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Fixed-Points of Interpolative Ćirić-Reich–Rus-Type Contractions in *b*-Metric Spaces

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Abstract: The concept of symmetry is inherent in the study of metric spaces due to the presence of the symmetric property of the metric. Significant results, such as with the Borsuk–Ulam theorem, make use of fixed-point arguments in their proofs to deal with certain symmetry principles. As such, the study of fixed-point results in metric spaces is highly correlated with the symmetry concept. In the current paper, we first define a new and modified Ćirić-Reich–Rus-type contraction in a b-metric space by incorporating the constant s in its definition and establish the corresponding fixed-point result. Next, we adopt an interpolative approach to establish some more fixed-point theorems. Existence of fixed points for ω-interpolative Ćirić-Reich–Rus-type contractions are investigated in this context. We also illustrate the validity of our results with some examples.

Keywords: fixed-point; b-metric space; interpolative contraction; Ćirić-Reich–Rus-type contractions

1. Introduction

In this introductory section, we present the literature review in the current context and motivate the present study.

The Banach Contraction principle [1] found its applications in several branches of mathematics, including other branches such as physics, chemistry, economics, computer science, and biology. As a result, investigation and generalization of this result turned out to be a prime area of research in nonlinear analysis. In this connection, we refer to the works of Geraghty [2], Rhoades [3], Altun et al. [4], Suzuki [5], Kadelburg and Radenović [6], and Chaipunya et al. [7].

It is a well-known fact that a map which satisfies the Banach Contraction principle is necessarily continuous. Thus, it was natural to ask the question whether in a complete metric space, a discontinuous map satisfying somewhat similar contractive conditions can posses a fixed point. Kannan [8] gave an affirmative answer to this question by introducing a new type of contraction.

Karapinar [9] defined the generalized Kannan-type contraction by adopting the interpolative approach in the following manner, and proved that such an interpolative Kannan-type contraction owns a fixed point in a complete metric space. Some more interesting results in this direction may be found in the work of Karapinar et al. [10,11].

Definition 1. *[9] Let M be a non-empty set and δ be a metric defined on it so that the metric space (M, δ) is complete. Also, let $I : M \to M$ be a self-mapping. Then, I is called an interpolative Kannan-type contraction if there exist constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, such that*

$$\delta(I\mu, I\nu) \leq \lambda[\delta(\mu, I\mu)]^{\alpha}.[\delta(\nu, I\nu)]^{1-\alpha},$$

for all $\mu, \nu \in M$ satisfying $\mu \neq I\mu$ and $\nu \neq I\nu$.

Other important variants of the Banach Contraction principle were independently studied by Ćirić-Reich and Rus [12–14]. A combined result due to them is given below, which is known as the Ćirić-Reich–Rus theorem:

Theorem 1. *If a self-map I, defined on a complete metric space (M, δ) satisfies the inequality:*

$$\delta(I\mu, I\nu) \leq \lambda[\delta(\mu, \nu) + \delta(\mu, I\mu) + \delta(\nu, I\nu)],$$

for all $\mu, \nu \in M$ and for $\lambda \in [0, \frac{1}{3})$, then I has a unique fixed point.

Recently, some important work has been carried out in this direction by Aydi et al. [15,16], Karapinar et al. [17], and Debnath et al. [18].

A class of non-decreasing, positive real functions denoted by Ξ has been used by Rus [14] in this connection.

Definition 2. *Let Ξ be denoted as the set of all non-decreasing functions $\xi : [0, \infty) \to [0, \infty)$, such that $\sum_{n=0}^{\infty} \xi^n(t) < \infty$ for each $t > 0$. Then, there are two important properties, $\xi(0) = 0$ and $\xi(t) < t$ for each $t > 0$.*

The concept of α-admissible maps was put forward by Samet et al. [19]. Popescu [20] modified it and introduced the concept of ω-orbital admissible maps to study fixed points of α-Geraghty contractive maps.

Definition 3. *Let $M \neq \phi$ and $\omega : M \times M \to [0, \infty)$ be a non-negative function. A self-map $I : M \to M$ is called ω-orbital admissible if, for all $\mu \in M$, we have $\omega(I\mu, \mu) \geq 1$ which implies $\omega(I^2\mu, I\mu) \geq 1$.*

The following condition is used (see [16]) when the continuity assumption of the underlying contractive map is to be avoided.

Definition 4. *The function $\omega : M \times M \to [0, \infty)$ owns the Condition (A) if for any convergent sequence $\{\mu_n\}$ in M, such that $\omega(\mu_n, \mu_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $\mu_n \to \mu \in M$ as $n \to \infty$, there exists a subsequence $\{\mu_{n(k)}\}$ of $\{\mu_n\}$, such that $\omega(\mu_{n(k)}, \mu) \geq 1$ for each $k \in \mathbb{N}$.*

Bakhtin [21] and Czerwik [22] independently defined the concept of a b-metric space.

Definition 5. *[21,22] Let M be a non-empty set, where the mapping $\delta : M \times M \to [0, \infty)$ satisfies the following:*
(1) $\delta(\mu, \nu) = 0$ if, and only if $\mu = \nu$;
(2) $\delta(\mu, \nu) = \delta(\nu, \mu)$ for all $\mu, \nu \in M$;
(3) There exists a real number $s \geq 1$, such that $\delta(\mu, \nu) \leq s[\delta(\mu, \gamma) + \delta(\gamma, \nu)]$ for all $\mu, \nu, \gamma \in M$.
Then, δ is called a b-metric on M and (M, δ) is called a b-metric space (in short bMS) with coefficient s.

Convergent sequences and Cauchy sequences in a bMS are defined exactly the same way as in the case of usual metric spaces.

For some recent significant developments in the area of bMS, we refer to the work of Kirk and Shahzad [23], Jleli et al. [24], Chifu and Petrusel [25], Debnath and de La Sen [26], and Hussain et al. [27].

In this paper, we present three results. Firstly, we define a new and modified Ćirić-Reich–Rus type contraction (in short, we call it the MCRR-type contraction) in a bMS by incorporating the constant s in its definition and discuss the corresponding fixed-point theorem. In our second result, an interpolative Ćirić-Reich–Rus type contraction (in short, CRR-type contraction) is defined and the existence of its fixed point is established assuming continuity of that self-map. In the third and final result, we show that continuity of the self-map may be dropped if it is replaced by a weaker condition.

2. MCRR-Type Contraction

One of our main results is presented in this section. Defined below is an MCRR-type contraction, followed by the corresponding theorem.

Definition 6. *Let (M, δ, s) be a bMS. A self-map $I : M \to M$ is called an interpolative MCRR-type contraction if there are constants $\lambda \in [0, 1)$ and $a, b \in (0, 1)$, such that*

$$\delta(I\mu, I\nu) \leq \lambda[\delta(\mu, \nu)]^b [\delta(\mu, I\mu)]^a [\frac{1}{s}(\delta(\nu, I\nu)]^{1-a-b} \tag{1}$$

for all $\mu, \nu \in M \setminus Fix(I)$.

Theorem 2. *Let (M, δ, s) be a complete bMS with continuous b-metric δ. If $I : M \to M$ is an interpolative MCRR-type contraction, then I has a fixed point in M.*

Proof. Let $\mu_0 \in M$ and define the iterative sequence $\{\mu_n\}$ by $\mu_n = I^n(\mu_0)$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\mu_{n_0} = \mu_{n_0+1}$, then μ_{n_0} is clearly a fixed point of I. Thus, assume that $\mu_n \neq \mu_{n+1}$ for all $n \geq 0$.

Substituting μ by μ_n and ν by μ_{n-1} in (1), we have

$$\begin{aligned}
\delta(\mu_{n+1}, \mu_n) &= \delta(I\mu_n, I\mu_{n-1}) \\
&\leq \lambda[\delta(\mu_n, \mu_{n-1})]^b [\delta(\mu_n, I\mu_n)]^a [\frac{1}{s} d(\mu_{n-1}, I\mu_{n-1})]^{1-a-b} \\
&= \lambda[\delta(\mu_n, \mu_{n-1})]^b [\delta(\mu_n, \mu_{n+1})]^a [\frac{1}{s}\delta(\mu_{n-1}, \mu_n)]^{1-a-b} \\
&\leq \lambda[\delta(\mu_n, \mu_{n-1})]^b [\delta(\mu_n, \mu_{n+1})]^a [\delta(\mu_{n-1}, \nu_n)]^{1-a-b} (\text{ for } s \geq 1) \\
&= \lambda[\delta(\mu_n, \mu_{n-1})]^{1-a} [\delta(\mu_n, \mu_{n+1})]^a.
\end{aligned} \tag{2}$$

From the above, we obtain

$$[\delta(\mu_n, \mu_{n+1})]^{1-a} \leq \lambda[\delta(\mu_n, \mu_{n-1})]^{1-a}, \tag{3}$$

which implies that

$$\delta(\mu_n, \mu_{n+1}) \leq \delta(\mu_n, \mu_{n-1}) \text{ for all } n \geq 0. \tag{4}$$

Combining (3) and (4), we have

$$\delta(\mu_n, \mu_{n+1}) < \lambda\delta(\mu_{n-1}, \mu_n) \text{ for all } n \geq 1. \tag{5}$$

However, we know from [28] that every sequence $\{\mu_n\}$ in a bMS satisfying the property (5) is Cauchy.

Hence, $\{\mu_n\}$ is a Cauchy sequence, and since (M, δ, s) is complete, we can obtain a $\theta \in M$ such that $\lim_{n\to\infty} \mu_n = \theta$.

Next, the fact that θ is a fixed point of I is proven. If possible, assume that $I\theta \neq \theta$, so that $\delta(I\theta, \theta) > 0$. Also, by hypothesis, $\mu_n \neq I\mu_n$ for all $n \geq 0$.

By substituting μ by μ_n and ν by θ in 1, we have

$$\begin{aligned} \delta(\mu_{n+1}, I\theta) &= \delta(I\mu_n, I\theta) \\ &\leq \lambda[\delta(\mu_n, \theta)]^b[\delta(\mu_n, I\mu_n)]^a[\frac{1}{s}d(\theta, I\theta)]^{1-a-b} \\ &= \lambda[\delta(\mu_n, \theta)]^b[\delta(\mu_n, \mu_{n+1})]^a[\frac{1}{s}\delta(\theta, I\theta)]^{1-a-b} \end{aligned} \tag{6}$$

Taking the limit as $n \to \infty$ in (6), since δ is continuous, we have $\delta(\theta, I\theta) = 0$, which contradicts our last hypothesis. Hence, $I\theta = \theta$. $\square$

Example 1. *Let $M = \{\mu, \nu, \gamma\}$ and $\delta : M \times M \to [0, \infty)$ be defined as $\delta(u, v) = 0$ if $u = v$, $\delta(u, v) = \delta(v, u)$ for all $u, v \in M$, $\delta(\mu, \nu) = 1$, $\delta(\mu, \gamma) = 2.2$, and $\delta(\nu, \gamma) = 1.1$. Then, it is easy to see that $(M, \delta, \frac{22}{21})$ is a complete bMS (but not a metric space).*

Define the self-map I on M by

u	μ	ν	γ
Iu	μ	μ	ν

Furthermore, we can see that

$$\delta(Iu, Iv) = \begin{cases} \delta(\mu, \mu) = 0, & \text{if } u \neq \gamma, v \neq \gamma \\ \delta(\nu, \mu) = 1, & \text{if } u = \gamma, v \neq \gamma \\ \delta(\mu, \nu) = 1, & \text{if } u \neq \gamma, v = \gamma \\ \delta(\nu, \nu) = 0, & \text{if } u = \gamma, v = \gamma. \end{cases}$$

Let $u, v \in M \setminus Fix(I)$. Clearly, the maximum value that $\delta(Iu, Iv)$ can attain is 1. Thus, the Inequality (1) together with all conditions of Theorem 2 are fulfilled if we choose $\lambda = 0.01, a = \frac{1}{2}, b = \frac{1}{3}$. Hence, I has a unique fixed point, μ.

3. CRR-Type Contraction

Here, we present two existence results for CRR-type contractions. First, we define an ω-interpolative CRR-type contraction.

Definition 7. *Let (M, δ, s) be a bMS. The self-map $I : M \to M$ is called an ω-interpolative CRR-type contraction map if there exists $\xi \in \Xi$, $\omega : M \times M \to [0, \infty)$ and $a, b > 0$ with $a + b < 1$, such that*

$$\omega(\mu, \nu)\delta(I\mu, I\nu) \leq \xi\Big([\delta(\mu, \nu)]^b[\delta(\mu, I\mu)]^a[\delta(\nu, I\nu)]^{1-a-b}\Big) \tag{7}$$

for all $\mu, \nu \in M \setminus Fix(I)$, where $Fix(I)$ denotes the set of fixed points of I.

Next, we prove an existence theorem for the aforementioned contraction where continuity of the self-map is assumed. It is to mention that the following theorem generalizes Theorem 1 due to Aydi et al. [16], which may be obtained by taking $s = 1$ in the definition of the concerned bMS. The first half of the proof adopts similar techniques as that of [16]. The similar portion of the proof is retained here verbatim due to clarity of presentation.

Theorem 3. *Let (M, δ, s) be a complete bMS and $I : M \to M$ be a continuous ω-orbital admissible and ω-interpolative CRR-type contraction. If there exists $\mu_0 \in M$ with $\omega(I\mu_0, \mu_0) \geq 1$, then $Fix(I) \neq \phi$.*

Proof. Let $\mu_0 \in M$ satisfying $\omega(I\mu_0, \mu_0) \geq 1$. Define the sequence $\{\mu_n\}$ by $\mu_n = I^n(\mu_0)$, for all $n \geq 0$. Without loss of generality, assume that $\mu_n \neq \mu_{n+1}$ for all $n \geq 0$. By the hypothesis, we can say that $\omega(\mu_1, \mu_0) \geq 1$ and also for I is w-orbital admissible, and we have

$$\omega(\mu_2, \mu_1) = w(I\mu_1, I\mu_0) \geq 1.$$

By repeating the above argument, we can assert that

$$\omega(\mu_{n+1}, \mu_n) \geq 1 \text{ for all } n \geq 0. \tag{8}$$

Replacing μ by μ_n and ν by μ_{n-1} in (7), we obtain

$$\begin{aligned}\delta(\mu_{n+1}, \mu_n) &\leq \omega(\mu_n, \mu_{n-1})\delta(I\mu_n, I\mu_{n-1}) \\ &\leq \xi([\delta(\mu_n, \mu_{n-1})]^b[\delta(\mu_n, I\mu_n)]^a[\delta(\mu_{n-1}, I\mu_{n-1})]^{1-a-b}) \\ &= \xi([\delta(\mu_n, \mu_{n-1})]^b[\delta(\mu_n, \mu_{n+1})]^a[\delta(\mu_{n-1}, \mu_n)]^{1-a-b}) \\ &= \xi([\delta(\mu_{n-1}, \mu_n)]^{1-a}[\delta(\mu_n, \mu_{n+1})]^a).\end{aligned} \tag{9}$$

Using the property $\xi(t) < t$ for each $t > 0$, we have

$$\begin{aligned}\delta(\mu_{n+1}, \mu_n) &\leq \xi\big([\delta(\mu_{n-1}, \mu_n)]^{1-a}[\delta(\mu_n, \mu_{n+1})]^a\big) \\ &< [\delta(\mu_{n-1}, \mu_n)]^{1-a}[\delta(\mu_n, \mu_{n+1})]^a.\end{aligned} \tag{10}$$

Further, we can derive that

$$[\delta(\mu_n, \mu_{n+1})]^{1-a} < [\delta(\mu_{n-1}, \mu_n)]^{1-a}. \tag{11}$$

This again implies

$$\delta(\mu_n, \mu_{n+1}) < \delta(\mu_{n-1}, \mu_n) \text{ for all } n \geq 1. \tag{12}$$

Thus, the sequence $\{\delta(\mu_{n-1}, \mu_n)\}$ of positive numbers is decreasing, and by the monotone convergence theorem there exists $l \geq 0$, such that $\lim_{n\to\infty} \delta(\mu_{n-1}, \mu_n) = l$.

Using (12), we have

$$\begin{aligned}[\delta(\mu_{n-1}, \mu_n)]^{1-a}[\delta(\mu_n, \mu_{n+1})]^a &\leq [\delta(\mu_{n-1}, \mu_n)]^{1-a}[\delta(\mu_{n-1}, \mu_n)]^a \\ &= \delta(\mu_{n-1}, \mu_n).\end{aligned} \tag{13}$$

Again, using (9) together with the non-decreasing property of ξ,

$$\delta(\mu_{n+1}, \mu_n) \leq \xi\big([\delta(\mu_{n-1}, \mu_n)]^{1-a}[\delta(\mu_n, \mu_{n+1})]^a\big) \leq \xi(\delta(\mu_{n-1}, \mu_n)).$$

By repeated application of the above argument, we obtain that

$$\delta(\mu_n, \mu_{n+1}) \leq \xi(\delta(\mu_{n-1}, \mu_n)) \leq \xi^2(\delta(\mu_{n-2}, \mu_{n-1})) \leq \ldots \leq \xi^n(\delta(\mu_0, \mu_1)). \tag{14}$$

Taking the limit in (14) as $n \to \infty$, and using the property that $\lim_{n\to\infty} \xi^n(t) = 0$ for each $t > 0$, we infer that $l = 0$.

Next, we show that $\{\mu_n\}$ is a Cauchy sequence in M by using the fact that "if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\delta(\mu_{n+k}, \mu_n) < \epsilon$ for each $k = 1, 2, 3, \ldots$, then $\{\mu_n\}$ is a Cauchy sequence".

From (14) and the triangle inequality of the bMS, for each $k \in \mathbb{N}$ we have that

$$\begin{aligned}\delta(\mu_n,\mu_{n+k}) &\le s\big[\delta(\mu_n,\mu_{n+1})+\delta(\mu_{n+1},\mu_{n+k})\big]\\ &\le s\big[\delta(\mu_n,\mu_{n+1})+s\big[\delta(\mu_{n+1},\mu_{n+2})+\delta(\mu_{n+2},\mu_{n+k})\big]\big]\\ &\vdots\\ &\le s\xi^n(\delta(\mu_0,\mu_1))+s^2\xi^{n+1}(\delta(\mu_0,\mu_1))+\cdots+s^k\xi^{n+k-1}(\delta(\mu_0,\mu_1))\end{aligned}$$

Using the fact that

$$1\le s\le s^2\le\cdots\le s^{k-1}\le s^k,$$

we obtain

$$\begin{aligned}s\xi^n(\delta(\mu_0,\mu_1))+s^2\xi^{n+1}(\delta(\mu_0,\mu_1))+\cdots+s^k\xi^{n+k-1}(\delta(\mu_0,\mu_1)) &\le s^k\sum_{i=n}^{n+k-1}\xi^i(\delta(\mu_0,\mu_1))\\ &\le s^k\sum_{i=n}^{\infty}\xi^i(\delta(\mu_0,\mu_1)).\end{aligned}$$

Thus, we have

$$\delta(\mu_n,\mu_{n+k})\le s^k\sum_{i=n}^{\infty}\xi^i(\delta(\mu_0,\mu_1)). \tag{15}$$

From Inequality (15), we can also have

$$\delta(\mu_{n+m},\mu_{n+m+k})\le s^k\sum_{i=m}^{\infty}\xi^i(\delta(\mu_n,\mu_{n+1})).$$

Now,

$$\begin{aligned}\lim_{m\to\infty,n\to\infty}\delta(\mu_{n+m},\mu_{n+m+k}) &\le s^k\lim_{m\to\infty}\sum_{i=m}^{\infty}\lim_{n\to\infty}\xi^i(\delta(\mu_n,\mu_{n+1}))\\ &=0.\end{aligned}$$

Therefore, $\lim_{n\to\infty}\delta(\mu_n,\mu_{n+k})=\lim_{m\to\infty,n\to\infty}\delta(\mu_{n+m},\mu_{n+m+k})=0$ for any finite integer $k\ge 1$ and, consequently, $\{\mu_n\}$ is a Cauchy sequence. Since the bMS (M,δ,s) is complete, we obtain $\mu\in M$ such that

$$\lim_{n\to\infty}\mu_n=\mu. \tag{16}$$

As I is continuous, we have $\mu=\lim_{n\to\infty}\mu_{n+1}=\lim_{n\to\infty}I\mu_n=I(\lim_{n\to\infty}\mu_n)=I\mu$. □

Next, we give an example for Theorem 3, where the self-map T is continuous.

Example 2. *Let $M=[0,1]$ and $\delta(\mu,\nu)=(\mu-\nu)^2$. Then, $(M,\delta,2)$ is a complete b-metric space.*
Define the self-map $I:M\to M$ by $I\mu=\frac{2}{3}\mu^2$.
Let

$$\omega(\mu,\nu)=\begin{cases}1, & \text{if } \mu,\nu\in(0,1]\\ 0, & \text{otherwise.}\end{cases}$$

Let $\mu,\nu\in M\setminus Fix(I)$ with $\omega(\mu,\nu)\ge 1$. Then, clearly, $\mu,\nu\in(0,1]$. Hence, (7) holds for $a=\frac{1}{3}$ and $\frac{1}{5}$. Also, for $\mu_0=1$ we have $\omega(1,I1)=\omega(1,\frac{2}{3})=1$.

Let $\gamma \in M$ such that $\omega(I\gamma, \gamma) \geq 1$. This implies $\gamma, I\gamma \in (0, 1]$. Thus, $\omega(I^2\gamma, I\gamma) \geq 1$. Hence, I is ω-orbital admissible.

Thus, all conditions of Theorem 3 are fulfilled, and we can see that 0 is the fixed point of I.

In our next result, we drop the continuity of the self-map I, but we assume that ω is satisfying the Condition (A). The following result may be considered as a variant of Theorem 2 due to Aydi et al. [16].

Theorem 4. *Let (M, δ, s) be a complete bMS and $I : M \to M$ be an ω-orbital admissible and ω-interpolative CRR-type contraction satisfying condition (A). If there exists $\mu_0 \in M$ such that $\omega(I\mu_0, \mu_0) \geq 1$, then $Fix(I) \neq \phi$.*

Proof. Invoking a similar procedure as in the proof of Theorem 3, we can assert that the condition (16) is true, that is, that the sequence $\{\mu_n\}$ constructed in such a way is convergent. Because of Condition (A), there exists a subsequence $\{\mu_{n_k}\}$ of μ_n such that $\omega(\mu_{n_k}, \mu) \geq 1$ for each k.

We claim that μ is a fixed point of I.

$$\begin{aligned}
\delta(\mu, I\mu) &\leq s\Big[(\delta(\mu, \mu_{n_k+1})) + \delta(I\mu_{n_k}, I\mu)\Big] \\
&= s\delta(\mu, \mu_{n_k+1})) + s\delta(I\mu_{n_k}, I\mu) \\
&\leq s\delta(\mu, \mu_{n_k+1})) + s\omega(\mu_{n_k}, \mu)\delta(I\mu_{n_k}, I\mu), \text{ (for } \omega(\mu_{n_k}, \mu) \geq 1 \forall k) \\
&\leq s\delta(\mu, \mu_{n_k+1})) + s\xi\Big([\delta(\mu_{n_k}, \mu)]^b [\delta(\mu_{n_k}, I\mu_{n_k})]^a [\delta(\mu, I\mu)]^{1-a-b}\Big) \\
&= s\delta(\mu, \mu_{n_k+1})) + s\xi\Big([\delta(\mu_{n_k}, \mu)]^b [\delta(\mu_{n_k}, \mu_{n_k+1})]^a [\delta(\mu, I\mu)]^{1-a-b}\Big).
\end{aligned}$$

Letting $k \to \infty$ in the last inequality, we have

$$\delta(\mu, I\mu) \leq 0 + s\xi(0).$$

Therefore, we conclude that $d(\mu, I\mu) = 0$. $\square$

We discuss an example for Theorem 4 where the self-map I is not continuous.

Example 3. *Let $M = [0, 1]$ and $\delta(\mu, \nu) = (\mu - \nu)^2$. Then, $(M, \delta, 2)$ is a complete bMS.*

Define the self-map $I : M \to M$ by

$$I\mu = \begin{cases} \frac{1}{5}, & \text{if } \mu \in [0, \frac{1}{2}) \\ \frac{2}{3}, & \text{if } \mu \in [\frac{1}{2}, 1]. \end{cases}$$

Again, let

$$\omega(\mu, \nu) = \begin{cases} 1, & \text{if } \mu, \nu \in [\frac{1}{2}, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mu, \nu \in M \setminus Fix(I)$ and $\omega(\mu, \nu) \geq 1$. Then, clearly, $\mu, \nu \in [\frac{1}{2}, 1]$ and $\mu, \nu \notin \{\frac{2}{3}\}$. Now, we have $I\mu = I\nu = \frac{2}{3}$. Hence, (7) holds. Also, for $\mu_0 = 1$ we have $\omega(1, I1) = \omega(1, \frac{2}{3}) = 1$.

Let $\gamma \in M$ such that $\omega(\gamma, I\gamma) \geq 1$. This implies $\gamma, I\gamma \in [\frac{1}{2}, 1]$. Thus, $\omega(I\gamma, I^2\gamma) \geq 1$. Hence, I is ω-orbital admissible.

Obviously, I is not continuous at $\mu = \frac{1}{2}$, but the Condition (A) holds for the ω defined above.

Indeed, if μ_n is a sequence in M such that $\omega(\mu_n, \mu_{n+1}) \geq 1$ for each n and $\mu_n \to \mu \in M$ as $n \to \infty$, we have $\delta(\mu_n, \mu) \to 0$ as $n \to \infty$. Therefore, we have $(\mu_n - \mu)^2 \to 0$ as $n \to \infty$, and consequently $|\mu_n - \mu| \to 0$ as $n \to \infty$. Hence, $\mu \in [\frac{1}{2}, 1]$ since $\mu_n \in [\frac{1}{2}, 1]$ for each n.

Therefore, $\omega(\mu_n, \mu) \geq 1$ for all $n \in \mathbb{N}$.

Thus, all conditions of Theorem 4 are fulfilled. Here, it is easy to see that $\frac{1}{5}$ and $\frac{2}{3}$ are two fixed points of I.

4. Conclusions

In this paper, we considered interpolative contractions in bMS. This new interpolative approach ensured the existence of fixed points for the contractive maps. One main contribution of this article is the introduction of a MCRR-type contraction by injecting the parameter s of the bMS concerned in its definition. Next, a CRR-type existence result was proved by assuming continuity of the self-map, and finally, a variant of that CRR-type contraction was obtained by dropping the continuity assumption of the self-map and replacing it by a weaker condition.

The study of the uniqueness of fixed points for those maps and their applications in the solution of nonlinear integral equations would be interesting topics for future work. Common fixed points for similar types of interpolative contractions may also be obtained in future, and further, similar results for non-self-mappings is another direction of future study. It is worth mentioning that a disadvantage of our results is that they are applicable to a limited class of contractive mappings satisfying several conditions.

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