangle\right) \\
|0,0,0;3\rangle &= \frac{1}{\sqrt{2}}\left(|3;4;0,0\rangle - |3;2;0,0\rangle\right)
\end{aligned}
$$

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