

# A Method of Generating Fuzzy Implications with Specific Properties

Dimitrios S. Grammatikopoulos  and Basil K. Papadopoulos \* 

Section of Mathematics and Informatics, Department of Civil Engineering, School of Engineering, Democritus University of Thrace, 67100 Kimeria, Greece; dimigram2@civil.duth.gr

\* Correspondence: papadob@civil.duth.gr

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**Abstract:** In this paper we introduce a new method of generating fuzzy implications via known fuzzy implications. We focus on the case of generating fuzzy implications via a fuzzy connective and at least one known fuzzy implication. We present some basic desirable properties of fuzzy implications that are invariant via this method. Furthermore, we suggest some ways of preservation or violation of these properties, based in this method. We show how we can generate not greater or not weaker fuzzy implications with specific properties. Finally, two subclasses of any fuzzy implication arise, the so called *T* and *S* subclasses.

**Keywords:** fuzzy implication; t-norm; t-conorm

## 1. Introduction

The generalization of the notion of implication from a classical to fuzzy topic is a known process. Generation methods of fuzzy implications are also known and proposed in the literature [1–7]. Specifically, the generation of fuzzy implications via known fuzzy implications has also been studied and many methods have been proposed (see [1] Chapter 6, [7] Subsection 12.2.3, [2,3,5]).

Moreover, many properties of fuzzy implications have been presented and their dependence or independence has also been studied ([1,4] Section 3). Many classes of fuzzy implications have also been proposed, and their properties studied extensively too [1,3].

Although, all of these are basically theoretical approaches, or they seem to be, they often have applicable extensions. These applicable extensions were the starting point of this work. For instance, if we need a fuzzy implication with a specific property (see [8] Equation (6)), how could we get it? Going one step further, how could we produce new fuzzy implications with only specific properties? Both of those questions are answered in this work.

In this work we dealt with another generation of fuzzy implications via known fuzzy implications. Particularly, we studied the case of using fuzzy implications and fuzzy connectives, such as t-norms and t-conorms. Two more construction methods of fuzzy implications are presented. One of their characteristics is that they preserve some properties, such as the left neutrality property, the identity principle, the ordering property, and the (left, right) contrapositive symmetry. Another, more important characteristic is that we studied and proved the conditions, such that these methods produced fuzzy implications with only specific properties from those of the left neutrality property, the identity principle, and the ordering property. In other words, not only the preservation, but also the violation of these properties can be controlled and achieved. One more characteristic of these methods is presented. That is the ability to generate not greater or not weaker fuzzy implications from the given ones with only specific properties of the aforementioned. Finally, two subclasses of any fuzzy implication arise, the so called *T* and *S* subclass, or the “not greater” and “not weaker” subclasses respectively.

## 2. Preliminaries

**Definition 1** ([1,9–11]). A decreasing function  $N : [0, 1] \rightarrow [0, 1]$  is called fuzzy negation, if  $N(0) = 1$  and  $N(1) = 0$ .

**Definition 2** (See [1] Definition 2.1.1). A function  $T : [0, 1]^2 \rightarrow [0, 1]$  is called a triangular norm (shortly  $t$ -norm), if it satisfies, for all  $x, y, z \in [0, 1]$ , the following conditions

$$T(x, y) = T(y, x), \quad (1)$$

$$T(x, T(y, z)) = T(T(x, y), z), \quad (2)$$

$$\text{if } y \leq z, \text{ then } T(x, y) \leq T(x, z), \text{ i.e., } T(x, \cdot) \text{ is increasing,} \quad (3)$$

$$T(x, 1) = x. \quad (4)$$

**Definition 3** (See [1] Definition 2.2.1). A function  $S : [0, 1]^2 \rightarrow [0, 1]$  is called a triangular conorm (shortly  $t$ -conorm), if it satisfies, for all  $x, y, z \in [0, 1]$ , the following conditions

$$S(x, y) = S(y, x), \quad (5)$$

$$S(x, S(y, z)) = S(S(x, y), z), \quad (6)$$

$$\text{if } y \leq z, \text{ then } S(x, y) \leq S(x, z), \text{ i.e., } S(x, \cdot) \text{ is increasing,} \quad (7)$$

$$S(x, 0) = x. \quad (8)$$

**Definition 4** (See [1] Definitions 2.1.2 and 2.2.2). A  $t$ -norm  $T$  (respectively a  $t$ -conorm  $S$ ) is called

(i) Idempotent, if

$$T(x, x) = x, \text{ for all } x \in [0, 1], \\ \text{(respectively } S(x, x) = x, \text{ for all } x \in [0, 1]),$$

(ii) Positive, if

$$T(x, y) = 0 \Leftrightarrow x = 0 \text{ or } y = 0, \\ \text{(respectively } S(x, y) = 1 \Leftrightarrow x = 1 \text{ or } y = 1).$$

**Definition 5** ([1,12]). By  $\Phi$  we denote the family of all increasing bijections from  $[0, 1]$  to  $[0, 1]$ . We say that functions  $f, g : [0, 1]^n \rightarrow [0, 1]$  are  $\Phi$ -conjugate, if there exists a  $\phi \in \Phi$  such that  $g = f_\phi$ , where

$$f_\phi(x_1, x_2, \dots, x_n) = \phi^{-1}(f(\phi(x_1), \phi(x_2), \dots, \phi(x_n))), x_1, x_2, \dots, x_n \in [0, 1].$$

**Remark 1** ([1]). It is easy to prove that if  $\phi \in \Phi$  and  $T$  is a  $t$ -norm,  $S$  is a  $t$ -conorm, and  $N$  is a fuzzy negation, then  $T_\phi$  is a  $t$ -norm,  $S_\phi$  is a  $t$ -conorm, and  $N_\phi$  is a fuzzy negation.

**Definition 6** ([1,9]). A function  $I : [0, 1]^2 \rightarrow [0, 1]$  is called a fuzzy implication if

$$I \text{ is decreasing with respect to the first variable,} \quad (9)$$

*I is increasing with respect to the second variable,* (10)

$$I(0, 0) = 1, \quad (11)$$

$$I(1, 1) = 1, \quad (12)$$

$$I(1, 0) = 0. \quad (13)$$

**Definition 7** (See [1] Definition 1.3.1). A fuzzy implication *I* is said to satisfy

(i) The left neutrality property, if

$$I(1, y) = y, y \in [0, 1]; \quad (14)$$

(ii) The identity principle, if

$$I(x, x) = 1, x \in [0, 1]; \quad (15)$$

(iii) The exchange principle, if

$$I(x, I(y, z)) = I(y, I(x, z)), x, y, z \in [0, 1]; \quad (16)$$

(iv) The ordering property, if

$$I(x, y) = 1 \Leftrightarrow x \leq y, x, y \in [0, 1]. \quad (17)$$

**Remark 2** ([1]). It is proven that, if  $\phi \in \Phi$  and  $I : [0, 1]^2 \rightarrow [0, 1]$  is a fuzzy implication, then  $I_\phi$  is also a fuzzy implication.

**Definition 8** (See [1] Definition 1.5.1). Let *I* be a fuzzy implication and *N* be a fuzzy negation. *I* is said to satisfy the

(i) Law of contraposition with respect to *N*, if

$$I(x, y) = I(N(y), N(x)), x, y \in [0, 1]; \quad (18)$$

(ii) Law of left contraposition with respect to *N*, if

$$I(N(x), y) = I(N(y), x), x, y \in [0, 1]; \quad (19)$$

(iii) Law of right contraposition with respect to *N*, if

$$I(x, N(y)) = I(y, N(x)), x, y \in [0, 1]. \quad (20)$$

If *I* satisfies the (left, right) contrapositive symmetry with respect to *N*, then we also denote this by *CP*(*N*) (respectively, by *L-CP*(*N*), *R-CP*(*N*)).

**Lemma 1** (See [1] Lemma 1.4.14). If a function  $I : [0, 1]^2 \rightarrow [0, 1]$  satisfies (9), (11) and (13), then the function  $N_I : [0, 1] \rightarrow [0, 1]$  is a fuzzy negation, where

$$N_I(x) = I(x, 0), x \in [0, 1].$$

**Definition 9** (See [1] Definition 1.4.15). Let  $I : [0, 1]^2 \rightarrow [0, 1]$  be a fuzzy implication. The function  $N_I$  defined by Lemma 1 is called the natural negation of *I*.

**Definition 10** (See [1] Definition 1.6.12). Let *N* be a fuzzy negation and *I* be a fuzzy implication. A function  $I_N : [0, 1]^2 \rightarrow [0, 1]$  defined by

$$I_N(x, y) = I(N(y), N(x)), x, y \in [0, 1],$$

is called the *N*-reciprocal of *I*.

### 3. The Main Results

#### 3.1. Fuzzy Implications Generated by Known Fuzzy Implications

As we mentioned before, there are many methods to generate fuzzy implications via one or two known fuzzy implications (see [1] Chapter 6, [7] Subsection 12.2.3, [2,3,5]). In this paper, firstly, we introduce another method that generates fuzzy implications via  $n$  known fuzzy implications, where  $n$  is any positive natural number.

**Theorem 1.** Let  $f : [0, 1]^n \rightarrow [0, 1]$  be an increasing function with respect to any of its variables, and moreover,  $f(0, 0, \dots, 0) = 0$  and  $f(1, 1, \dots, 1) = 1$ . If  $I_{(i)}, i = 1, 2, \dots, n$  are fuzzy implications, then the function that is defined by

$$I(x, y) = f[I_{(1)}(x, y), I_{(2)}(x, y), \dots, I_{(n)}(x, y)], \quad (21)$$

is a fuzzy implication.

**Proof.**  $I$  satisfies (9) since for all  $x_1, x_2, y \in [0, 1]$  with

$$\begin{aligned} x_1 \leq x_2 &\Rightarrow I_{(i)}(x_1, y) \geq I_{(i)}(x_2, y), i = 1, 2, \dots, n \\ &\Rightarrow f[I_{(1)}(x_1, y), I_{(2)}(x_1, y), \dots, I_{(n)}(x_1, y)] \geq f[I_{(1)}(x_2, y), I_{(2)}(x_2, y), \dots, I_{(n)}(x_2, y)] \\ &\Rightarrow I(x_1, y) \geq I(x_2, y). \end{aligned}$$

$I$  satisfies (10) since for all  $x, y_1, y_2 \in [0, 1]$  with

$$\begin{aligned} y_1 \leq y_2 &\Rightarrow I_{(i)}(x, y_1) \leq I_{(i)}(x, y_2), i = 1, 2, \dots, n \\ &\Rightarrow f[I_{(1)}(x, y_1), I_{(2)}(x, y_1), \dots, I_{(n)}(x, y_1)] \leq f[I_{(1)}(x, y_2), I_{(2)}(x, y_2), \dots, I_{(n)}(x, y_2)] \\ &\Rightarrow I(x, y_1) \leq I(x, y_2). \end{aligned}$$

$I$  satisfies (11) since  $I(0, 0) = f[I_{(1)}(0, 0), I_{(2)}(0, 0), \dots, I_{(n)}(0, 0)] = f(0, 0, \dots, 0) = 0$ .

$I$  satisfies (12) since  $I(1, 1) = f[I_{(1)}(1, 1), I_{(2)}(1, 1), \dots, I_{(n)}(1, 1)] = f(1, 1, \dots, 1) = 1$ .

$I$  satisfies (13) since  $I(1, 0) = f[I_{(1)}(1, 0), I_{(2)}(1, 0), \dots, I_{(n)}(1, 0)] = f(0, 0, \dots, 0) = 0$ .

Thus,  $I$  is a fuzzy implication.  $\square$

**Remark 3.** The fuzzy implication  $I$  defined in (21) is denoted by  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$ . So in the rest of this paper when we use the symbolisms  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  and  $f$ , it will be understood that we refer to Theorem 1.

**Proposition 1.** Let  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  be a fuzzy implication. Then its natural negation is

$$N_{I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}}(x) = f[N_{I_{(1)}}(x), N_{I_{(2)}}(x), \dots, N_{I_{(n)}}(x)].$$

**Proof.** It is deduced by Lemma 1, Definition 9 and Theorem 1.  $\square$

**Proposition 2.** Let  $I_{(i)}, i = 1, 2, \dots, n$  be fuzzy implications which satisfy (15) (respectively (18)–(20) with respect to  $N$ ). Then, the obtained fuzzy implication  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  satisfies (15) (respectively (18)–(20) with respect to  $N$ ).

**Proof.** Let  $I_{(i)}, i = 1, 2, \dots, n$  are fuzzy implications which satisfy (15), then  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  satisfies (15) since for all  $x \in [0, 1]$  it is

$$I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}(x, x) = f[I_{(1)}(x, x), I_{(2)}(x, x), \dots, I_{(n)}(x, x)] = f(1, 1, \dots, 1) = 1.$$

If  $I_{(i)}, i = 1, 2, \dots, n$  are fuzzy implications which satisfy the contrapositive symmetry (18) with respect to  $N$ , then  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  satisfies (18) with respect to  $N$  since for all  $x, y \in [0, 1]$  it is

$$\begin{aligned} I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}(x, y) &= f[I_{(1)}(x, y), I_{(2)}(x, y), \dots, I_{(n)}(x, y)] \\ &= f[I_{(1)}(N(y), N(x)), I_{(2)}(N(y), N(x)), \dots, I_{(n)}(N(y), N(x))] \\ &= I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}(N(y), N(x)). \end{aligned}$$

The proof is similar for the left contrapositive symmetry (19) and the right contrapositive symmetry (20) with respect to  $N$ .  $\square$

**Remark 4.** Similar to the previous proof we deduce that the  $N$ -reciprocal of  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  is

$$(I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}})_N = I_{f, (I_{(1)})_N, (I_{(2)})_N, \dots, (I_{(n)})_N}.$$

On the other hand, if  $I_{(i)}, i = 1, 2, \dots, n$  are fuzzy implications which satisfy (14) (respectively (16) and (17)), then it is not ensured that  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  satisfies (14) (respectively (16) and (17)) as it is presented to the following examples.

**Example 1.** Consider the Łukasiewicz's implication  $I_{LK}(x, y) = \min\{1, 1 - x + y\}$  and Gödel's implication  $I_{GD}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$  (see [1] Table 1.3) and the function  $f(x, y) = x \cdot y, x, y \in [0, 1]$ . It is known that  $I_{LK}$  and  $I_{GD}$  satisfy (14) and (16) (see [1] Table 1.4). On the other hand,  $I_{f, I_{LK}, I_{GD}}(x, y) = I_{LK}(x, y) \cdot I_{GD}(x, y)$  does not satisfy (14), since for all  $y \in [0, 1]$  it is

$$I_{f, I_{LK}, I_{GD}}(1, y) = I_{LK}(1, y) \cdot I_{GD}(1, y) = y \cdot y = y^2.$$

The same holds for (16), since

$$I_{f, I_{LK}, I_{GD}}(0.9, I_{f, I_{LK}, I_{GD}}(1, 0.9)) = 0.7371 \neq 1 = I_{f, I_{LK}, I_{GD}}(1, I_{f, I_{LK}, I_{GD}}(0.9, 0.9)).$$

**Example 2.** Consider the function  $f(x, y) = \begin{cases} 1, & \text{if } (x, y) \in (0.5, 1]^2 \\ x \cdot y, & \text{otherwise} \end{cases}$ . It is known that  $I_{LK}$  and  $I_{GD}$  satisfy (17) (see [1] Table 1.4). On the other hand,

$$I_{f, I_{LK}, I_{GD}}(x, y) = \begin{cases} 1, & \text{if } (I_{LK}(x, y), I_{GD}(x, y)) \in (0.5, 1]^2 \\ I_{LK}(x, y) \cdot I_{GD}(x, y), & \text{otherwise} \end{cases},$$

it does not satisfy (17), since  $I_{f, I_{LK}, I_{GD}}(0.7, 0.6) = 1$ .

**Theorem 2.** If  $\phi \in \Phi$  and  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  is a fuzzy implication, then  $(I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}})_\phi$  is a fuzzy implication, and moreover

$$(I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}})_\phi = I_{f_\phi, (I_{(1)})_\phi, (I_{(2)})_\phi, \dots, (I_{(n)})_\phi}.$$

**Proof.** Let  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$  be a fuzzy implication; then,  $(I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}})_\phi$  is a fuzzy implication, according to the Remark 2. So, for all  $x, y \in [0, 1]$ , we deduce that

$$\begin{aligned}
(I_{f,I_{(1)},I_{(2)},\dots,I_{(n)}})_\phi(x,y) &= \phi^{-1}I_{f,I_{(1)},I_{(2)},\dots,I_{(n)}}(\phi(x),\phi(y)) \\
&= \phi^{-1}(f(I_{(1)}(\phi(x),\phi(y)), I_{(2)}(\phi(x),\phi(y)), \dots, I_{(n)}(\phi(x),\phi(y)))) \\
&= \phi^{-1}(f(\phi(\phi^{-1}(I_{(1)}(\phi(x),\phi(y)))), \dots, \phi(\phi^{-1}(I_{(n)}(\phi(x),\phi(y))))) \\
&= f_\phi((I_{(1)})_\phi(x,y), (I_{(2)})_\phi(x,y), \dots, (I_{(n)})_\phi(x,y)) \\
&= I_{f_\phi, (I_{(1)})_\phi, (I_{(2)})_\phi, \dots, (I_{(n)})_\phi}(x,y).
\end{aligned}$$

□

### 3.2. Fuzzy Implications Generated by Fuzzy Connectives and Fuzzy Implications

In this section we will study the special case, where  $n = 2$  and  $f$  is a fuzzy connective, i.e. a t-norm or a t-conorm. So firstly we must prove that a t-norm and a t-conorm are suitable functions to replace  $f$ ; i.e. they satisfy the properties of the function  $f$ .

**Corollary 1.** Let  $T$  be a t-norm, and  $I_{(1)}$  and  $I_{(2)}$  be two fuzzy implications. Then, the function that is defined by  $I_{T,I_{(1)},I_{(2)}}(x,y) = T[I_{(1)}(x,y), I_{(2)}(x,y)]$  is a fuzzy implication.

**Proof.** Since  $T$  is a t-norm, it is increasing with respect to both of its variables. This is deduced by (3) and (1). Furthermore, by (4) it is  $T(1,1) = 1$  and  $T(0,1) = 0$ . So, since  $0 \leq 1$ , by (3) we deduce that  $T(0,0) \leq T(0,1) = 0 \Rightarrow T(0,0) = 0$ . By Theorem 1 we deduce that

$$I_{T,I_{(1)},I_{(2)}}(x,y) = T[I_{(1)}(x,y), I_{(2)}(x,y)]$$

is a fuzzy implication. □

**Corollary 2.** Let  $S$  be a t-conorm and  $I_{(1)}, I_{(2)}$  two fuzzy implications. Then, the function that is defined by  $I_{S,I_{(1)},I_{(2)}}(x,y) = S[I_{(1)}(x,y), I_{(2)}(x,y)]$  is a fuzzy implication.

**Proof.** Since  $S$  is a t-conorm, it is increasing with respect to both of its variables. This is deduced by (7) and (5). Furthermore, by (8) it is  $S(0,0) = 0$  and  $S(0,1) = 1$ . So, since  $0 \leq 1$ , by (7) we deduce that  $S(0,0) = 1 \leq S(1,1) \Rightarrow S(1,1) = 1$ . By Theorem 1 we deduce that

$$I_{S,I_{(1)},I_{(2)}}(x,y) = S[I_{(1)}(x,y), I_{(2)}(x,y)]$$

is a fuzzy implication. □

**Corollary 3.** Let  $I_{(1)}, I_{(2)}$  be two fuzzy implications. Then the fuzzy implications  $I_{T,I_{(1)},I_{(2)}}$  and  $I_{S,I_{(1)},I_{(2)}}$  have, respectively, the following natural negations

$$N_{I_{T,I_{(1)},I_{(2)}}}(x) = T[N_{I_{(1)}}(x), N_{I_{(2)}}(x)] \text{ and } N_{I_{S,I_{(1)},I_{(2)}}}(x) = S[N_{I_{(1)}}(x), N_{I_{(2)}}(x)].$$

**Proof.** It is deduced by Proposition 1. □

**Corollary 4.** Let  $I_{(1)}, I_{(2)}$  be two fuzzy implications that satisfy (15) (respectively (18)–(20) with respect to  $N$ ). Then the fuzzy implications  $I_{T,I_{(1)},I_{(2)}}$  and  $I_{S,I_{(1)},I_{(2)}}$  satisfy (15) (respectively (18)–(20) with respect to  $N$ ).

**Proof.** It is deduced by Proposition 2. □

**Proposition 3.** Let  $I_{(1)}, I_{(2)}$  be two fuzzy implications that satisfy (17). Then the fuzzy implication  $I_{T, I_{(1)}, I_{(2)}}$  satisfies (17).

**Proof.** Let  $I_{(1)}, I_{(2)}$  be two fuzzy implications that satisfy (17), then

$$I_{(i)}(x, y) = 1 \Leftrightarrow x \leq y, x, y \in [0, 1], i = 1, 2.$$

Thus, for all  $x, y \in [0, 1]$ , if  $x \leq y$  then  $I_{T, I_{(1)}, I_{(2)}}(x, y) = T[I_{(1)}(x, y), I_{(2)}(x, y)] = T(1, 1) = 1$ .  
Vice versa; if  $T$  is a t-norm then it satisfies the equivalence

$$T(x, y) = 1 \Leftrightarrow x = y = 1.$$

Thus,

$$\begin{aligned} I_{T, I_{(1)}, I_{(2)}}(x, y) = 1 &\Rightarrow T[I_{(1)}(x, y), I_{(2)}(x, y)] = 1 \\ &\Rightarrow I_{(1)}(x, y) = I_{(2)}(x, y) = 1 \\ &\Rightarrow x \leq y. \end{aligned}$$

□

Moreover, for properties (15), (17) and (14) we prove the following propositions.

**Proposition 4.** Let  $I_{(1)}, I_{(2)}$  are two fuzzy implications.

- (i)  $I_{(1)}, I_{(2)}$  satisfy (15) if the fuzzy implication  $I_{T, I_{(1)}, I_{(2)}}$  satisfies (15).
- (ii)  $I_{(1)}, I_{(2)}$  satisfy (17) if the fuzzy implication  $I_{T, I_{(1)}, I_{(2)}}$  satisfies (17).

**Proof.** (i) If  $I_{(1)}, I_{(2)}$  satisfy (15), then  $I_{T, I_{(1)}, I_{(2)}}$  satisfies (15) and the proof is deduced by Proposition 2.  
Vice versa; if  $I_{T, I_{(1)}, I_{(2)}}$  satisfies (15), then for all  $x \in [0, 1]$  it is

$$\begin{aligned} I_{T, I_{(1)}, I_{(2)}}(x, x) = 1 &\Leftrightarrow T[I_{(1)}(x, x), I_{(2)}(x, x)] = 1 \\ &\Leftrightarrow I_{(i)}(x, x) = 1, i = 1, 2. \end{aligned}$$

Thus,  $I_{(i)}, i = 1, 2$  satisfy (15).

- (ii) If  $I_{(1)}, I_{(2)}$  satisfy (17), then  $I_{T, I_{(1)}, I_{(2)}}$  satisfies (17) and the proof is deduced by Proposition 3.  
Vice versa; if  $I_{T, I_{(1)}, I_{(2)}}$  satisfies (17), then for all  $x, y \in [0, 1]$  it is

$$\begin{aligned} x \leq y &\Leftrightarrow I_{T, I_{(1)}, I_{(2)}}(x, y) = 1 \\ &\Leftrightarrow T[I_{(1)}(x, y), I_{(2)}(x, y)] = 1 \\ &\Leftrightarrow I_{(i)}(x, y) = 1, i = 1, 2. \end{aligned}$$

Thus,  $I_{(i)}, i = 1, 2$  satisfy (17). □

**Proposition 5.** Let  $I_{(1)}, I_{(2)}$  be two fuzzy implications and  $S$  is a positive t-conorm. If the fuzzy implication  $I_{S, I_{(1)}, I_{(2)}}$  satisfies (15), then for all  $x \in [0, 1]$  it is

$$I_{(1)}(x, x) = 1 \text{ or } I_{(2)}(x, x) = 1.$$

**Proof.** If  $I_{S,I(1),I(2)}$  satisfies (15), then for all  $x \in [0, 1]$  it is

$$\begin{aligned} I_{S,I(1),I(2)}(x, x) &= 1 \Leftrightarrow S[I(1)(x, x), I(2)(x, x)] = 1 \\ &\Leftrightarrow I(1)(x, x) = 1 \text{ or } I(2)(x, x) = 1, \end{aligned}$$

since  $S$  is a positive t-conorm.  $\square$

**Proposition 6.** Let  $I(1)$  and  $I(2)$  be two fuzzy implications that satisfy (17) and  $S$  is a positive t-conorm. Then the fuzzy implication  $I_{S,I(1),I(2)}$  satisfies (17).

**Proof.** Let  $I(1)$  and  $I(2)$  be two fuzzy implications that satisfy (17); then,

$$I(i)(x, y) = 1 \Leftrightarrow x \leq y, x, y \in [0, 1], i = 1, 2.$$

Thus, for all  $x, y \in [0, 1]$  it is

$$\begin{aligned} I_{S,I(1),I(2)}(x, y) &= 1 \Leftrightarrow S[I(1)(x, y), I(2)(x, y)] = 1 \\ &\Leftrightarrow I(1)(x, y) = 1 \text{ or } I(2)(x, y) = 1 \\ &\Leftrightarrow x \leq y, \end{aligned}$$

since  $S$  is a positive t-conorm.  $\square$

**Proposition 7.** Let  $I(1)$  and  $I(2)$  be two fuzzy implications and  $S$  is a positive t-conorm. If the fuzzy implication  $I_{S,I(1),I(2)}$  satisfies (17), then for any  $x, y \in [0, 1]$  it is

$$x \leq y \Leftrightarrow I(1)(x, y) = 1 \text{ or } I(2)(x, y) = 1.$$

**Proof.** If  $I_{S,I(1),I(2)}$  satisfies (17), then for all  $x, y \in [0, 1]$  it is

$$\begin{aligned} x \leq y &\Leftrightarrow I_{S,I(1),I(2)}(x, y) = 1 \\ &\Leftrightarrow S[I(1)(x, y), I(2)(x, y)] = 1 \\ &\Leftrightarrow I(1)(x, y) = 1 \text{ or } I(2)(x, y) = 1, \end{aligned}$$

since  $S$  is a positive t-conorm.  $\square$

**Proposition 8.** Let  $I(1)$  and  $I(2)$  be two fuzzy implications that satisfy (14). Then the fuzzy implication

- (i)  $I_{T,I(1),I(2)}$  satisfies (14) if  $T = T_M = \min\{x, y\}$  and
- (ii)  $I_{S,I(1),I(2)}$  satisfies (14) if  $S = S_M = \max\{x, y\}$ .

**Proof.** (i) Let  $I(1)$  and  $I(2)$  be two fuzzy implications that satisfy (14); then,

$$I(i)(1, y) = y, y \in [0, 1], i = 1, 2.$$

So, for all  $y \in [0, 1]$ , we have

$$I_{T,I(1),I(2)}(1, y) = T[I(1)(1, y), I(2)(1, y)] = T(y, y).$$

Thus,  $I_{T,I(1),I(2)}$  satisfies (14), when  $T$  is idempotent. Moreover the only idempotent t-norm is  $T = T_M = \min\{x, y\}$  (see [1] Remark 2.1.4(ii), [11] Proposition 1.9).



Vice versa; if  $T = T_M = \min\{x, y\}$ , then

$$I_{T_M, I_{(1)}, I_{(2)}}(1, y) = T_M[I_{(1)}(1, y), I_{(2)}(1, y)] = \min\{y, y\} = y, y \in [0, 1].$$

(ii) Similarly, for all  $y \in [0, 1]$ , we have

$$I_{S, I_{(1)}, I_{(2)}}(1, y) = S[I_{(1)}(1, y), I_{(2)}(1, y)] = S(y, y).$$

Thus,  $I_{S, I_{(1)}, I_{(2)}}$  satisfies (14), when  $S$  is idempotent. Moreover the only idempotent t-conorm is  $S_M = \max\{x, y\}$  (see [1] Remark 2.2.5(ii)).

Vice versa; if  $S_M = \max\{x, y\}$ , then

$$I_{S_M, I_{(1)}, I_{(2)}}(1, y) = S_M[I_{(1)}(1, y), I_{(2)}(1, y)] = \max\{y, y\} = y, y \in [0, 1].$$

□

**Example 3.** Consider the Łukasiewicz's implication  $I_{LK}(x, y) = \min\{1, 1 - x + y\}$ , Gödel's implication  $I_{GD}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$  (See [1] Table 1.3) and the positive t-conorm  $S_P(x, y) = x + y - x \cdot y$  (see [1] Table 2.2). It is known that  $I_{LK}$  and  $I_{GD}$  satisfy (16) (see [1] Table 1.4). On the other hand,

$$I_{S_P, I_{LK}, I_{GD}}(x, y) = I_{LK}(x, y) + I_{GD}(x, y) - I_{LK}(x, y) \cdot I_{GD}(x, y)$$

does not satisfy (16), since

$$I_{S_P, I_{LK}, I_{GD}}(0.97, I_{S_P, I_{LK}, I_{GD}}(1, 0.8)) = 0.9996 \neq 0.998844 = I_{S_P, I_{LK}, I_{GD}}(1, I_{S_P, I_{LK}, I_{GD}}(0.97, 0.8)).$$

We have to notice at this point that the same result for the violation of (16) holds if we use a t-norm. This is clear in Example 1, where  $f = T_M$ .

**Corollary 5.** (i) If  $\phi \in \Phi$  and  $I_{T, I_{(1)}, I_{(2)}}$  is a fuzzy implication, then  $(I_{T, I_{(1)}, I_{(2)}})_\phi$  is a fuzzy implication, and moreover,

$$(I_{T, I_{(1)}, I_{(2)}})_\phi = I_{T_\phi, (I_{(1)})_\phi, (I_{(2)})_\phi}.$$

(ii) If  $\phi \in \Phi$  and  $I_{S, I_{(1)}, I_{(2)}}$  is a fuzzy implication, then  $(I_{S, I_{(1)}, I_{(2)}})_\phi$  is a fuzzy implication, and moreover,

$$(I_{S, I_{(1)}, I_{(2)}})_\phi = I_{S_\phi, (I_{(1)})_\phi, (I_{(2)})_\phi}.$$

**Proof.** It is deduced by Theorem 2 and Corollaries 1 and 2. □

Now let us explain a difference between these methods, the one with the t-norms and the other with the t-conorms. Firstly, we prove the following proposition.

**Proposition 9.** For all  $x, y \in [0, 1]$  it is

$$T(x, y) \leq x \leq S(x, y) \text{ and } T(x, y) \leq y \leq S(x, y).$$

**Proof.** For all  $x, y \in [0, 1]$  it is  $0 \leq y \leq 1$  and by (3) and (4) we deduce that

$$T(x, y) \leq T(x, 1) \Rightarrow T(x, y) \leq x$$

and similarly by (7), (8), and (5) we deduce that

$$S(x, 0) \leq S(x, y) \Rightarrow S(0, x) \leq S(x, y) \Rightarrow x \leq S(x, y).$$

Thus, for all  $x, y \in [0, 1]$  it is

$$T(x, y) \leq x \leq S(x, y) \text{ and (obviously) } T(x, y) \leq y \leq S(x, y).$$

□

Proposition 9 testifies to the importance of the method presented, since if we have two fuzzy implications  $I_{(1)}, I_{(2)}$  and we want to generate a not greater fuzzy implication of them, a solution is the fuzzy implication  $I_{T, I_{(1)}, I_{(2)}}$ . On the other hand, if we want a not weaker fuzzy implication, then the solution is  $I_{S, I_{(1)}, I_{(2)}}$ . Moreover, since  $T_D \leq T \leq T_M \leq S_M \leq S \leq S_D$  (see [1] Remarks 2.1.4(ix) and 2.2.5(viii)), where  $T_D$  is the drastic product t-norm (see [1] Table 2.1) and  $S_D$  the drastic sum t-conorm (see [1] Table 2.2), we deduce that

$$I_{T_D, I_{(1)}, I_{(2)}} \leq I_{T, I_{(1)}, I_{(2)}} \leq I_{T_M, I_{(1)}, I_{(2)}} \leq I_{(i)} \leq I_{S_M, I_{(1)}, I_{(2)}} \leq I_{S, I_{(1)}, I_{(2)}} \leq I_{S_D, I_{(1)}, I_{(2)}}, i = 1, 2.$$

### 3.3. Fuzzy Connectives' Classes of Fuzzy Implications

In this section in an attempt to simplify a previous theoretical approach; we show the special case, where  $I_{(1)} = I_{(2)} = \dots = I_{(n)}$ . Then the corresponding fuzzy implication is denoted by  $I_{(1)}^f$  instead of  $I_{f, I_{(1)}, I_{(2)}, \dots, I_{(n)}}$ . Moreover, if  $f$  is a fuzzy connective, i.e., a t-norm or a t-conorm, then the corresponding fuzzy implication is denoted by  $I_{(1)}^T$  and respectively  $I_{(1)}^S$ .

It is obvious that all the previous Theorems, Propositions, Corollaries, and results hold case-by-case for these implications, since they are special cases of the previous we have mentioned. We just mentioned these cases due to their simplicity, since we used only one and not two fuzzy implications. The previous results of those cases, when we used a fuzzy connective, were transformed to the following corollaries, which are presented without proofs due to their simplicity.

**Corollary 6.** Let  $I_{(1)}$  be a fuzzy implication that satisfies (15) (respectively (18)–(20) with respect to  $N$ ). Then, the fuzzy implications  $I_{(1)}^T$  and  $I_{(1)}^S$  satisfy (15) (respectively (18)–(20) with respect to  $N$ ).

**Corollary 7.** Let  $I_{(1)}$  be a fuzzy implication.

- (i)  $I_{(1)}$  satisfies (15) if the fuzzy implication  $I_{(1)}^T$  satisfies (15).
- (ii)  $I_{(1)}$  violates (15) if the fuzzy implication  $I_{(1)}^T$  violates (15).
- (iii)  $I_{(1)}$  satisfies (17) if the fuzzy implication  $I_{(1)}^T$  satisfies (17).
- (iv)  $I_{(1)}$  violates (17) if the fuzzy implication  $I_{(1)}^T$  violates (17).

**Corollary 8.** Let  $I_{(1)}$  be a fuzzy implication and  $S$  be a positive t-conorm.

- (i)  $I_{(1)}$  satisfies (15) if the fuzzy implication  $I_{(1)}^S$  satisfies (15).
- (ii)  $I_{(1)}$  violates (15) if the fuzzy implication  $I_{(1)}^S$  violates (15).
- (iii)  $I_{(1)}$  satisfies (17) if the fuzzy implication  $I_{(1)}^S$  satisfies (17).
- (iv)  $I_{(1)}$  violates (17) if the fuzzy implication  $I_{(1)}^S$  violates (17).

**Corollary 9.** Let  $I_{(1)}$  be a fuzzy implication that satisfies (14). Then the fuzzy implication

- (i)  $I_{(1)}^T$  satisfies (14) if  $T = T_M$ .
- (ii)  $I_{(1)}^T$  violates (14) if  $T \neq T_M$ .
- (iii)  $I_{(1)}^S$  satisfies (14) if  $S = S_M$ .
- (iv)  $I_{(1)}^S$  violates (14) if  $S \neq S_M$ .

Furthermore, two subclasses of every fuzzy implication were created. The first one is the  $T$  subclass of a fuzzy implication  $I_{(1)}$ . If we consider as  $\mathbb{T}$  the set of t-norms, then the  $T$  subclass of  $I_{(1)}$  is defined as  $I_{(1)}^{\mathbb{T}} = \cup_{T \in \mathbb{T}} \{I_{(1)}^T\}$ . We must notice that  $I_{(1)}^{\mathbb{T}}$  contains fuzzy implications that are not greater than  $I_{(1)}$ . Moreover, since  $I_{(1)} = I_{(1)}^M$ ,  $I_{(1)} \in I_{(1)}^{\mathbb{T}} \Rightarrow I_{(1)}^{\mathbb{T}} \neq \emptyset$ , and  $I_{(1)}$  is the greatest fuzzy implication that is contained in  $I_{(1)}^{\mathbb{T}}$ . On the other hand it is obvious that the weakest fuzzy implication that is contained in  $I_{(1)}^{\mathbb{T}}$  is  $I_{(1)}^{T_D}$ .

The second one is the  $S$  subclass of a fuzzy implication  $I_{(1)}$ . If we consider as  $\mathbb{S}$  the set of t-conorms, then the  $S$  subclass of  $I_{(1)}$  is defined as  $I_{(1)}^{\mathbb{S}} = \cup_{S \in \mathbb{S}} \{I_{(1)}^S\}$ . We must notice that  $I_{(1)}^{\mathbb{S}}$  contains fuzzy implications that are not weaker than  $I_{(1)}$ . Moreover, since  $I_{(1)} = I_{(1)}^M$ ,  $I_{(1)} \in I_{(1)}^{\mathbb{S}} \Rightarrow I_{(1)}^{\mathbb{S}} \neq \emptyset$  and  $I_{(1)}$  is the weakest fuzzy implication that is contained in  $I_{(1)}^{\mathbb{S}}$ . On the other hand, it is obvious that the greatest fuzzy implication that is contained in  $I_{(1)}^{\mathbb{S}}$  is  $I_{(1)}^{S_D}$ .

By the previous results it is obvious that  $I_{(1)}^{\mathbb{T}} \cap I_{(1)}^{\mathbb{S}} = \{I_{(1)}\}$ . Furthermore these two subclasses construct the fuzzy connectives' class of a fuzzy implication  $I_{(1)}$ , which is defined by  $I_{(1)}^{\mathbb{C}} = I_{(1)}^{\mathbb{T}} \cup I_{(1)}^{\mathbb{S}} = \cup_{C \in \mathbb{C}} \{I_{(1)}^C\}$ , where  $\mathbb{C}$  is the set of fuzzy connectives.

Our interest is focused on  $T$  and  $S$  subclass of a fuzzy implication. Firstly we must notice that if we use two valued fuzzy implications, such as  $I_{RS}$ ,  $I_0$ ,  $I_1$ ,  $I_3$ ,  $I_4$ ,  $I_6$ ,  $I_{10}$ ,  $I_{15}$ ,  $I_{18}$  (see [1] Table 1.3, Proposition 1.1.7 and [13]), then  $I_{(1)}^{\mathbb{C}} = I_{(1)}^{\mathbb{T}} = I_{(1)}^{\mathbb{S}} = \{I_{(1)}\}$ . This means that these fuzzy implications are invariant via this method and there is nothing to study and mention about these cases.

Moreover, according to Corollary 9, (14) is invariant only if we use an idempotent t-norm or t-conorm. On the other hand, as we mentioned before for any fuzzy implication  $I_{(1)}$  it is  $I_{(1)} = I_{(1)}^{T_M} = I_{(1)}^{S_M}$ . So another characteristic of these three sets is that if  $I_{(1)}$  satisfies (14), then  $I_{(1)}^{\mathbb{C}} - \{I_{(1)}\}$ ,  $I_{(1)}^{\mathbb{T}} - \{I_{(1)}\}$  and  $I_{(1)}^{\mathbb{S}} - \{I_{(1)}\}$ , when they are not empty, they are sets that contain only fuzzy implications that violate (14).

At this point let us give an example which explains the aforementioned theoretical approach.

**Example 4.** Consider the Łukasiewicz's implication  $I_{LK}(x, y) = \min\{1, 1 - x + y\}$  that satisfies (14), (16), (15), and (17) (see [1] Table 1.4). If we are looking for a weaker fuzzy implication that satisfies (15), (17) and violates (14), this could be  $I_{LK}^{T_{LK}}$ , where  $T_{LK}(x, y) = \max\{x + y - 1, 0\}$  (see [1] Table 2.1). So, it is

$$\begin{aligned} I_{LK}^{T_{LK}}(x, y) &= T_{LK}(I_{LK}(x, y), I_{LK}(x, y)) \\ &= \max\{I_{LK}(x, y) + I_{LK}(x, y) - 1, 0\} \\ &= \max\{2I_{LK}(x, y) - 1, 0\} \\ &= \max\{2\min\{1, 1 - x + y\} - 1, 0\} \\ &= \max\{\min\{1, 1 - 2x + 2y\}, 0\} \\ &= \begin{cases} 0, & \text{if } x - y \geq 0.5 \\ 1 - 2x + 2y, & \text{if } 0 \leq x - y \leq 0.5 \\ 1, & \text{if } x \leq y \end{cases} \end{aligned}$$

Moreover,  $I_{LK}^{T_{LK}}$  violates (16) since

$$I_{LK}^{T_{LK}}(0.9, I_{LK}^{T_{LK}}(1, 0.8)) = 0.4 \neq 0.6 = I_{LK}^{T_{LK}}(1, I_{LK}^{T_{LK}}(0.9, 0.8)).$$

If we consider  $T_D(x, y) = \begin{cases} 0, & \text{if } x, y \in [0, 1) \\ \min\{x, y\}, & \text{otherwise} \end{cases}$  (see [1] Table 2.1), then the weakest fuzzy implication we can generate with this method is

$$\begin{aligned} I_{LK}^{T_D}(x, y) &= T_D(I_{LK}(x, y), I_{LK}(x, y)) \\ &= \begin{cases} 0, & \text{if } I_{LK}(x, y) \in [0, 1) \\ \min\{I_{LK}(x, y), I_{LK}(x, y)\}, & \text{otherwise} \end{cases} \\ &= \begin{cases} 0, & \text{if } x > y \\ I_{LK}(x, y), & \text{otherwise} \end{cases} \\ &= \begin{cases} 0, & \text{if } x > y \\ 1, & \text{if } x \leq y \end{cases} = I_{RS}(x, y), \end{aligned}$$

where  $I_{RS}$  is the Rescher's fuzzy implication (see [1] Table 1.3). Also,  $I_{RS}$  satisfies (15), (17) and does not satisfy (14) and (16) (see [1] Table 1.4).

On the other hand, if we are looking for a greater fuzzy implication that satisfies (15) and does not satisfy (14), this could be  $I_{LK}^{S_{LK}}$ , where  $S_{LK}(x, y) = \min\{x + y, 1\}$  the Lukasiewicz's t-conorm (see [1] Table 2.2). So, it is

$$\begin{aligned} I_{LK}^{S_{LK}}(x, y) &= S_{LK}(I_{LK}(x, y), I_{LK}(x, y)) \\ &= \min\{I_{LK}(x, y) + I_{LK}(x, y), 1\} \\ &= \min\{2I_{LK}(x, y), 1\} \\ &= \min\{2\min\{1, 1 - x + y\}, 1\} \\ &= \min\{\min\{2, 2 - 2x + 2y\}, 1\} \\ &= \min\{2 - 2x + 2y, 1\} \\ &= \begin{cases} 2 - 2x + 2y, & \text{if } x - y \geq 0.5 \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

Moreover,  $I_{LK}^{S_{LK}}$  violates (17) and (16), since

$$I_{LK}^{S_{LK}}(1, 0.8) = 1 \text{ and } I_{LK}^{S_{LK}}(0.9, I_{LK}^{S_{LK}}(0.95, 0.1)) = 0.8 \neq 0.9 = I_{LK}^{S_{LK}}(0.95, I_{LK}^{S_{LK}}(0.9, 0.1)).$$

If we consider  $S_D(x, y) = \begin{cases} 1, & \text{if } x, y \in (0, 1] \\ \max\{x, y\}, & \text{otherwise} \end{cases}$  (see [1] Table 2.1), then the greatest fuzzy implication we can generate with this method is

$$\begin{aligned} I_{LK}^{S_D}(x, y) &= S_D(I_{LK}(x, y), I_{LK}(x, y)) \\ &= \begin{cases} 1, & \text{if } I_{LK}(x, y) \in (0, 1] \\ \max\{I_{LK}(x, y), I_{LK}(x, y)\}, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } x - y < 1 \\ I_{LK}(x, y), & \text{if } x - y = 1 \end{cases} \\ &= \begin{cases} 0, & \text{if } x = 1 \text{ and } y = 0 \\ 1, & \text{otherwise} \end{cases} = I_1(x, y), \end{aligned}$$

where  $I_1$  is the greatest fuzzy implication (see [1] Proposition 1.1.7 and [13]).  $I_1$  obviously satisfies (15) and does not satisfy (14). Moreover, it does not satisfy (17) since  $I_1(0.9, 0.8) = 1$ , and it is easy to prove that satisfies (16).

Because of the previous example, we have to notice that since we do not use positive t-conorms, (17) must be checked by the result every time and we cannot predict it a priori. Nevertheless, this check is an easy process.

#### 4. Conclusions

We believe that the above production machine of fuzzy implications will play a crucial role in many areas, theoretical and applied ones. For instance, we refer to the theoretical topic of subethood measures and applied topics such as artificial intelligence and pattern recognition (see [8,14,15]).

Moreover, many properties of fuzzy implications are proposed by the literature (see [1,3,4]). All these properties and many of the construction methods of fuzzy implications are generalizations from classical to fuzzy topic. We could claim in a point of view that we think classically and we apply fuzzy methods. A classical thinking of a part of our study in this paper may be the classical tautologies

$$p \equiv p \wedge p \text{ and } p \equiv p \vee p. \quad (22)$$

The real question we asked ourselves to begin this research was whether or not some properties of fuzzy implications are desirable. Moreover, we asked—how do we totally control them? For instance, as we mentioned in the Introduction in [8], (17) was desirable in the construction of the fuzzy implication  $I_Z$  (see [8] Equation (6)). The truth is that there are a lot of fuzzy implications that satisfy (17), but are they enough? What if we want only one or some of (14), (15) and (17)?

All the previous thoughts lead us to introduce the aforementioned method of generating fuzzy implications via known fuzzy implications and a function  $f$ , which has some properties as they are mentioned in Theorem 1. The properties of fuzzy implications that are preserved via this method were also presented.

Moreover, the special case of generating fuzzy implications via two known fuzzy implications and a fuzzy connective, such as a t-norm or a t-conorm, was studied. This method is very important, since it gives us a tool to generate not greater or not weaker fuzzy implications than the preliminaries we use. Another advantage is that we can control many properties of these induced implications, such as (14), (15) and (17).

As it is proven in Corollary 4, (18)–(20) with respect to  $N$  and (15) are preserved by this production of fuzzy implications. The same happens for (17), when the applying fuzzy connective is any t-norm or positive t-conorm, according to Propositions 4 and 6. Moreover, the same happens for (14) only when we use  $T_M$  or  $S_M$ , according to Proposition 8. On the other hand, (16) is not generally preserved by this method, at least when we use a fuzzy connective of the Definition 4 (see Examples 1 and 3; and in [1] Tables 2.1 and 2.2, and Remark 6.1.5).

We have to note that these are important results, but Propositions 5–8 are also very important. All these Propositions in other words give us the following statements:

- If we want a fuzzy implication that satisfies the (left, right) contrapositive symmetry (18)–(20) with respect to  $N$  we construct it by two fuzzy implications  $I_{(1)}, I_{(2)}$  that respectively satisfy (18)–(20) with respect to  $N$  and any t-norm or t-conorm.
- If we want a fuzzy implication that satisfies (15), its construction is completed by two fuzzy implications  $I_{(1)}, I_{(2)}$  that satisfy (15) and any t-norm or t-conorm. On the other hand, if we want to construct a fuzzy implication that violates (15), we can construct it by two ways. The first way is to consider  $I_{T, I_{(1)}, I_{(2)}}$ , where  $T$  is any t-norm and at least one of  $I_{(1)}, I_{(2)}$  violate (15), according to Proposition 4. The second way is to consider  $I_{T, I_{(1)}, I_{(2)}}$ , where  $S$  is any positive

t-conorm and the choice of the fuzzy implications  $I_{(1)}, I_{(2)}$  is made, such that there exists at least one  $x \in (0, 1)$ , such that

$$I_{(1)}(x, x) \neq 1 \text{ and } I_{(2)}(x, x) \neq 1,$$

according to Proposition 5.

- If we want a fuzzy implication that satisfies (17), its construction is achieved similar to the previous ones, if we consider two fuzzy implications  $I_{(1)}, I_{(2)}$  that satisfy (17) and any t-norm or any positive t-conorm. On the other hand, the construction of a fuzzy implication that does not satisfy (17) is achieved by two ways. The first way is to consider  $I_{T, I_{(1)}, I_{(2)}}$ , where  $T$  is any t-norm and at least one of  $I_{(1)}, I_{(2)}$  violate (17), according to Proposition 4. The second way is to consider  $I_{S, I_{(1)}, I_{(2)}}$ , where  $S$  is any positive t-conorm and the choice of the fuzzy implications  $I_{(1)}, I_{(2)}$  will be done, such that there exist at least one  $(x, y) \in [0, 1]^2$ , such

$$x \leq y \Leftrightarrow I_{(1)}(x, y) \neq 1 \text{ and } I_{(2)}(x, y) \neq 1,$$

according to Proposition 7.

- If we want a fuzzy implication that satisfies (14) its construction is given in Proposition 8. On the other hand, the construction of a fuzzy implication that does not satisfy (14) is given by the same Proposition. This construction is achieved by using two implications  $I_{(i)}, i = 1, 2$  that satisfy (14) and any t-norm or t-conorm, except  $T_M$  and  $S_M$ .

Another characteristic of this method is that if we use a t-norm, we achieve the construction of a not greater fuzzy implication than the preliminaries. On the other hand, if we use a t-conorm we achieve the construction of a not weaker fuzzy implication than the preliminaries.

Finally, the simpler case we use one preliminary fuzzy implication, and a fuzzy connective is studied too. This case lead us to two subclasses of fuzzy implications the so called T and S subclasses, or the not greater and not weaker subclasses respectively, of a fuzzy implication. The findings of this case are presented in detail in Section 3.3.

This theoretical approach gives us many advantages, since a priori we can construct fuzzy implications from known ones that violate or preserve any property of (14), (15) and (17) we want. At this point we must note that  $(17) \Rightarrow (15)$ . In other words, a fuzzy implication that satisfies (17) and violates (15) is impossible by definition to be constructed. Moreover, another characteristic of this approach is that if the preliminary fuzzy implications satisfy (14), we have a generator that violates it, except in the case we use  $T_M$  or  $S_M$ .

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## References

1. Baczyński, M.; Jayaram, B. *Fuzzy Implications*; Springer: Berlin/Heidelberg, Germany, 2008.
2. Baczyński, M.; Drewniak, J. Conjugacy Classes of Fuzzy Implications. In *Lecture Notes in Computer Science*; Springer: Basel, Switzerland, 2019; Volume 1625, pp. 287–298.34. [CrossRef]
3. Massanet, S.; Torrens, J. An Overview of Construction Methods of Fuzzy Implications. In *Advances in Fuzzy Implication Functions*; Springer: Berlin/Heidelberg, Germany, 2013; Volume 300, pp. 1–30.1. [CrossRef]

4. Shi, Y.; Van Gasse, B.; Kerre, E.E. Fuzzy Implications: Classification and a New Class. In *Advances in Fuzzy Implication Functions*; Springer: Berlin/Heidelberg, Germany, 2013; Volume 300, pp. 31–51.\_2. [[CrossRef](#)]
5. Drewniak, J.; Sobera, J. Compositions of Fuzzy Implications. In *Advances in Fuzzy Implication Functions*; Springer: Berlin/Heidelberg, Germany, 2013; Volume 300, pp. 155–176.\_7. [[CrossRef](#)]
6. Hliněná, D.; Kalina, M.; Král', P. Implication Functions Generated Using Functions of One Variable. In *Advances in Fuzzy Implication Functions*; Springer: Berlin/Heidelberg, Germany, 2013; Volume 300, pp. 125–153.\_6. [[CrossRef](#)]
7. Baczynski, M.; Jayaram, B.; Massanet, S.; Torrens, J. Fuzzy Implications: Past, Present, and Future. In *Springer Handbook of Computational Intelligence*; Springer: Berlin/Heidelberg, Germany, 2015; pp. 183–202.\_12. [[CrossRef](#)]
8. Bogiatzis, A.; Papadopoulos, B. Global Image Thresholding Adaptive Neuro-Fuzzy Inference System Trained with Fuzzy Inclusion and Entropy Measures. *Symmetry* **2019**, *11*, 286. [[CrossRef](#)]
9. Fodor, J.C.; Roubens, M. *Fuzzy Preference Modelling and Multicriteria Decision Support*; Kluwer: Dordrecht, The Netherlands, 1994.
10. Gottwald, S. *A Treatise on Many-Valued Logics*; Research Studies Press: Baldock, UK, 2001.
11. Klement, E.P.; Mesiar, R.; Pap, E. *Triangular Norms*; Kluwer: Dordrecht, The Netherlands, 2000.
12. Kuczma, M. *Functional Equations in a Single Variable*; PWN—Polish Scientific Publishers: Warsaw, Poland, 1968.
13. Drewniak, J. Invariant Fuzzy Implications. *Soft Comput.* **2006**, *10*, 506–513. [[CrossRef](#)]
14. Bogiatzis, A.C.; Papadopoulos, B.K. Producing fuzzy inclusion and entropy measures and their application on global image thresholding. *Evol. Syst.* **2018**, *9*, 331–353. [[CrossRef](#)]
15. Bogiatzis, A.C.; Papadopoulos, B.K. Local thresholding of degraded or unevenly illuminated documents using fuzzy inclusion and entropy measures. *Evol. Syst.* **2019**, *10*, 593–619. [[CrossRef](#)]



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