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Symmetry Group Classification and Conservation Laws of the Nonlinear Fractional Diffusion Equation with the Riesz Potential

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Abstract: Symmetry properties of a nonlinear two-dimensional space-fractional diffusion equation with the Riesz potential of the order $\alpha \in (0,1)$ are studied. Lie point symmetry group classification of this equation is performed with respect to diffusivity function. To construct conservation laws for the considered equation, the concept of nonlinear self-adjointness is adopted to a certain class of space-fractional differential equations with the Riesz potential. It is proved that the equation in question is nonlinearly self-adjoint. An extension of Ibragimov's constructive algorithm for finding conservation laws is proposed, and the corresponding Noether operators for fractional differential equations with the Riesz potential are presented in an explicit form. To illustrate the proposed approach, conservation laws for the considered nonlinear space-fractional diffusion equation are constructed by using its Lie point symmetries.

Keywords: space-fractional filtration equation; Riesz potential; Lie point symmetry group; group classification; nonlinear self-adjointness; conservation laws

1. Introduction

Fractional differential equations (FDEs) with multi-dimensional spatial fractional differential operators have attracted considerable attention during the last decade due to the possibility to describe power-law long-range interactions in complex systems [1–3]. In particular, such equations can be efficiently used for modelling a fluid flow in naturally fractured porous media, which is a very important problem for the oil industry. Examples of such fractional differential models can be found in [4,5]. Usually, the integer-order derivatives of the Riesz potential or fractional Laplacian (see, e.g., [6,7]) are used as fractional differential operators in these models. These fractional operators are well studied [7–9] and have a lot of similar properties. Nevertheless, there are some differences in classes of functions for which these operators exist. As a result, FDEs corresponding to these two types of fractional spatial operators will also have different qualitative properties (see, e.g., discussion in [10]). Note that nowadays FDEs with the Riesz potential are much less studied than FDEs containing fractional Laplacian. Therefore, in this paper, we restrict our attention by equations with the Riesz potential.

Finding exact solutions to nonlinear space-fractional FDEs is a sufficiently complex problem. Nevertheless, this problem can be significantly simplified if symmetry properties of the considered equation are known. These properties can be studied by the methods of Lie group analysis of differential equations [11–13]. Recently, some basic Lie symmetry group methods have been extended to fractional differential equations with the Riemann–Liouville and Caputo fractional derivatives (see the survey papers [14,15] and references therein). In [16], the algorithm for finding the Lie point symmetry group of FDEs with the Riesz potential was firstly proposed, and symmetries of the linear two-dimensional space-fractional filtration equation with the Riesz potential were obtained.

Classification of nonlinear equations belonging to a certain class with respect to symmetry groups is an important task of modern Lie group analysis. Fundamentals of the group classification originated in works by Sophus Lie. An efficient approach to symmetry group classification was developed by Ovsyannikov [17] (see also [11]). He first performed a complete group classification of the nonlinear heat equation with the thermal conductivity treated as a function of the temperature. In [18], the group classification of time-fractional analogues of this equation with the Riemann–Liouville and Caputo fractional derivatives was performed. Nevertheless, the problem of group classification for the space-fractional FDEs with the Riesz potential has never been considered. In this paper, we present the results of Lie point symmetry group classification for a nonlinear space-fractional diffusion equation containing the Riesz potential with respect to diffusivity function.

It is well known that there is a close connection between symmetries and conservation laws (see, e.g., [19,20]). In 1918, Emmy Noether proved [21] that a conservation law can be obtained by the group invariance principle from the variational integral with a Lagrangian function as an integrand. An efficient constructive algorithm for finding conservation laws of a differential equation possesses a Lagrangian was proposed by Ibragimov and can be found in [22]. Recently, this algorithm was enhanced to fractional differential equations with Lagrangians depending on fractional differential variables formed by arbitrary compositions of fractional integral operators and integer-order differential operators [23].

To find conservation laws for differential equations without Lagrangians, the general concept of nonlinear self-adjointness of integer-order differential equations was proposed by Ibragimov [24–26]. He also proved that the constructive algorithm proposed earlier is also applicable to find conservation laws for nonlinear self-adjoint equations with formal Lagrangians. In [27–29], it is shown that the concept of nonlinear self-adjointness can be enhanced to FDEs with the Riemann–Liouville and Caputo fractional derivatives. In this paper, we extend this approach to FDEs with the Riesz potential.

The paper is organized as follows. In Section 2, we recall necessary definitions of the Lie symmetry group theory and give essential theorems on the Riesz potential properties. In Section 3, the results of group classification are presented for a nonlinear space-fractional diffusion equation with the Riesz potential. Section 4 is devoted to considering the nonlinear self-adjointness of FDEs containing the Riesz potential. In Section 5, we present a technique of finding conservation laws for such equations using their symmetries.

2. Preliminaries

This section provides a brief introduction to the basic principles of Lie group analysis and its application to fractional differential equations with the Riesz potential. First of all, we give necessary definitions.

Let $x^0 \in [0,T]$ $(T > 0)$ and $x = (x^1, \dots, x^n) \in R^n$ be the time variable and the vector of spatial variables, respectively. The function $u = u(x^0, x)$ will be considered as a dependent variable.

We will deal with *one-parameter Lie groups of point transformations* [13] given by

$$\begin{aligned} &\bar{x}^i = f^i(x^0, x, u, a), \quad f^i|_{a=0} = x^i, \quad i = 0, 1, \dots, n; \\ &\bar{u} = g(x^0, x, u, a), \qquad g|_{a=0} = u, \end{aligned} \tag{1}$$

depending on a continuous parameter a. The *infinitesimal transformations* of group (1) can be written as

$$\bar{x}^i \approx x^i + a\xi^i(x^0, x, u), \quad \bar{u} \approx u + a\eta(x^0, x, u), \tag{2}$$

where

$$\xi^i(x^0,x,u) = \frac{\partial f^i(x^0,x,u,a)}{\partial a}\Big|_{a=0}, \quad \eta(x^0,x,u) = \frac{\partial g(x^0,x,u,a)}{\partial a}\Big|_{a=0}.$$

The approximate equality $f \approx g$ means that $f = g + o(a)$.

The *infinitesimal generator* of group (1) is the linear first-order differential operator

$$X = \xi^i(x^0,x,u)\frac{\partial}{\partial x^i} + \eta(x^0,x,u)\frac{\partial}{\partial u}. \tag{3}$$

Summation over repeated indices is implied in this paper.

Any group (1) with

$$\xi^i = \xi^i(x^0,x), \qquad \eta(x^0,x,u) = \eta^0(x^0,x) + \eta^1(x^0,x)u \tag{4}$$

is called a *linear autonomous* one-parameter Lie group of point transformations [30] (see also [14,15]).

The *Riesz potential* [7] in n-dimensional space is defined by

$$R^\alpha u(x^0,x) = \frac{1}{\gamma_n(\alpha)}\int\limits_{R^n} \frac{u(x^0,\mu)}{|x-\mu|^{n-\alpha}}d\mu, \tag{5}$$

where $\mu = (\mu^1,\dots,\mu^n)$ and

$$\gamma_n(\alpha) = 2^\alpha \pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)/\Gamma\left(\frac{n-\alpha}{2}\right).$$

Remark 1. *In a 1D case, the Riesz potential can be rewritten as a sum of the left-sided $I^\alpha_+ u$ and the right-sided $I^\alpha_- u$ Liouville fractional integrals on R:*

$$R^\alpha u(x^0,x) = \frac{\Gamma(\alpha)}{\gamma_n(\alpha)}\left(I^\alpha_+ u + I^\alpha_- u\right), \quad x \in R, \tag{6}$$

where

$$I^\alpha_+ u = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}\frac{u(x^0,\mu)}{(x-\mu)^{1-\alpha}}d\mu, \qquad I^\alpha_- u = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}\frac{u(x^0,\mu)}{(\mu-x)^{1-\alpha}}d\mu.$$

In addition, in [31], it was proved that, if u is a radial function, i.e., $u = u(x^0,|x|)$, then the Riesz potential can be written as

$$R^\alpha u(x^0,x) = 2^{-\alpha}|x|^{2-n}\left(I^{\frac{\alpha}{2}}_{0+} s^{\frac{n-\alpha}{2}-1} I^{\frac{\alpha}{2}}_{-} u(x^0,\sqrt{s})\right)\Big|_{s=x^2}. \tag{7}$$

In special cases corresponding to representations (6) and (7), equations with the integer-order derivatives of the Riesz potential are equivalent to fractional differential equations with the Riemann–Liouville fractional derivatives. Such equations will not be considered in this paper.

A group (1) can be prolonged to the Riesz potential (5). The corresponding prolongation formula is given by the following theorem.

Theorem 1 ([16])**.** *The infinitesimal transformation of the Riesz potential* (5) *induced by the group* (1) *has the form*

$$\bar{R}^\alpha \bar{u} \approx R^\alpha u + a\zeta^\alpha, \tag{8}$$

where ζ^α is given by the prolongation formula

$$\zeta^\alpha = R^\alpha(\eta - \xi^i u_i) + \xi^i D_i(R^\alpha u).$$

Here, $\bar{R}^\alpha$ is the Riesz operator with respect to $\bar{x}$, $u_i = \frac{\partial u}{\partial x^i}$, and D_i denotes the operator of total differentiation with respect to x^i.

We will consider fractional differential equations with integer-order derivatives of the Riesz potential (5). If such equation does not change the form when it is written in the new variables $\bar{x}^i$ $(i = 0, 1, \ldots, n)$ and $\bar{u}$ defined by group (1), then the corresponding one-parameter group is called *Lie symmetry group of point transformations* for this equation. The infinitesimal generator of such a group is called an *infinitesimal symmetry* of this equation.

By using transformation (8), the prolongation formula for any integer-order derivative of the Riesz potential can be constructed. We introduce the notation

$$(R^\alpha u)_{i_1 \ldots i_s} = \frac{\partial^s (R^\alpha u)}{\partial x^{i_1} \ldots \partial x^{i_s}},$$

where $i_1, \ldots, i_s = 0, 1, \ldots, n$.

Theorem 2. *The infinitesimal transformation of $(R^\alpha u)_{i_1 \ldots i_s}$ induced by the group (1) can be written as*

$$(\bar{R}^\alpha \bar{u})_{i_1 \ldots i_s} \approx (R^\alpha u)_{i_1 \ldots i_s} + a \zeta^\alpha_{i_1 \ldots i_s}, \tag{9}$$

where $\zeta^\alpha_{i_1 \ldots i_s}$ is given by the prolongation formula

$$\zeta^\alpha_{i_1 \ldots i_s} = D_{i_s} \ldots D_{i_1} R^\alpha (\eta - \xi^i u_i) + \xi^i D_i (R^\alpha u)_{i_1 \ldots i_s}. \tag{10}$$

Let $F = 0$ be a fractional differential equation with the Riesz potential. We denote by $\tilde{X}$ an infinitesimal generator of the Lie group of point transformations prolonged to all integer-order and fractional-order differential variables included into the function F. Then, the necessary condition of X to be a symmetry of the equation $F = 0$ can be written as

$$(\tilde{X} F)|_{F=0} = 0. \tag{11}$$

Note that, contrary to integer-order differential equations, the invariance condition (11) is a necessary but not sufficient condition for fractional differential equations (the detailed discussion of this fact can be found in [14]).

Now we give some useful properties of the Riesz potential in two-dimensional space. We introduce the integral operator $I^{m,n}_\beta$ by

$$I^{m,n}_\beta f(x,y) = \frac{1}{\gamma_2(\alpha)} \int\limits_{-\infty}^{\infty} \frac{(\mu - x)^m (\nu - y)^n f(\mu, \nu)}{\left[(\mu - x)^2 + (\nu - y)^2\right]^{\frac{\beta}{2}}} d\mu d\nu. \tag{12}$$

Proposition 1 ([16])**.** *The integral operator (12) possesses the property*

$$I^{m+2,n}_{4-\alpha} = I^{m,n}_{2-\alpha} - I^{m,n+2}_{4-\alpha}. \tag{13}$$

The following theorem gives a generalization of the Leibniz rule for the two-dimensional Riesz potential [16].

Theorem 3. *Let $f(x,y)$ be an analytic function in R^2 and $g(x,y)$ be a function such that integrals $I^{m,n}_{2-\alpha} g$ exist for any $m, n \in N \cup \{0\}$. Then,*

$$R^\alpha (fg) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m! n!} \frac{\partial^{m+n} f}{\partial x^m \partial y^n} I^{m,n}_{2-\alpha} g. \tag{14}$$

From Theorem 3 and the identity (13), we infer the following.

Corollary 1. *Under assumptions of Theorem 3, the following equalities hold:*

$$D_x^2 R^\alpha (fg) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\partial^{m+n} f}{\partial x^m \partial y^n} J_{x,2-\alpha}^{m,n} g, \qquad D_y^2 R^\alpha (fg) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\partial^{m+n} f}{\partial x^m \partial y^n} J_{y,2-\alpha}^{m,n} g, \tag{15}$$

where

$$J_{x,2-\alpha}^{m,n} = \frac{1}{m!n!}\left(m(m-1) I_{2-\alpha}^{m-2,n} + 2m D_x I_{2-\alpha}^{m-1,n} + D_x^2 I_{2-\alpha}^{m,n}\right),$$
$$J_{y,2-\alpha}^{m,n} = \frac{1}{m!n!}\left(n(n-1) I_{2-\alpha}^{m,n-2} + 2n D_y I_{2-\alpha}^{m,n-1} + D_y^2 I_{2-\alpha}^{m,n}\right).$$

3. Group Classification of the Nonlinear Space-Fractional Porous Medium Equation

In [16], it was shown that one phase flow of viscous compressible fluid through a naturally fractured oil reservoir can be modeled by a diffusion-type fractional differential equation with the Riesz potential. If the viscosity of fluid depends on pressure, then this equation becomes nonlinear. In this paper, we restrict our attention by the two-dimensional case. For convenience, we denote $x^0 = t$, $x^1 = x$, $x^2 = y$. Then, the equation in question has the form

$$u_t = \nabla\left(k(u)\nabla R^\alpha u\right), \tag{16}$$

where $u = u(t,x,y)$, $t > 0$, $(x,y) \in R^2$, and $\alpha \in (0,1)$. In the linear case ($k(u) = 1$), the symmetry properties of this equation has been investigated in [16]. In this paper, a group classification of Equation (16) with respect to the function $k(u)$ is performed.

We note that, if $\alpha = 0$, then Equation (16) coincides with the classical nonlinear heat equation. The group classification of this integer-order partial differential equation is well-known and can be found in [11]. It is very important that this classical nonlinear heat equation has only linear autonomous symmetries for any $k(u)$. Numerous calculations show that, if an integer-order partial differential equation has only linear autonomous symmetries, then the related partial fractional differential equation inherits this property. Therefore, we will perform group classification of (16) with respect to Lie point linear autonomous symmetries. The corresponding group generator has the form

$$X = \xi^0\frac{\partial}{\partial t} + \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} \tag{17}$$

with

$$\xi^0 = \xi^0(t,x,y),\ \ \xi^1 = \xi^1(t,x,y),\ \ \xi^2 = \xi^2(t,x,y),\ \ \eta = \eta^0(t,x,y) + \eta^1(t,x,y)u.$$

It can be easily verified that Equation (16) admits the following five-parameter group of equivalence transformations:

$$\bar{u} = B_1 u, \quad \bar{t} = B_2 t + B_1, \quad \bar{k} = B_2^{-1} B_3^{2-\alpha} k, \quad \bar{x} = B_3 x + B_4, \quad \bar{y} = B_3 x + B_5. \tag{18}$$

These transformations preserve the fractional differential structure of Equation (16) but change the form of classifying function $k(u)$. The symmetry group classification of Equation (16) will be performed up to these transformations.

For convenience, we rewrite Equation (16) as

$$u_t = k(u)[D_x^2(R^\alpha u) + D_y^2(R^\alpha u)] + k'(u) u_x D_x(R^\alpha u) + k'(u) u_y D_y(R^\alpha u). \tag{19}$$

It can be seen that the prolongation of the generator (17) to all derivatives included in (16) has the form

$$
\begin{aligned}
\bar{X} = {} & \xi^0 \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} \\
& + \zeta_x^\alpha \frac{\partial}{\partial D_x R^\alpha u} + \zeta_y^\alpha \frac{\partial}{\partial D_y R^\alpha u} + \zeta_{xx}^\alpha \frac{\partial}{\partial D_x^2 R^\alpha u} + \zeta_{yy}^\alpha \frac{\partial}{\partial D_y^2 R^\alpha u}.
\end{aligned} \tag{20}
$$

Here,

$$
\begin{aligned}
\zeta_t &= D_t \left(\eta - \xi^0 u_t - \xi^1 u_x - \xi^2 u_y\right) + \xi^0 u_{tt} + \xi^1 u_{tx} + \xi^2 u_{ty}, \\
\zeta_x &= D_x \left(\eta - \xi^0 u_t - \xi^1 u_x - \xi^2 u_y\right) + \xi^0 u_{tx} + \xi^1 u_{xx} + \xi^2 u_{yx}, \\
\zeta_y &= D_y \left(\eta - \xi^0 u_t - \xi^1 u_x - \xi^2 u_y\right) + \xi^0 u_{ty} + \xi^1 u_{xy} + \xi^2 u_{yy},
\end{aligned} \tag{21}
$$

and ζ_{xx}^α, ζ_{yy}^α, ζ_x^α, ζ_y^α are obtained from (10) as

$$
\begin{aligned}
\zeta_x^\alpha &= D_x R^\alpha \left[\eta - \xi^0 u_t - \xi^1 u_x - \xi^2 u_y\right] + \xi^0 D_x D_t (R^\alpha u) + \xi^1 D_x^2 (R^\alpha u) + \xi^2 D_x D_y (R^\alpha u), \\
\zeta_y^\alpha &= D_y R^\alpha \left[\eta - \xi^0 u_t - \xi^1 u_x - \xi^2 u_y\right] + \xi^0 D_y D_t (R^\alpha u) + \xi^1 D_y D_x (R^\alpha u) + \xi^2 D_y^2 (R^\alpha u), \\
\zeta_{xx}^\alpha &= D_x^2 R^\alpha \left[\eta - \xi^0 u_t - \xi^1 u_x - \xi^2 u_y\right] + \xi^0 D_x^2 D_t (R^\alpha u) + \xi^1 D_x^3 (R^\alpha u) + \xi^2 D_x^2 D_y (R^\alpha u), \\
\zeta_{yy}^\alpha &= D_y^2 R^\alpha \left[\eta - \xi^0 u_t - \xi^1 u_x - \xi^2 u_y\right] + \xi^0 D_y^2 D_t (R^\alpha u) + \xi^1 D_y^2 D_x (R^\alpha u) + \xi^2 D_y^3 (R^\alpha u).
\end{aligned} \tag{22}
$$

Acting by the generator (20) on the Equation (19), we obtain the determining equation:

$$
\begin{aligned}
& \zeta_t = k(u) \left(\zeta_{xx}^\alpha + \zeta_{yy}^\alpha\right) + k'(u) \eta [D_x^2 (R^\alpha u) + D_y^2 (R^\alpha u)] \\
& + k'(u) [\zeta_x D_x (R^\alpha u) + \zeta_y D_y (R^\alpha u) + u_x \zeta_x^\alpha + u_y \zeta_y^\alpha] + k''(u) \eta [u_x D_x (R^\alpha u) + u_y D_y (R^\alpha u)].
\end{aligned} \tag{23}
$$

We substitute the prolongation formulae (21), (22) into Equation (23) and replace $D_y^2(R^\alpha u)$ in view of Equation (19) as

$$
D_y^2 (R^\alpha u) = k^{-1}(u) \left[u_t - k(u) D_x^2 (R^\alpha u) - k'(u) u_x D_x (R^\alpha u) - k'(u) u_y D_y (R^\alpha u)\right].
$$

Then, we use the generalized Leibniz rule (14) and its consequences (15). Taking into account that $R^\alpha u = I_{2-\alpha}^{0,0} u$, we represent all terms with $R^\alpha u$ and their derivatives via integral operators (12). By applying the recurrence relation (13), we rewrite all integrals $I_{k-\alpha}^{m,n} u$, $I_{k-\alpha}^{m,n} u_t$ in terms of $I_{8-\alpha}^{m,n} u$ and $I_{8-\alpha}^{m,n} u_t$. As a result, we obtain the determining equation in which $u_t, u_x, u_y, D_x^2(R^\alpha u), I_{8-\alpha}^{m,n} u, I_{8-\alpha}^{m,n} u_t$ can be considered as independent variables. Splitting the obtained equation by all these variables, we get an infinite system of integer-order partial differential equations and one fractional differential equation. We do not write here the whole obtained system due to its large size, but we present a particular result of the splitting which leads to the classifying relation.

By equating to zero the coefficients for u_t, we obtain

$$
\xi_x^0 = 0, \qquad \xi_y^0 = 0, \qquad \xi_t^0 = (2 - \alpha)\, \xi_y^2 - \frac{k'(u)}{k(u)} \eta. \tag{24}
$$

Thus, $\xi^0 = \xi^0(t)$. Since ξ^0 and ξ^2 do not depend on u, and η is a linear function with respect to u, from the last equation in (24), we obtain the classifying relation for $k(u) \neq const$:

$$
\left(\frac{1}{K(u)}\right)'' = 0, \qquad K(u) = \frac{k'(u)}{k(u)}, \qquad k'(u) \neq 0.
$$

This relation is exactly the same as for the integer-order nonlinear heat equation [11] and as for the time-fractional nonlinear diffusion equation [18]. From this classifying relation, in view of the equivalence transformations (18), we obtain that the following cases should be distinguished:

1. $k(u)$ is an arbitrary function;
2. $k(u) = e^u$;
3. $k(u) = (u+A)^\sigma, \sigma \neq 0, A = const$;
4. $k(u) = 1$.

The subsequent analysis shows that there is no extension of the symmetry group for $k(u) = e^u$, and for $k(u) = (u+A)^\sigma$ the symmetry group is extended only with $A = 0$. The main reason for these results is that the integral $R^\alpha(1)$ diverges.

The final results of group classification for Equation (16) are summarized in the following theorem.

Theorem 4. *The nonlinear Equation (16) with an arbitrary functions $k(u)$ has a five-parameter Lie point symmetry group spanned by the generators*

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial y}, \\ X_4 = (2-\alpha)t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \end{gathered} \tag{25}$$

This symmetry group is extended only for $k(u) = u^\sigma$ $(\sigma \in R)$:

$$X_6 = \sigma t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} \tag{26}$$

for arbitrary σ;

$$X_\infty^1 = g(t,x,y)\frac{\partial}{\partial u}$$

for $\sigma = 0$ (linear case), where $g(t,x,y)$ is an arbitrary solution of the linear equation $g_t = \Delta R^\alpha g$;

$$X_\infty^2 = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} - 2A_x u\frac{\partial}{\partial u} \tag{27}$$

for $\sigma = -1$, where $A(x,y)$, $B(x,y)$ are the arbitrary solutions of the system

$$A_x = B_y, \qquad A_y = -B_x.$$

The symmetries given in this theorem can be used for finding group invariant solutions and conservation laws of the equation in question. We do not present here invariant solutions because their construction is a problem for future research. Now, we focus on the problem of conservation laws finding.

4. Nonlinear Self-Adjointness

In this section, we extend the basic notions of the concept of nonlinear self-adjointness to FDEs with the Riesz potential and prove that Equation (16) is nonlinearly self-adjoint. We will assume that $n > 1$ and u is not a radial function since otherwise known results [15,23,27] for fractional differential equations with the Riemann–Liouville fractional derivatives can be used (see Remark 1).

Let us consider the function

$$F = F\left(x^0, x, u, u_{(1)}, \ldots, u_{(k)}, (R^\alpha u)_{(1)}, \ldots, (R^\alpha u)_{(m)}\right), \tag{28}$$

where $u = u(x^0, x)$, $x^0 \in [0, T]$ is the time variable and $x = (x^1, \ldots, x^n) \in R^n$ is the vector of spatial variables. In (28), we use the following notation of differential algebra (see, e.g., [13]):

$$v_{(1)} = \{v_{i_1}\},\ \ldots,\ v_{(k)} = \{v_{i_1 \ldots i_k}\}; \qquad v_{i_1} = D_{i_1}(v),\ \ldots,\ v_{i_1 \ldots i_k} = D_{i_k}(v_{i_1 \ldots i_{k-1}}) = D_{i_k} \ldots D_{i_1}(v)$$

with $i_1, \ldots, i_k = 0, \ldots, n$. Here, D_i denotes the operator of total differentiation with respect to x^i.

First of all, we obtain an explicit representation for the variational derivative of the function F in (28). The corresponding variational integral has the form

$$S[u] = \int_0^T \int_{R^n} F\left(x^0, x, u, u_{(1)}, \ldots, u_{(k)}, (R^\alpha u)_{(1)}, \ldots, (R^\alpha u)_{(m)}\right) dx dx^0. \tag{29}$$

The first variation of the functional $S[u]$ can be found as

$$\delta S[u] = \left(\frac{\partial}{\partial \varepsilon} S[u + \varepsilon \delta u]\right)\Big|_{\varepsilon=0},$$

where δu is a variation of the function u. As usual in calculus of variations, we will assume that δu, $R^\alpha(\delta u)$ and all their derivatives with respect to all variables x^i $(i = 0, 1, \ldots, n)$ are equal to zero for $|x| \to \infty$, $x^0 = 0$ and $x^0 = T$. Since the Riesz potential and any differential operator are linear, after simple algebra, we obtain

$$\delta S[u] = \int_0^T \int_{R^n} \left[\frac{\partial F}{\partial u}\delta u + \sum_{s=1}^{k} \frac{\partial F}{\partial u_{i_1 \ldots i_s}} D_{i_s} \ldots D_{i_1}(\delta u) + \sum_{r=1}^{m} \frac{\partial F}{\partial (R^\alpha u)_{j_1 \ldots j_r}} D_{j_r} \ldots D_{j_1}(R^\alpha \delta u)\right] dx dx^0.$$

The multiple integration by parts yields

$$\delta S[u] = \int_0^T \int_{R^n} \left[\frac{\partial F}{\partial u}\delta u + \sum_{s=1}^{k} (-1)^s D_{i_1} \ldots D_{i_s}\left(\frac{\partial F}{\partial u_{i_1 \ldots i_s}}\right)\delta u \right.$$
$$\left. + \sum_{r=1}^{m} (-1)^r D_{j_1} \ldots D_{j_r}\left(\frac{\partial F}{\partial (R^\alpha u)_{j_1 \ldots j_r}}\right)(R^\alpha \delta u)\right] dx dx^0.$$

Let $f(x)$ and $g(x)$ be such functions that $R^\alpha f$, $R^\alpha g$, and the integral $\int_{R^n} f R^\alpha g dx$ exist. Then, it is easy to prove that

$$\int_{R^n} f R^\alpha g dx = \int_{R^n} g R^\alpha f dx.$$

Using this property of the Riesz potential, we get

$$\begin{aligned} \delta S[u] = \int_0^T \int_{R^n} &\left[\frac{\partial F}{\partial u} + \sum_{s=1}^{k} (-1)^s D_{i_1} \ldots D_{i_s}\left(\frac{\partial F}{\partial u_{i_1 \ldots i_s}}\right)\right. \\ &\left. + \sum_{r=1}^{m} (-1)^r R^\alpha D_{j_1} \ldots D_{j_r}\left(\frac{\partial F}{\partial (R^\alpha u)_{j_1 \ldots j_r}}\right)\right](\delta u) dx dx^0. \end{aligned} \tag{30}$$

If, for any δu we have $\delta S[u] = 0$, then the function u extremize the functional $S[u]$. It follows from (30) that this function can be found as a solution of the fractional generalization of the Euler–Lagrange equation

$$\frac{\delta F}{\delta u} = 0,$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{k} (-1)^s D_{i_1} \ldots D_{i_s}\left(\frac{\partial}{\partial u_{i_1 \ldots i_s}}\right) + \sum_{r=1}^{m} (-1)^r R^\alpha D_{j_1} \ldots D_{j_r}\left(\frac{\partial}{\partial (R^\alpha u)_{j_1 \ldots j_r}}\right) \tag{31}$$

is the Euler–Lagrange operator (operator of variational derivative) for the integral (29).

Now, let us consider a fractional differential equation

$$F\left(x^0, x, u, u_{(1)}, \ldots, u_{(k)}, (R^\alpha u)_{(1)}, \ldots, (R^\alpha u)_{(m)}\right) = 0. \tag{32}$$

Following Ibragimov [25,26], we introduce the *formal Lagrangian*

$$\mathcal{L} = v(x^0, x) F\left(x^0, x, u, u_{(1)}, \ldots, u_{(k)}, (R^\alpha u)_{(1)}, \ldots, (R^\alpha u)_{(m)}\right),$$

where $v(x^0, x)$ is a new dependent variable, and define the function

$$F^*\left(x^0, x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(k)}, v_{(k)}, (R^\alpha u)_{(1)}, R^\alpha(v_{(1)}), \ldots, (R^\alpha u)_{(m)}, R^\alpha(v_{(m)})\right) = \frac{\delta(vF)}{\delta u}.$$

Then,

$$F^*\left(x^0, x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(k)}, v_{(k)}, (R^\alpha u)_{(1)}, R^\alpha(v_{(1)}), \ldots, (R^\alpha u)_{(m)}, R^\alpha(v_{(m)})\right) = 0 \tag{33}$$

is the *adjoint equation* to Equation (32).

Similarly to [25,26], Equation (32) will be called *nonlinearly self-adjoint* if the adjoint Equation (33) will be satisfied for all solutions $u(x^0, x)$ of Equation (32) upon a substitution

$$v = \varphi(x^0, x, u), \quad \varphi \neq 0. \tag{34}$$

Theorem 5. *The nonlinear Equation (16) is nonlinearly self-adjoint.*

Proof. It is easy to see that Equation (16) is a particular case of the equation

$$F\left(t, x, y, u, u_t, u_x, u_y, D_x(R^\alpha u), D_y(R^\alpha u), D_x^2(R^\alpha u), D_y^2(R^\alpha u)\right) = 0.$$

Then, the Euler–Lagrange operator (31) has the form

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} - R^\alpha D_x \frac{\partial}{\partial(D_x R^\alpha u)} - R^\alpha D_y \frac{\partial}{\partial(D_y R^\alpha u)} + R^\alpha D_x^2 \frac{\partial}{\partial(D_x^2 R^\alpha u)} + R^\alpha D_y^2 \frac{\partial}{\partial(D_y^2 R^\alpha u)}.$$

The corresponding adjoint equation

$$\frac{\delta(vF)}{\delta u} = 0$$

can be written as

$$-v_t + k'(u)[v_x D_x(R^\alpha u) + v_y D_y(R^\alpha u)] - R^\alpha[D_x(kv_x) + D_y(kv_y)] = 0. \tag{35}$$

The substitution (34) takes the form $v = \varphi(t, x, y, u)$. Then, Equation (35) is transformed into

$$-\varphi_t - \varphi_u u_t + k'(u)[(\varphi_x + \varphi_u u_x) D_x(R^\alpha u) + (\varphi_y + \varphi_u u_y) D_y(R^\alpha u)] - R^\alpha[D_x(k(\varphi_x + \varphi_u u_x)) + D_y(k(\varphi_y + \varphi_u u_y))] = 0.$$

It is evident that this equation holds identically for $\varphi = c$ $(c = const)$. Therefore, Equation (16) is nonlinearly self-adjoint. □

5. Conservation Laws

We will use the classical definition of a conservation law (see, e.g., [11]) since it is suitable for most applications. In such a way, the conservation law for Equation (32) can be written as

$$D_i(C^i) = 0, \quad i = 0, 1, \ldots, n, \tag{36}$$

where $C = (C^0, \ldots, C^n)$ is the so-called *conserved vector*. Other approaches for fractional differential equations can be found in [32–34].

In [22], it is shown that the components of a conserved vector can be found as

$$C^i = \mathcal{N}^i \mathcal{L}, \tag{37}$$

where $\mathcal{L}$ is a classical or formal Lagrangian, and $\mathcal{N}^i$ $(i = 0, 1, \ldots, n)$ are the so-called *Noether operators*. These operators are defined by the fundamental operator identity

$$\tilde{X} + D_i(\xi^i)\mathcal{I} = W\frac{\delta}{\delta u} + D_i(\mathcal{N}^i), \tag{38}$$

where $\tilde{X}$ is an appropriate prolongation of the Lie point group generator to all dependent variables in the considered equation, $\mathcal{I}$ is the identity operator, $W = \eta - \xi^i u_i$, and $\frac{\delta}{\delta u}$ is the Euler–Lagrange operator.

For Equation (32), the prolonged generator $\tilde{X}$ can be written in the form

$$\tilde{X} = \xi^i \frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \sum_{s=1}^{k} \zeta_{i_1\ldots i_s}\frac{\partial}{\partial u_{i_1\ldots i_s}} + \sum_{r=1}^{m} \zeta^{\alpha}_{j_1\ldots j_r}\frac{\partial}{\partial (R^\alpha u)_{j_1\ldots j_r}}, \tag{39}$$

where $i_1, \ldots, i_k = 0, 1, \ldots, n$ and $j_1, \ldots, j_m = 0, 1, \ldots, n$. The functions $\zeta_{i_1\ldots i_s}$ and $\zeta^{\alpha}_{j_1\ldots j_r}$ are given by the prolongation formulae

$$\zeta_{i_1\ldots i_s} = D_{i_s}\ldots D_{i_1}(W) + \xi^l D_l(u_{i_1\ldots i_s}),$$
$$\zeta^{\alpha}_{j_1\ldots j_r} = D_{j_r}\ldots D_{j_1}(R^\alpha W) + \xi^l D_l((R^\alpha u)_{j_1\ldots j_r}).$$

By applying (38) to the function F defined in (28) and using (31), we get

$$\begin{aligned}
D_i(\mathcal{N}^i F) = \tilde{X}F + D_i(\xi^i)F - W\frac{\delta F}{\delta u} = \xi^i\frac{\partial F}{\partial x^i} + \eta\frac{\partial F}{\partial u} + \sum_{s=1}^{k}[D_{i_s}\ldots D_{i_1}(W) + \xi^l D_l(u_{i_1\ldots i_s})]\frac{\partial F}{\partial u_{i_1\ldots i_s}} \\
+ \sum_{r=1}^{m}[D_{j_r}\ldots D_{j_1}(R^\alpha W) + \xi^l D_l((R^\alpha u)_{j_1\ldots j_r})]\frac{\partial F}{\partial (R^\alpha u)_{j_1\ldots j_r}} + D_i(\xi^i)F - (\eta - \xi^i u_i)\frac{\partial F}{\partial u} \\
- W\sum_{s=1}^{k}(-1)^s D_{i_1}\ldots D_{i_s}\left(\frac{\partial F}{\partial u_{i_1\ldots i_s}}\right) - W\sum_{r=1}^{m}(-1)^r R^\alpha D_{j_1}\ldots D_{j_r}\left(\frac{\partial F}{\partial (R^\alpha u)_{j_1\ldots j_r}}\right).
\end{aligned}$$

Note that

$$D_i(F) = \frac{\partial F}{\partial x^i} + u_i\frac{\partial F}{\partial u} + \sum_{s=1}^{k} u_{i_1\ldots i_s i}\frac{\partial F}{\partial u_{i_1\ldots i_s}} + \sum_{r=1}^{m}(R^\alpha u)_{j_1\ldots j_r i}\frac{\partial F}{\partial (R^\alpha u)_{j_1\ldots j_r}}$$

and

$$\xi^i D_i(F) + D_i(\xi^i)F = D_i(\xi^i F).$$

Then, we have

$$D_i(\mathcal{N}^i F) = D_i(\xi^i F) + \sum_{s=1}^{k}\left[D_{i_s}\dots D_{i_1}(W)\cdot\frac{\partial F}{\partial u_{i_1\dots i_s}} - (-1)^s W\cdot D_{i_1}\dots D_{i_s}\left(\frac{\partial F}{\partial u_{i_1\dots i_s}}\right)\right]$$
$$+\sum_{r=1}^{m}\left[D_{j_r}\dots D_{j_1}(R^\alpha W)\cdot\frac{\partial F}{\partial (R^\alpha u)_{j_1\dots j_r}} - (-1)^r W\cdot R^\alpha D_{j_1}\dots D_{j_r}\left(\frac{\partial F}{\partial (R^\alpha u)_{j_1\dots j_r}}\right)\right]. \quad (40)$$

For the first sum on the right-hand side of Equation (40), we can use the known representation [22,26]

$$\sum_{s=1}^{k}\left[D_{i_s}\dots D_{i_1}(W)\cdot\frac{\partial F}{\partial u_{i_1\dots i_s}} - (-1)^s W\cdot D_{i_1}\dots D_{i_s}\left(\frac{\partial F}{\partial u_{i_1\dots i_s}}\right)\right]$$
$$= D_i\left[W\left(\frac{\partial F}{\partial u_i} + \sum_{s=1}^{k-1}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial F}{\partial u_{ii_1\dots i_s}}\right)\right. \quad (41)$$
$$\left.+\sum_{r=1}^{k-1} D_{l_1}\dots D_{l_r}(W)\left(\frac{\partial F}{\partial u_{il_1\dots l_r}} + \sum_{s=1}^{k-1-r}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial F}{\partial u_{il_1\dots l_r i_1\dots i_s}}\right)\right].$$

The second sum on the right-hand side of Equation (40) contains the Riesz operator and therefore a new approach is needed. In [23,27], it is shown that, for equations with the Riemann–Liouville and Caputo fractional derivatives, the corresponding Noether operators can be written in an explicit form by using special integral operators. Similar but more complex operators can be introduced for the equations with the Riesz potential.

We define the operator $J^\alpha_{(i)}$ $(\alpha \in (0,1))$ acting on an ordered pair of functions $\{f(x^0,x), g(x^0,x)\}$ $(x^0 \in [0,T],\ x \in R^n)$ by

$$J^\alpha_{(i)}\{f(x^0,x), g(x^0,x)\} = \frac{1}{\gamma_n(\alpha)}\int_{-\infty}^{x^i}\int_{R^n}\frac{f(x^0,x)|_{x^i=\rho}\, g(x^0,\mu)}{(|x-\mu|^{n-\alpha})\,|_{x^i=\rho}}d\mu d\rho$$
$$+\frac{1}{\gamma_n(\alpha)}\int_{x^i}^{\infty}\int_{R^n}\frac{f(x^0,\mu)\, g(x^0,x)|_{x^i=\rho}}{(|x-\mu|^{n-\alpha})\,|_{x^i=\rho}}d\mu d\rho \quad (42)$$

for $i = 1,\dots,n$, and by

$$J^\alpha_{(0)}\{f(x^0,x), g(x^0,x)\} = \frac{1}{\gamma_n(\alpha)}\int_0^{x^0}\int_{R^n}\frac{f(\rho,x)g(x^0,\mu)}{|x-\mu|^{n-\alpha}}d\mu d\rho + \frac{1}{\gamma_n(\alpha)}\int_{x^0}^{T}\int_{R^n}\frac{f(x^0,\mu)g(\rho,x)}{|x-\mu|^{n-\alpha}}d\mu d\rho, \quad (43)$$

for $i = 0$. It is easy to show by the direct computation that the following equality holds:

$$D_i(J^\alpha_{(i)}\{f(x), g(x)\}) = fR^\alpha g - gR^\alpha f \quad (44)$$

(here (i) means that there is no summation with respect to i). In Equation (44), we assume that $f(x^0,x)$ and $g(x^0,x)$ belong to an appropriate class of functions such that all integrals exist. For completeness, we will assume that

$$J^0_{(i)}\{f(x^0,x), g(x^0,x)\} = 0, \quad i = 0,1,\dots,n.$$

By using (42) and (44), we obtain

$$D_i(R^\alpha W)\cdot\frac{\partial F}{\partial (R^\alpha u)_i} + W\cdot R^\alpha D_i\left(\frac{\partial F}{\partial (R^\alpha u)_i}\right) = D_i\left(R^\alpha W\cdot\frac{\partial F}{\partial (R^\alpha u)_i} + J^\alpha_{(i)}\left\{W, D_i\left(\frac{\partial F}{\partial (R^\alpha u)_i}\right)\right\}\right). \quad (45)$$

By using (41), we can rewrite the second sum on the right-hand side of Equation (40) in the form

$$\sum_{r=1}^{m}\left[D_{j_r}\dots D_{j_1}(R^\alpha W)\cdot\frac{\partial F}{\partial(R^\alpha u)_{j_1\dots j_r}}-(-1)^r W\cdot R^\alpha D_{j_1}\dots D_{j_r}\left(\frac{\partial F}{\partial(R^\alpha u)_{j_1\dots j_r}}\right)\right]$$
$$=D_i\left[(R^\alpha W)\left(\frac{\partial F}{\partial(R^\alpha u)_i}+\sum_{r=1}^{m-1}(-1)^r D_{j_1}\dots D_{j_r}\frac{\partial F}{\partial(R^\alpha u)_{ij_1\dots j_r}}\right)\right.$$
$$\left.+\sum_{s=1}^{m-1}D_{l_1}\dots D_{l_s}(R^\alpha W)\left(\frac{\partial F}{\partial(R^\alpha u)_{il_1\dots l_s}}+\sum_{r=1}^{m-1-s}(-1)^r D_{j_1}\dots D_{j_r}\frac{\partial F}{\partial(R^\alpha u)_{il_1\dots l_s j_1\dots j_r}}\right)\right]$$
$$+\sum_{r=1}^{m}(-1)^r R^\alpha W\cdot D_{j_1}\dots D_{j_r}\left(\frac{\partial F}{\partial(R^\alpha u)_{j_1\dots j_r}}\right)-\sum_{r=1}^{m}(-1)^r W\cdot R^\alpha D_{j_1}\dots D_{j_r}\left(\frac{\partial F}{\partial(R^\alpha u)_{j_1\dots j_r}}\right).$$

In view of Equality (44), the two last terms in this expression can be written as

$$\sum_{r=1}^{m}(-1)^r R^\alpha W\cdot D_{j_1}\dots D_{j_r}\left(\frac{\partial F}{\partial(R^\alpha u)_{j_1\dots j_r}}\right)-\sum_{r=1}^{m}(-1)^r W\cdot R^\alpha D_{j_1}\dots D_{j_r}\left(\frac{\partial F}{\partial(R^\alpha u)_{j_1\dots j_r}}\right)$$
$$=-D_i\left[\sum_{r=1}^{m}(-1)^r J^\alpha_{(i)}\left\{W,D_{j_1}\dots D_{j_r}\left(\frac{\partial F}{\partial(R^\alpha u)_{j_1\dots j_r}}\right)\right\}\right].$$

By substitution of all obtained expressions for sums into the right-hand side of the Equality (40), we derive the following explicit representations for the Noether operators:

$$\mathcal{N}^i F=\xi^i F+W\left(\frac{\partial F}{\partial u_i}+\sum_{s=1}^{k-1}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial F}{\partial u_{ii_1\dots i_s}}\right)$$
$$+\sum_{r=1}^{k-1}D_{l_1}\dots D_{l_r}(W)\left(\frac{\partial F}{\partial u_{il_1\dots l_r}}+\sum_{s=1}^{k-1-r}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial F}{\partial u_{il_1\dots l_r i_1\dots i_s}}\right)$$
$$+(R^\alpha W)\left(\frac{\partial F}{\partial(R^\alpha u)_i}+\sum_{r=1}^{m-1}(-1)^r D_{j_1}\dots D_{j_r}\frac{\partial F}{\partial(R^\alpha u)_{ij_1\dots j_r}}\right)$$
$$+\sum_{s=1}^{m-1}D_{l_1}\dots D_{l_s}(R^\alpha W)\left(\frac{\partial F}{\partial(R^\alpha u)_{il_1\dots l_s}}+\sum_{r=1}^{m-1-s}(-1)^r D_{j_1}\dots D_{j_r}\frac{\partial F}{\partial(R^\alpha u)_{il_1\dots l_s j_1\dots j_r}}\right)$$
$$-\sum_{r=1}^{m}(-1)^r J^\alpha_{(i)}\left\{W,D_{j_1}\dots D_{j_r}\left(\frac{\partial F}{\partial(R^\alpha u)_{j_1\dots j_r}}\right)\right\}. \quad (46)$$

Thus, the conservation laws for any fractional differential equation of the form (32) can be obtained by (37) with (46). To illustrate this approach, we construct conservation laws for the nonlinear fractional diffusion Equation (16).

The conservation law (36) for Equation (16) takes the form

$$D_t(C^t)+D_x(C^x)+D_y(C^y)=0.$$

In the previous section, it is proved that Equation (16) is nonlinearly self-adjoint and the substitution (34) is $v=c=const$. Without loss of generality, we can set $c=1$ because any conservation law is linear with respect to conserved vector's components. Then, the formal Lagrangian coincides with the equation:

$$\mathcal{L}=F\equiv u_t-k(u)[D_x^2(R^\alpha u)+D_y^2(R^\alpha u)]-k'(u)u_x D_x(R^\alpha u)-k'(u)u_y D_y(R^\alpha u). \quad (47)$$

For this function, the Noether operators (46) takes a more simple form:

$$\begin{aligned}
\mathcal{N}^t F &= \xi^0 F + W\frac{\partial F}{\partial u_t},\\
\mathcal{N}^x F &= \xi^1 F + W\frac{\partial F}{\partial u_x} + R^\alpha W\left[\frac{\partial F}{\partial(D_x R^\alpha u)} - D_x\left(\frac{\partial F}{\partial(D_x^2 R^\alpha u)}\right)\right] + D_x(R^\alpha W)\frac{\partial F}{\partial(D_x^2 R^\alpha u)}\\
&\quad + J^\alpha_{(1)}\left\{W, D_x\left(\frac{\partial F}{\partial(D_x R^\alpha u)}\right)\right\} - J^\alpha_{(1)}\left\{W, D_x^2\left(\frac{\partial F}{\partial(D_x^2 R^\alpha u)}\right)\right\},\\
\mathcal{N}^y F &= \xi^2 F + W\frac{\partial F}{\partial u_y} + R^\alpha W\left[\frac{\partial F}{\partial(D_y R^\alpha u)} - D_y\left(\frac{\partial F}{\partial(D_y^2 R^\alpha u)}\right)\right] + D_y(R^\alpha W)\frac{\partial F}{\partial(D_y^2 R^\alpha u)}\\
&\quad + J^\alpha_{(2)}\left\{W, D_y\left(\frac{\partial F}{\partial(D_y R^\alpha u)}\right)\right\} - J^\alpha_{(2)}\left\{W, D_y^2\left(\frac{\partial F}{\partial(D_y^2 R^\alpha u)}\right)\right\}.
\end{aligned} \tag{48}$$

Substituting (47) and (48) into (37), in view of the equation $F = 0$, we obtain

$$\begin{aligned}
C^t &= W,\\
C^x &= -[k'(u)WD_x(R^\alpha u) + k(u)D_x(R^\alpha W)],\\
C^y &= -[k'(u)WD_y(R^\alpha u) + k(u)D_y(R^\alpha W)].
\end{aligned}$$

Here, we use the linearity of operators $J^\alpha_{(i)}$:

$$\begin{aligned}
&J^\alpha_{(1)}\left\{W, D_x\left(\frac{\partial F}{\partial(D_x R^\alpha u)}\right)\right\} - J^\alpha_{(1)}\left\{W, D_x^2\left(\frac{\partial F}{\partial(D_x^2 R^\alpha u)}\right)\right\}\\
&\qquad = J^\alpha_{(1)}\left\{W, D_x\left(\frac{\partial F}{\partial(D_x R^\alpha u)}\right) - D_x^2\left(\frac{\partial F}{\partial(D_x^2 R^\alpha u)}\right)\right\}\\
&\qquad = J^\alpha_{(1)}\left\{W, D_x(-k'(u)u_x) - D_x^2(-k(u))\right\} = J^\alpha_{(1)}\{W, 0\} = 0.
\end{aligned}$$

The similar equality holds for $J^\alpha_{(2)}$.

Let us consider the case when $k(u)$ is an arbitrary function. Then, Equation (16) admits five-parameter Lie point symmetry group with the basis (25). For these basis operators, we have

$$W_1 = -u_t, \qquad W_2 = -u_x, \qquad W_3 = -u_y, \qquad W_4 = (\alpha-2)tu_t - xu_x - yu_y, \qquad W_5 = xu_y - yu_x.$$

The conservation law corresponding to W_1 is trivial. Indeed, in view of the Equation (16), we have

$$\begin{aligned}
C_1^t &= -u_t = -D_x(k(u)D_x R^\alpha u) - D_y(k(u)D_y R^\alpha u),\\
C_1^x &= k'(u)u_t D_x(R^\alpha u) + k(u)D_x(R^\alpha u_t),\\
C_1^y &= k'(u)u_t D_y(R^\alpha u) + k(u)D_y(R^\alpha u_t).
\end{aligned}$$

Then, the conservation law reads

$$\begin{aligned}
D_t(C_1^t) + D_x(C_1^x) + D_y(C_1^y) &= D_t[-D_x(k(u)D_x R^\alpha u) - D_y(k(u)D_y R^\alpha u)]\\
&\quad + D_x[k'(u)u_t D_x(R^\alpha u) + k(u)D_x(R^\alpha u_t)] + D_y([k'(u)u_t D_y(R^\alpha u) + k(u)D_y(R^\alpha u_t)])\\
&= D_x[k'(u)u_t D_x(R^\alpha u) + k(u)D_x(R^\alpha u_t) - D_t(k(u)D_x R^\alpha u)]\\
&\quad + D_y[k'(u)u_t D_y(R^\alpha u) + k(u)D_y(R^\alpha u_t) - D_t(k(u)D_y R^\alpha u)] \equiv 0
\end{aligned}$$

because $D_t R^\alpha u = R^\alpha u_t$.

Similarly, it is easy to prove that conservation laws corresponding to W_2 and W_3 are trivial too.

For W_4 after equivalent transformations, we find

$$C_4^t = u, \qquad C_4^x = k(u)D_x(R^\alpha u), \qquad C_4^y = k(u)D_y(R^\alpha u). \tag{49}$$

The corresponding conservation law is non-trivial, and it coincides with the considered Equation (16).

For W_5, we have

$$C_5^t = xu_y - yu_x = D_y(xu) - D_x(yu).$$

Then,

$$\begin{aligned} D_t(C_5^t) + D_x(C_5^x) + D_y(C_5^y) &= D_t[D_y(xu) - D_x(yu)] + D_x(C_5^x) + D_y(C_5^y) \\ &= D_x(C_5^x - yu_t) + D_y(C_5^y + xu_t). \end{aligned}$$

Now, we can replace u_t in view of Equation (16). Since the components C_5^x and C_5^y do not depend on u_t, we obtain that $C^t = 0$. After transformations, we find

$$\begin{aligned} C^x &= k[D_y(R^\alpha u) - D_x(R^\alpha(xu_y - yu_x)) - yD_x^2(R^\alpha u) + xD_xD_y(R^\alpha u)], \\ C^y &= k[-D_x(R^\alpha u) - D_y(R^\alpha(xu_y - yu_x)) + xD_y^2(R^\alpha u) + yD_xD_y(R^\alpha u)]. \end{aligned}$$

Now, let $k(u)$ be a power function: $k(u) = u^\gamma$. Then, Equation (16) has one additional symmetry X_6 defined by (26). For this symmetry, we have $W_6 = -u - \gamma t u_t$. It can be shown that, in this case after transformations, one can get a conserved vector with the components (49).

For $k(u) = u^{-1}$, we have an infinite number of symmetries X_∞ defined by (27). In this case, we have $W_\infty = -2A_x u - Au_x - Bu_y$. Then,

$$C_\infty^t = W_\infty = -2A_x u - D_x(Au) + A_x u - D_y(Bu) + B_y u = -D_x(Au) - D_y(Bu),$$

since $A_x = B_y$. Thus, we have $C_\infty^t = 0$ and

$$\begin{aligned} C^x &= u^{-2}u_y[AD_y(R^\alpha u) - BD_x(R^\alpha u)] + u^{-1}[D_x(R^\alpha(2A_x u + Au_x + Bu_y)) - A\Delta R^\alpha u - 2A_x D_x(R^\alpha u)], \\ C^y &= u^{-2}u_x[BD_x(R^\alpha u) - AD_y(R^\alpha u)] - u^{-1}[D_y(R^\alpha(2A_x u + Au_x + Bu_y)) + B\Delta R^\alpha u + 2A_x D_y(R^\alpha u)]. \end{aligned}$$

Thus, several different conservation laws have been found for the nonlinear space-fractional diffusion equation with the Riesz potential.

In conclusion, we note that, despite the fact that we consider nonlinear space-fractional diffusion Equation (16) only in two-dimensional space, all results for nonlinear self-adjointness and explicit forms of the Noether operators are valid for arbitrary dimension n. The proposed technique gives one the opportunity to construct conservation laws for a wide class of fractional differential equations with the Riesz potential.

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