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A New Version of Schauder and Petryshyn Type Fixed Point Theorems in S-Modular Function Spaces

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Received: 2 December 2019; Accepted: 10 December 2019; Published: 19 December 2019

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Abstract: In this paper, using the conditions of Taleb-Hanebaly's theorem in a modular space where the modular is s-convex and symmetric with respect to the ordinate axis, we prove a new generalized modular version of the Schauder and Petryshyn fixed point theorems for nonexpansive mappings in s-convex sets. Our results can be applied to a nonlinear integral equation in Musielak-Orlicz space L^p where $0 < p \leq 1$ and $0 < s \leq p$.

Keywords: s-convex set; s-modular function space; fixed point; Musielak-Orlicz space

MSC: 47H10; 54H25

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1. Introduction

In 1950, Nakano [1] initiated the concept of modular spaces which are natural generalizations of L^p spaces where $p > 0$. Then Musielak and Orlicz [2] refined and generalized these spaces in 1959. This idea has been studied for almost sixty years and there is a large set of known applications of them in various parts of analysis.

The monographic exposition of the theory of Orlicz spaces may be found in the book of Krasnoselskii and Rutickii [3]. For a current review of the theory of Musielak-Orlicz spaces and modular spaces, the reader is referred to the book of Kozlowski [4] and the most recent paper of Khamsi et al. [5], also see [6,7].

As a generalization of the Banach contraction principle, Taleb and Hanebaly [7] presented a fixed point theorem of the Banach type in a modular space where the modular is s-convex, having the Fatou property and satisfying the Δ_2-condition as follows.

Theorem 1 ([7]). *Let X_ρ be a ρ-complete modular space. Assume that ρ is an s-convex modular satisfying the Δ_2-condition and having the Fatou property. Let B be a ρ-closed subset of X_ρ and $T : B \to B$ a mapping such that:*

$$\exists c, k \in \mathbb{R}^+ : c > \max\{1, k\},\ \rho(c(Tx - Ty)) \leq k^s \rho(x - y)\ \forall x, y \in B. \tag{1}$$

Then T has a fixed point.

In this paper, by means of [7], we prove the existence of fixed points for a general class of contractive mappings satisfying Schauder and Petryshyn conditions in s-modular function spaces. We give an application of our result to a nonlinear integral equation in Musielak-Orlicz spaces.

2. Preliminaries

We begin by recalling some definitions. Let X be a linear space over $\mathbb{C}$. Then we have the following.

(1) A function $\rho : X \to [0, +\infty]$ is said to be modular if

(a) $\rho(x) = 0$ if and only if $x = 0$;

(b) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;

(c) for all $x, y \in X$, $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ for any $\alpha, \beta \geq 0$;

(2) If (c) is replaced by

(c') $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$ if $\alpha^s + \beta^s = 1$ for any $\alpha, \beta \geq 0$,
where, if $0 \leq s < 1$, then we say that ρ is an s-convex modular and if $s = 1$, then ρ is convex modular;

(3) A modular ρ defines a corresponding modular space, i.e., the vector space X_ρ given by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \; as \; \lambda \to 0\}.$$

(4) The modular space X_ρ can be equipped with the $F-$norm defined by $|x|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq \alpha\}$. If ρ is convex, then the functional

$$\|x\|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq 1\}$$

is a norm called the Luxemburg norm in X_ρ which is equivalent to the F-norm $|.|_\rho$.

Note that, by taking $\alpha = -1$ in 1(b), it follows that $y = \rho(x) = \rho(-x)$, so that a modular is symmetric with respect to the y-axis meaning that its graph remains unchanged under reflection about the y-axis. It turns out that an s-convex modular keeps the same property.

Definition 1. *Let X_ρ be a modular space.*

(a) A sequence $\{x_n\}$ in X_ρ is said to be:

$$(i) \; \rho - convergent \; to \; x, \; denoted \; by \; x_n \overset{\rho}{\to} x, \; if \; \rho(x_n - x) \to 0 \; as \; n \to \infty.$$
$$(ii) \; \rho - Cauchy \; if \; \rho(x_n - x_m) \to 0 \; as \; n, m \to \infty.$$

(b) X_ρ is $\rho-$complete if every $\rho-$Cauchy sequence is $\rho-$convergent.

(c) A subset $B \subseteq X_\rho$ is said to be $\rho-$closed if for any sequence $\{x_n\} \subset B$ with $x_n \overset{\rho}{\to} x$, $x \in B$. Also, B is ρ-open if B^c is ρ-closed.

(d) We say that $\partial_\rho(B)$ is the bound of a subset B of X_ρ, whenever

$$\partial_\rho(B) = \overline{B}^\rho - int(B)$$

where $\overline{B}^\rho$ is the closure of B and $int(B)$ is the interior of B in the sense of ρ.

(e) A subset $B \subseteq X_\rho$ is said to be $\rho-$compact if every sequence in B has a convergent subsequence.

(f) A subset $B \subseteq X_\rho$ is called $\rho-$bounded if

$$\delta_\rho(B) = \sup\{\rho(x - y) : \; x, y \in B\} < \infty,$$

where $\delta_\rho(B)$ is called the $\rho-$diameter of B.

(g) ρ is said to satisfy the Δ_2-condition if $2x_n \overset{\rho}{\to} 0$ whenever $x_n \overset{\rho}{\to} 0$.

(h) We say that ρ has the Fatou property if $\rho(x-y) \leq \liminf \rho(x_n - y_n)$ whenever, $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$.

3. Main Results

Now, we start our work with the following definitions.

Definition 2. *Let X_ρ be a modular space and $C \subseteq X_\rho$. A mapping $T : C \to X_\rho$ is said to be ρ-nonexpansive if $\rho(Tx - Ty) \leq \rho(x-y)$ for all $x, y \in C$.*

Definition 3. *A set C of a modular space X is said to be s-convex, where $0 < s \leq 1$ if the following condition is satisfied*

$$\alpha^{\frac{1}{s}} x + \beta^{\frac{1}{s}} y \in C \ \text{ whenever } \ x, y \in C \ , \quad \alpha + \beta = 1.$$

We first prove a Schauder type fixed point theorem when the mapping T is ρ-nonexpansive.

Theorem 2. *Let ρ be an s-convex modular that satisfies the Δ_2-condition and Fatou property, X_ρ be a ρ-complete modular space and B be a nonempty, s-convex, and ρ-closed subset of X_ρ. Assume that $T : B \to B$ is a ρ-nonexpansive operator and $T(B)$ is a subset of ρ-compact set of B. Then T has a fixed point.*

Proof. For every $n \in \mathbb{N}$, define $T_n = {t_n}^{\frac{1}{s}} T$, where $\{t_n\} \subseteq (0,1)$, $t_n \to 1$ as $n \to \infty$. If $s = 1$, then B is convex set. Without loss of generality, we assume that $0 \in B$. If $s < 1$, then $0 \in B$. Thus for each $n \in \mathbb{N}$, $T_n : B \to B$. There are two cases:

Case-1: Let $s = 1$. By putting $c = {t_n}^{\frac{-1}{2}}$ and $k = {t_n}^{\frac{1}{2}}$, we have

$$\begin{aligned}
\rho(c(T_n x - T_n y)) &= \rho({t_n}^{\frac{-1}{2}}(t_n(Tx - Ty))) \\
&= \rho({t_n}^{\frac{1}{2}}(Tx - Ty)) \\
&\leq {t_n}^{\frac{1}{2}} \rho(Tx - Ty) \\
&\leq {t_n}^{\frac{1}{2}} \rho(x - y) \\
&= k\rho(x - y),
\end{aligned}$$

for all $x, y \in X_\rho$.

Case-2: Let $0 < s < 1$. Set $c = t_n^{-1}$ and $k = {t_n}^{\frac{1-s}{s}}$, we obtain

$$\begin{aligned}
\rho(c(T_n x - T_n y)) &= \rho({t_n}^{-1}({t_n}^{\frac{1}{s}}(Tx - Ty))) \\
&= \rho({t_n}^{\frac{1-s}{s}}(Tx - Ty)) \\
&\leq {t_n}^{1-s} \rho(Tx - Ty) \\
&\leq {t_n}^{1-s} \rho(x - y) \\
&= k^s \rho(x - y),
\end{aligned}$$

for all $x, y \in X_\rho$.

Therefore, all of the assumptions of Theorem 1 hold. Thus for each $n \in \mathbb{N}$, T_n has a fixed point $x_n \in B$, that is, $x_n = T_n x_n = {t_n}^{\frac{1}{s}} T x_n$.

Since $T(B)$ lies in a ρ-compact subset of B, we assume without loss of generally that there exists $u \in B$ such that $\rho(Tx_n - u) \to 0$ as $n \to \infty$. Δ_2-condition follows that $\rho(2^{\frac{1}{s}}(Tx_n - u)) \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}\rho(x_n - Tx_n) &= \rho({t_n}^{\frac{1}{s}}Tx_n - Tx_n)\\ &= \rho((1-{t_n}^{\frac{1}{s}})Tx_n) \leq (1-{t_n}^{\frac{1}{s}})^s \rho(Tx_n) \to 0 \; as \; n \to \infty.\end{aligned}$$

Again Δ_2- condition implies that $\rho(2^{\frac{1}{s}}(x_n - Tx_n)) \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned}\rho(x_n - u) &= \rho(x_n - Tx_n + Tx_n - u)\\ &\leq \frac{1}{2}\rho(2^{\frac{1}{s}}(x_n - Tx_n)) + \frac{1}{2}\rho(2^{\frac{1}{s}}(Tx_n - u)) \to 0 \; as \; n \to \infty.\end{aligned}$$

Since T is ρ-nonexpansive,

$$\rho(Tx_n - Tu) \leq \rho(x_n - u) \;\; as \;\; n \to \infty.$$

Therefore,

$$\rho(u - Tu) \leq \frac{1}{2}\rho(2^{\frac{1}{s}}(u - Tx_n)) + \frac{1}{2}\rho(2^{\frac{1}{s}}(Tu - Tx_n)) \to 0 \;\; as \;\; n \to \infty.$$

This implies that $u = Tu$. □

Theorem 3. *Let ρ be an s-convex modular that satisfies the Δ_2-condition and Fatou property, X_ρ be a ρ-complete modular space and B be a nonempty, s-convex, and ρ-closed subset of X_ρ. Assume that $T : B \to B$ is a ρ-nonexpansive and I is an identity operator, and $(I - T)(B)$ is ρ-closed. Then T has a fixed point.*

Proof. Proceeding as in the proof of Theorem 2, one can prove that for each $n \in \mathbb{N}$, T_n has a fixed point x_n. Thus

$$\begin{aligned}\rho(x_n - Tx_n) &= \rho({t_n}^{\frac{1}{s}}Tx_n - Tx_n)\\ &= \rho((1-{t_n}^{\frac{1}{s}})Tx_n)\\ &\leq (1-{t_n}^{\frac{1}{s}})^s \rho(Tx_n) \to 0 \;\; as \;\; n \to \infty.\end{aligned}$$

The closedness of $(I - T)(B)$ implies that $0 \in (I - T)(B)$. Therefore, there exists $u \in B$ such that $Tu = u$. □

The following theorem is a new version of the Petryshyn theorem in s-modular function spaces.

Theorem 4. *Let X_ρ be a complete modular space and B a ρ-bounded, ρ-open, s-convex subset of X_ρ with $0 \in B$. Assume that ρ is an s-convex modular satisfying the Δ_2-condition and Fatou property, $T : \overline{B}^\rho \to X_\rho$ is a mapping satisfying (1) and the following condition:*

$$x \neq \lambda Tx \;, \qquad \forall x \in \partial_\rho(B), \; \lambda \in (0,1). \tag{2}$$

Then T has a fixed point.

Proof. Consider $A := \{\lambda \in [0,1] : x = \lambda Tx$ for some $x \in B\}$. Notice A is nonempty since $0 \in B$. We will show that A is both open and closed in $[0,1]$ and hence $A = [0,1]$.

Let $\alpha^{\frac{1}{s}}$ be the s-conjugate of c, i.e., $\frac{1}{c^s} + \frac{1}{\alpha} = 1$. We first show that A is closed. To see this let $\{\lambda_n\} \subseteq A$ with $\lambda_n \to \lambda$ as $n \to \infty$. There exists $\{x_n\} \subseteq B$ with $x_n = \lambda_n Tx_n$. Since $0 \in B$ and B is ρ-bounded, there exists positive real number M_0 such that $\rho(x) \leq M_0$ for all $x \in B$. Condition (1) follows that for all $x \in B$, $\rho(Tx) \leq \rho(x) + \frac{1}{\alpha}\rho(\alpha^{\frac{1}{s}}T0)$. By putting $M = M_0 + \frac{1}{\alpha}\rho(\alpha^{\frac{1}{s}}T0)$, we see that M is an upper bound for the set $\{\rho(Tx); \;\; x \in B\}$. For any $n, m \in \mathbb{N}$, we have

$$\begin{aligned}\rho(x_n - x_m) &= \rho(\lambda_n Tx_n - \lambda_m Tx_m) \\ &= \rho\left(\frac{c\lambda_n(Tx_n - Tx_m)}{c} + \frac{\alpha^{\frac{1}{s}}(\lambda_n - \lambda_m)Tx_m}{\alpha^{\frac{1}{s}}}\right) \\ &\le \frac{1}{c^s}\rho(c\lambda_n(Tx_n - Tx_m)) + \frac{1}{\alpha}\rho(\alpha^{\frac{1}{s}}(\lambda_n - \lambda_m)Tx_m).\end{aligned}$$

For large enough numbers n, m, we have $\alpha^{\frac{1}{s}}|\lambda_n - \lambda_m| < 1$ and so

$$\rho(x_n - x_m) \le (\frac{k}{c})^s \rho(x_n - x_m) + |\lambda_n - \lambda_m|^s\, M.$$

Hence,

$$\rho(x_n - x_m) \le \frac{M}{\alpha(1 - (\frac{k}{c})^s)}\ |\lambda_n - \lambda_m|^s \to 0 \ as\ n, m \to \infty.$$

Since X_ρ is complete, we deduce that there exists $x \in \overline{B}^\rho$ with $\rho(x_n - x) \to 0$ as $n \to \infty$. It follows from Δ_2-condition and $\lambda_n \to \lambda$ that

$$\begin{aligned}\rho(x - \lambda Tx) &= \rho(x - x_n + x_n - \lambda Tx) \\ &= \rho\Big(\frac{c\lambda_n(Tx_n - Tx)}{c} + \frac{\alpha^{\frac{1}{s}}(\lambda_n - \lambda)Tx}{\alpha^{\frac{1}{s}}} + \frac{\alpha^{\frac{1}{s}}(x - x_n)}{\alpha^{\frac{1}{s}}}\Big) \\ &\le (\frac{k}{c})^s\rho(x_n - x) + |\lambda_n - \lambda|^s\rho(Tx) + \frac{1}{2\alpha}\rho((2\alpha)^{\frac{1}{s}}(x_n - x)) \to 0 \ as\ n \to \infty.\end{aligned}$$

Thus $\lambda \in A$ and A is closed in $[0, 1]$.

Now, let $\lambda_0 \in A$. Then there exists $x_0 \in B$ with $x_0 = \lambda_0 Tx_0$. Choose $\epsilon > 0$ such that

$$\epsilon^s \le \min\{\alpha^{-1}, \frac{1}{M}(1 - (\frac{k}{c})^s)r\},$$

where $r = \inf\{\rho(x - x_0);\quad x \in \partial_\rho(B)\}$. If $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, then for $x \in \overline{B(x_0, r)}^\rho$ we have

$$\begin{aligned}\rho(x_0 - \lambda Tx) &= \rho\Big(\frac{c\lambda(Tx_0 - Tx)}{c} + \frac{\alpha^{\frac{1}{s}}(\lambda_0 - \lambda)Tx_0}{\alpha^{\frac{1}{s}}}\Big) \\ &\le (\frac{k}{c})^s\rho(x_0 - x) + |\lambda_0 - \lambda|^s\rho(Tx_0) \\ &\le (\frac{k}{c})^s r + \epsilon^s M \\ &\le (\frac{k}{c})^s r + (1 - (\frac{k}{c})^s)r = r.\end{aligned}$$

Therefore $\lambda T : \overline{B(x_0, r)}^\rho \to \overline{B(x_0, r)}^\rho$. It is easy to show that λT satisfies the condition (1) and by applying Theorem 1, we can deduce that λT has a fixed point. Thus there exists $x \in B$ for which $x = \lambda Tx$. This shows that $\lambda \in A$ and hence A is open in $[0, 1]$. □

4. Application

In this section, we give an application of Theorem 4 to the following integral equation:

$$u(t) = u_0 + \int_0^t G(t, r) f(r, u(r))dr; \quad t \in I = [0, 1], \tag{3}$$

in a modular space $C = C([0,1], L^p)$, where $0 < p \leq 1, 0 < s \leq p$ and

$$\begin{aligned} L^p &= L^p([0,1]) \\ &= \{f : \ f : [0,1] \to \mathbb{R} \ \text{ is measurable and } \rho(f) = \int_0^1 |f(t)|^p dt < \infty\}, \end{aligned}$$

$u_0 \in B$ and B is an s-convex, ρ-closed, ρ-bounded subset of L^p with $0 \in B$.

Notice that the s-convexity of ρ implies the following lemma.

Lemma 1 ([8]). *Let $0 < p \leq 1$, $a \geq 0$, $b \geq 0$, then $(a+b)^p \leq a^p + b^p$ and $|a^p - b^p| \leq |a-b|^p$.*

We denote by $X = C(I, B)$ the space of all ρ-continuous functions from I to B, endowed with the modular ρ_X defined by $\rho_X(u) = \sup_{t \in I} \rho(u(t))$. Using Proposition 2.1 of [7], one can show that X is s-convex, ρ_X-bounded, ρ_X-closed of ρ_X-complete space $C = C(I, L^p)$ and ρ_X satisfies the Δ_2-condition and Fatou property.

Consider the following assumptions:

(i) $f : I \times B \to B$ is ρ-continuous and satisfies

$$\exists \gamma > 1 \quad \rho(f(t,u) - f(t,v)) \leq \gamma \rho(u - v)\,; \quad \forall t \in I \text{ and } u, v \in B.$$

(ii) $G : I \times I \to \mathbb{R}$ is a measurable mapping such that the map $r \mapsto G(t,r)$ is continuous for almost all $t \in I$, and also $\int_0^1 |G(t,r)|dr < 1$ for all $t \in I$.

Theorem 5. *Under the conditions (i) and (ii), if for some positive number $\lambda > 1$ we have $\lambda \int_0^1 |G(t,r)|dr < \gamma^{\frac{-1}{s}}$, then a mapping F defined on X as*

$$Fu(t) = u_0 + \int_0^t G(t,r) f(r, u(r))\, dr$$

is a self-adjoint operator which satisfies (1).

Proof. First we show that $F : X \to X$ is a self-adjoint operator. Suppose $t_n, t_0 \in [0,1]$ and $t_n \to t_0$ as $n \to \infty$. Since u is ρ-continuous in t_0 and condition (i) holds, by Δ_2-condition f is $|.|_\rho$-continuous where $|.|_\rho$ is the F-norm generated by modular ρ. Hence, Fu is $|.|_\rho$-continuous. On the other hand, the topologies generated by $|.|_\rho$ and ρ are equivalent, therefore, Fu is ρ-continuous at t_0.

Fix $t \in [0,1]$. Let $\lambda > 0$ and $T = \{t_0, t_1, \ldots, t_n\}$ be any subdivision of $[0,t]$. It can be seen that $\sum_{i=1}^{n-1} \lambda(t_{i+1} - t_i) G(t, t_i) u(t_i)$ is ρ-convergent to $\int_0^1 \lambda G(t,r) u(r) dr$ when

$$|T| = \sup\{|t_{i+1} - t_i|;\ \ 0 \leq i \leq n-1\} \to 0 \ \text{ as } \ n \to \infty.$$

By the Fatou property, condition (ii) and s-convexity of ρ, we have

$$\begin{aligned} \rho\Big(\int_0^t \lambda G(t,r) u(r) dr \Big) &\leq \liminf \rho\Big(\sum_{i=1}^{n-1} \lambda(t_{i+1} - t_i) G(t, t_i) u(t_i) \Big) \\ &\leq \lambda^s \Big(\int_0^1 |G(t,r)| dr \Big)^s \rho(u(t_i)) \\ &\leq \lambda^s \Big(\int_0^1 |G(t,r)| dr \Big)^s \rho_X(u). \end{aligned}$$

This implies that

$$\begin{aligned}\rho(\lambda(Fu-Fv)(t)) &= \rho\Big(\int_0^t \lambda G(t,r)\Big(f(r,u(r))-f(r,v(r))\Big)\,dr\Big)\\ &\le \lambda^s\Big(\int_0^1 |G(t,r)|dr\Big)^s \rho\big(f(t,u(t))-f(t,v(t))\big)\\ &\le \lambda^s\Big(\int_0^1 |G(t,r)|dr\Big)^s \gamma\,\rho(u(t)-v(t))\\ &= \lambda^s\Big(\int_0^1 |G(t,r)|dr\Big)^s \gamma\,\rho_X(u-v).\end{aligned}$$

Therefore,

$$\rho_X(\lambda(Fu-Fv)) \le \lambda^s\Big(\int_0^1 |G(t,r)|dr\Big)^s \gamma\,\rho_X(u-v).$$

By putting $c=\lambda$ and $k=\lambda\Big(\int_0^1 |G(t,r)|dr\Big)\gamma^{\frac{1}{s}}$, the operator F satisfies the condition (1). □

Theorem 6. *Under the conditions (i) and (ii), suppose there exists ρ_X-bounded open $U\subseteq X$ with $0\in U$ such that if u solves the integral equation*

$$u(t)=ku_0+\int_0^t kG(t,r)f(r,u(r))\,dr \tag{4}$$

for some $k\in(0,1)$, then $u\notin \partial_{\rho_X}(U)$. Then (3) has a unique solution in $\overline{U}^{\rho_X}$.

Proof. Evidently $F:\overline{U}^{\rho_X}\to X$ satisfies in (1). If we apply Theorem 4 and consider the fact that condition (2) occurs because of (4), we get the required result. □

Author Contributions: M.R. contributed in conceptualization, investigation, methodology, validation and writing the original draft; H.B. contributed in conceptualization, investigation, methodology, validation and writing the original draft; O.E. contributed in conceptualization, investigation, methodology, validation and writing the original draft; M.D.l.S. contributed in funding acquisition, methodology, project administration, supervision, validation, visualization and editing. All authors agree and approve the final version of this manuscript.

Funding: The authors thank the Basque Government for its support of this work through Grant IT1207-19.

Acknowledgments: The authors thank the Spanish Government and the European Fund of Regional Development FEDER for Grant RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE) and the Basque Government for Grant IT1207-19. We would like to express our gratitude to the anonymous referees for their helpful suggestions and corrections.

Conflicts of Interest: The authors declare that they have no competing interests concerning the publication of this article.

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