

Article

The Weighted Surplus Division Value for Cooperative Games

Hui Yang ^{1,2} , Wenna Wang ^{1,*}  and Zhengsheng Ding ²

¹ Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China; yangh816@mail.nwpu.edu.cn

² Department of Science, Xi'an University of Science and Technology, Xi'an 710054, China; dingzs@xust.edu.cn

* Correspondence: wenna_wang@foxmail.com; Tel.: +86-155-9183-1216

Received: 27 July 2019; Accepted: 12 September 2019; Published: 16 September 2019



Abstract: The weighted surplus division value is defined in this paper, which allocates to each player his individual worth and then divides the surplus payoff with respect to the weight coefficients. This value can be characterized from three different angles. First, it can be obtained analogously to the scenario of getting the procedural value whereby the surplus is distributed among all players instead of among the predecessors. Second, endowing the exogenous weight to the surplus brings about the asymmetry of the distribution. We define the disweighted variance of complaints to remove the effect of the weight and prove the weighted surplus division value is the unique solution of an optimization model. Lastly, the paper offers axiomatic characterizations of the weighted surplus division value through proposing new properties, including the ω -symmetry for zero-normalized game and individual equity.

Keywords: cooperative games; weighted surplus division value; procedural interpretation; optimization implementation; axiomatization

1. Introduction

Cooperative game theory offers an effective model of cooperation between rational persons [1]. It is widely used in economics, wireless networks, political science, and so on. The critical issue of cooperative games is how to distribute social worth among all players. Various solutions in cooperative games embody different criteria. Egalitarianism and utilitarianism are two key concepts related to distribution preference in games. Generally, egalitarianism aims at making sure players get equal treatment, while utilitarianism reflects distributing the payoff based on the players' contribution and ability [2,3].

The Shapley value [4] allocates the payoff based on players' productivity and gives everyone his expected marginal contribution, presuming that all permutations of players' entrances be equal possible. In the process, each new entrant does not share his marginal contribution with others, which embodies utilitarianism. The equal division value (ED value) shares the worth equally among all players no matter how much their contributions are, which reflects egalitarianism. The equal surplus division value (ES value, [5]) gives everyone his individual worth at first and then divides the surplus evenly among all players. The equal allocation of non-separable costs value (EANS value, [6]) is also egalitarian in some sense because it first assigns to every player his grand marginal contribution and average the remainder to everyone. Balancing the relationship between utilitarianism and egalitarianism is a main issue of allocation problems.

Malawski [7] puts forward a family of procedural values, which are determined by a procedure of dividing marginal contributions among players' predecessors. Players share their marginal contributions with their predecessors in the permutation, which is exactly a way to balance

utilitarianism and egalitarianism. The convex combinations of solutions also efficiently reconcile these two major economic allocation thoughts. Wang et al. [8] designs a process to achieve the convex values of the Shapley value and the ED value. van den Brink et al. [9] discuss all convex combinations of the ED value, the ES value and the EANS value.

Another method for balancing the relationship between utilitarianism and egalitarianism is endowing weights to a value. The weighted division values treat every participant differently to embody exogenous characteristics. Béal et al. [10] characterize the weighted ED values and Shapley exhibits a family of weighted Shapley values [11]. Kalai and Samet [12] extend the “weights” to “weight system” and enable a weight of zero for some players.

In this paper, we follow Shapley’s work and consider the positive weights to defining the weighted surplus division value, which presents a new distribution rule for the marginal contributions. The philosophical idea of the weighted surplus division value is similar but different to the procedural values. The former studies how the marginal contributions are shared by every player, while the latter discusses how the marginal contributions are divided among the predecessors. Through designing the process of obtaining an extended procedural value, we determine that this extended procedural value coincides with the weighted surplus division value. Therefore, we endow a reasonable procedural interpretation to the weighted surplus division value.

Optimality theory is proposed to allocations for the first time by Ruiz et al. [13]. The least square value is put forward as the unique value that minimizes the variance of the total complaints for coalitions. Wang et al. [14] recommend the ideal values by minimizing the variance of all players between the payoff and the individual expected reward. Taking into account the disweighted variance between the payoff and individual worth, this paper proves the unique optimum solution of an optimization model is exactly the weighted surplus division value.

Furthermore, we characterize the weighted surplus division value by some new properties. The first axiomatization concerns an axiom called the ω -symmetry for zero-normalized game, which implies that the disweighted payoffs of two different players are symmetric in a zero-normalized game. Then, we replace the inessential game property with individual rationality to describe the weighted surplus division value. Individual equity implies the inessential game property, so we obtain an obvious corollary. Finally, the covariance is introduced to axiomatize the weighted surplus division value.

The organization of the article is as follows. The primary notions of cooperative games are briefly presented in Section 2. Section 3 proposes the definition of the weighted surplus division value and designs a procedure to achieve this value. Section 4 provides the optimization implementation for the weighted surplus division value. Section 5 axiomatically characterizes the weighted surplus division value using several newly defined properties from the cooperative viewpoint and gives an example to validate the new defined value. We summarize the paper and exhibit further interesting aspect of the weighted surplus division value in Section 6.

2. Preliminaries

A cooperative game on a finite participant set N is an ordered pair $\langle N, v \rangle$, where characteristic function $v : 2^N \rightarrow \mathbb{R}$ assigns to each coalition $S \in 2^N \setminus \{\emptyset\}$ the worth $v(S)$, satisfying $v(\emptyset) = 0$, in short denoted as a game v . For each coalition S , $v(S)$ represents the reward that coalition S can be guaranteed by just itself without collaboration of others. The cardinality of a set S is represented by s . $\mathcal{G}^N$ denotes the linear space of all games with player set N .

For any games $v, w \in \mathcal{G}^N$ and $a \in \mathbb{R}$, the games $v + w$ and $a \cdot v$ are given by $(v + w)(S) = v(S) + w(S)$ and $(a \cdot v)(S) = a \cdot v(S)$ for all $S \subseteq N$. A game $v \in \mathcal{G}^N$ is zero-normalized, if $v(\{i\}) = 0$ for all $i \in N$. A game $v \in \mathcal{G}^N$ is weakly essential, if it satisfies $\sum_{i \in N} v(\{i\}) \leq v(N)$. A game $v \in \mathcal{G}^N$ is

inessential, if $v(S) = \sum_{i \in S} v(\{i\})$ holds, for each $S \subseteq N$. For any $\emptyset \neq T \subseteq N$, the unanimity game u_T and standard game b_T are respectively defined as

$$u_T(S) = \begin{cases} 1, & S \supseteq T, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad b_T(S) = \begin{cases} 1, & S = T, \\ 0, & \text{otherwise.} \end{cases}$$

Any game $v \in \mathcal{G}^N$ can be represented employing unanimity games: $v = \sum_{T \subseteq N, T \neq \emptyset} \Delta_v(T) u_T$, where $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{t-s} v(S)$, which means the games $\{u_T\}_{T \subseteq N, T \neq \emptyset}$ make up a basis of $\mathcal{G}^N$. Similarly, v can be expressed by standard games: $v = \sum_{T \subseteq N, T \neq \emptyset} v(T) b_T$, which implies $\{b_T\}_{T \subseteq N, T \neq \emptyset}$ also constitute a basis of $\mathcal{G}^N$.

Any vector $x \in \mathbb{R}^N$ is called a payoff vector (or payoff), and $x(S) = \sum_{i \in S} x_i$ for any coalition S . A payoff x is efficient or a pre-imputation if $x(N) = v(N)$. We denote the pre-imputation set of a game v with $I^*(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$. Formally, a value on $\mathcal{G}^N$ is a function ϕ that gives each game $v \in \mathcal{G}^N$ a payoff, $\phi(v) = (\phi_i(v))_{i \in N} \in \mathbb{R}^N$. The coordinate $\phi_i(v)$ gives an evaluation of i 's gains when participating in the game v .

The equal surplus division value (ES value) [5] allocates the surplus evenly after gives every player his individual worth.

$$ES_i(v) = v(\{i\}) + \frac{1}{n} [v(N) - \sum_{j \in N} v(\{j\})], \text{ for all } i \in N.$$

The equal division value (ED value) averages the worth of the grand coalition to all players,

$$ED_i(v) = \frac{v(N)}{n}, \text{ for all } i \in N.$$

Given any $\alpha \in [0, 1]$, the α -ES value is defined as the convex of the ED value and the ES value [9], i.e.,

$$ES_i^\alpha(v) = \alpha v(\{i\}) + \frac{1}{n} [v(N) - \sum_{j \in N} \alpha v(\{j\})], \text{ for all } i \in N.$$

First, each player gets a fraction α of his own worth, then all players divides up the surplus equally. Xu et al. [15] present several axiomatizations and a bidding mechanism interpretation for the α -ES value.

Let $\phi : \mathcal{G}^N \rightarrow \mathbb{R}^N$ be a value. Some well-known properties of ϕ are listed.

- Efficiency: For any game $v \in \mathcal{G}^N$, $\sum_{i \in N} \phi_i(v) = v(N)$.
- Symmetry: For any game $v \in \mathcal{G}^N$, if players $i, j \in N$ are symmetric, then $\phi_i(v) = \phi_j(v)$.
- Individual rationality: For any weakly essential game $v \in \mathcal{G}^N$, $\phi_i(v) \geq v(\{i\})$, for all $i \in N$.
- Additivity: For any games $v, w \in \mathcal{G}^N$, $\phi(v + w) = \phi(v) + \phi(w)$.
- Linearity: For any games $v, w \in \mathcal{G}^N$ and $p, q \in \mathbb{R}$, $\phi(pv + qw) = p\phi(v) + q\phi(w)$.
- Inessential game property: For any inessential game $v \in \mathcal{G}^N$, $\phi_i(v) = v(\{i\})$ for all $i \in N$.

3. The Weighted Surplus Division Value and Its Procedural Interpretation

Procedural values [7] are obtained through a procedure that stipulating how to distribute the marginal contributions of coalitions formed by players joining randomly. If a player wants to enter a coalition already formed, he must take out part of his marginal contribution as an “admission fee” to his predecessors. As a new viewpoint, a player who hopes to enter a coalition would like to share his extra contribution with all players rather than only with his predecessors after getting his guaranteed return. Based on this view, we can define a new value employing the weight and give it a procedural explain.

Let $\Delta^n := \{\omega \in \mathbb{R}_{++}^N \mid \sum_{i \in N} \omega_i = 1\}$ be the positive exogenous given weight system to all players.

Definition 1. Given a game $v \in \mathcal{G}^N$ and weight vector $\omega \in \Delta^n$, the weighted surplus division value is defined as

$$SD_i^\omega(v) = v(\{i\}) + \omega_i \left[v(N) - \sum_{j \in N} v(\{j\}) \right], \text{ for } i \in N. \quad (1)$$

This value is a new generalization of the ES value, associates every player $i \in N$ with a positive real number ω_i . Please note that each weight coefficient ω_i is exogenously given, which means it is independent to the game v under consideration. The weight of every player can be explained as a measure of his negotiating ability or player's probability of forming coalitions with other players.

Given a game $v \in \mathcal{G}^N$, let $\Pi(N)$ be the set including all permutations on N . Given a permutation $\pi \in \Pi(N)$, denote $P_i^\pi := \{\pi(k) \in N | k \leq \pi^{-1}(i)\}$ as player i 's predecessors (containing player i). In the following, we design a procedure in terms of a weight system $\omega \in \Delta^n$ to obtain the weighted surplus division value.

PROCEDURE 1

- Step 1 Each player goes into a permutation π randomly and all orders in $\Pi(N)$ have the same probability.
- Step 2 Every arriving player, $i \in N$, joins the coalition of his predecessors to form a new coalition P_i^π and brings his marginal contribution $v(P_i^\pi) - v(P_i^\pi \setminus \{i\})$ to the coalition of his predecessors.
- Step 3 The arriving player $i \in N$ claims his individual worth $v(\{i\})$ and the surplus (or deficit), $v(P_i^\pi) - v(P_i^\pi \setminus \{i\}) - v(\{i\})$, is shared (or afforded) by all players according to the weight system ω .
- Step 4 A player's value is the expected payoff of his part of $v(N)$ (over all orders of arrival).

In PROCEDURE 1, given an order, players go into the game one after another. Each newcomer asks for his individual worth and the remaining marginal contributions are put into a common pool. Afterwards, the worth in the common pool is divided according to the exogenous weight ω_i , $i \in N$ among all players. Lastly, compute the expected payoff over all permutations.

Theorem 1. For any game $v \in \mathcal{G}^N$ and weight vector $\omega \in \Delta^n$, the expected value in PROCEDURE 1 coincides with the weighted surplus division value $SD^\omega(v)$.

Proof. In light of PROCEDURE 1, for any game $v \in \mathcal{G}^N$ and any order π in $\Pi(N)$, given the weight vector $\omega \in \Delta^n$, player i 's payoff $\varphi_{i,\pi}^\omega(v)$ is determined as

$$\varphi_{i,\pi}^\omega(v) = v(\{i\}) + \omega_i \sum_{k=1}^n [v(P_{\pi(k)}^\pi) - v(P_{\pi(k)}^\pi \setminus \pi(k)) - v(\pi(k))]. \quad (2)$$

The value $\varphi^\omega(v)$ defined in the above procedure, i.e., the expected payoff is

$$\varphi_i^\omega(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \varphi_{i,\pi}^\omega(v), \text{ for } i \in N. \quad (3)$$

Computing Equation (2), we can get

$$\varphi_{i,\pi}^\omega(v) = v(\{i\}) + \omega_i \left[v(N) - \sum_{k=1}^n v(\pi(k)) \right].$$

Please note that the payoff $\varphi_{i,\pi}^\omega(v)$ is independent to the permutation π , hence

$$\varphi_i^\omega(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \varphi_{i,\pi}^\omega(v) = v(\{i\}) + \omega_i \left[v(N) - \sum_{j \in N} v(\{j\}) \right] = SD_i^\omega(v).$$

□

Observing the definition of the weighted surplus division value, the following conclusion can be deduced.

Remark 1. For any game $v \in \mathcal{G}^N$, if we perform the procedure by assuming that every player has the same weight, i.e., $\omega_i = \frac{1}{n}$, the expected value will coincide with the ES value.

Remark 2. For any game $v \in \mathcal{G}^N$ and $0 \leq \alpha \leq 1$, if in Step 2, the player $i \in N$ entering the coalition wants to take $\alpha v(\{i\})$, and assuming that every player has the same weight, i.e., $\omega_i = \frac{1}{n}$, the expected value will coincide with the α -ES value.

4. Optimization Implementation

The least square value [13] is testified to be the only pre-imputation that minimizes the variance of the excess of coalitions by optimality theory. This section focuses on studying the optimization implementation of the weighted surplus division value.

In cooperative games, every participant is reluctant to accept the payoff that is lower than his individual worth. Therefore, given any payoff $x \in I^*(v)$, the excess of player $i \in N$ at the payoff, which is expressed as $e(i, x) = v(\{i\}) - x_i$, is an effective measure of i 's complaint. The larger $e(i, x)$ is, the more dissatisfied i would feel.

In general, the sum of all the excesses is constant. It is impossible to decreasing everyone's excess altogether. To look for an allocation in which all the dissatisfactions are relatively balanced, the least Euclidean distance function can be used to minimize the variance of the excesses of every player according to the egalitarian philosophy.

Furthermore, in the definition of the weighted surplus division value, every player has a different level of importance, so we need to take the weight coefficients ω_i , $i \in N$ into account when evaluating the variance of excesses of every player. If a player's weight coefficient is relatively larger, then his ability or significance is greater, correspondingly his allowed variance of complaints should be bigger than that of the others. We construct the following problem to equilibrate the complaints of all players. For any game $v \in \mathcal{G}^N$ and weight vector $\omega \in \Delta^n$,

$$\begin{aligned} \text{Problem X : } \min_{x \in \mathbb{R}^N} \quad & \sum_{i \in N} \frac{1}{\omega_i} [v(\{i\}) - x_i]^2 \\ \text{s.t. } \quad & \sum_{i \in N} x_i = v(N). \end{aligned} \quad (4)$$

Theorem 2. For any game $v \in \mathcal{G}^N$ and weight vector $\omega \in \Delta^n$, the optimal solution of Problem X uniquely exists and coincides with the weighted surplus division value, i.e.,

$$x_i^* = v(\{i\}) + \omega_i \left[v(N) - \sum_{j \in N} v(\{j\}) \right] = SD_i^\omega(v), \text{ for all } i \in N. \quad (5)$$

Proof. This problem's optimal solution is unique if it exists because this is a convex optimization model. We just need to verify its Lagrange conditions. The Lagrange function of this problem is

$$L(x, \lambda) = \sum_{i \in N} \frac{1}{\omega_i} [v(\{i\}) - x_i]^2 + \lambda \left[\sum_{i \in N} x_i - v(N) \right].$$

Then, compute the partial derivative of $L(x, \lambda)$ with respect to x_i , $i \in N$, the Lagrange conditions are obtained as following

$$L_{x_i}(x, \lambda) = -\frac{2}{\omega_i} [v(\{i\}) - x_i] + \lambda = 0.$$

Obviously, the partial derivative with respect to λ actually states the efficiency constraint

$$L_{\lambda}(x, \lambda) = \sum_{i \in N} x_i - v(N) = 0.$$

Through solving this linear problem, we find that the optimal solution is $SD^{\omega}(v)$ given by (5). $\square$

Ignoring the weight coefficient, we naturally deduce the following corollaries from the proof of Theorem 2. Given any game $v \in \mathcal{G}^N$,

$$\begin{aligned} \text{Problem } X' : \min_{x \in \mathbb{R}^N} \quad & \sum_{i \in N} [v(\{i\}) - x_i]^2 \\ \text{s.t.} \quad & \sum_{i \in N} x_i = v(N). \end{aligned}$$

Corollary 1. For any game $v \in \mathcal{G}^N$, the optimal solution of Problem X' is uniquely exists and coincides with the ES value,

$$x_i^* = v(\{i\}) + \frac{1}{n} [v(N) - \sum_{j \in N} v(\{j\})] = ES_i(v), \text{ for all } i \in N.$$

Similarly, for any game $v \in \mathcal{G}^N$ and $\alpha \in [0, 1]$,

$$\begin{aligned} \text{Problem } X'' : \min_{x \in \mathbb{R}^N} \quad & \sum_{i \in N} [\alpha v(\{i\}) - x_i]^2 \\ \text{s.t.} \quad & \sum_{i \in N} x_i = v(N). \end{aligned}$$

Corollary 2. For any game $v \in \mathcal{G}^N$, the optimal solution of Problem X'' is uniquely exists and coincides with the α -ES value, i.e.,

$$x_i^* = \alpha v(\{i\}) + \frac{1}{n} [v(N) - \sum_{j \in N} \alpha v(\{j\})] = ES_i^{\alpha}(v), \text{ for all } i \in N.$$

The definition of the weighted surplus value creatively incorporates the weight into consideration. As a result, the excess (or complaint) of every player may be imbalanced. We offer a new complaint related with the weight as below.

For any game $v \in \mathcal{G}^N$, weight vector $\omega \in \Delta^n$ and a payoff $x \in I^*(v)$, define the disweighted complaint of $i \in N$ as

$$e^{\omega}(i, x) = \frac{1}{\omega_i} [v(\{i\}) - x_i].$$

- Equal disweighted complaint property: For any game $v \in \mathcal{G}^N$ and weight vector $\omega \in \Delta^n$, the value $\phi : \mathcal{G}^N \rightarrow \mathbb{R}^N$ verifies that

$$e^{\omega}(i, \phi) = e^{\omega}(j, \phi), \text{ for } i, j \in N.$$

According to Definition 1, it is explicit that

Theorem 3. For any game $v \in \mathcal{G}^N$ and weight vector $\omega \in \Delta^n$, the weighted surplus division value is the unique efficient value that possesses the equal disweighted complaint property.

5. Axiomatizations

Axiomatization is a normal tool to depict the reasonability of solutions in cooperative games. The α -ES value is characterized with a series of α -version of the classical properties [15]. In this section, we will offer a new property concerning the weight to describe the weighted surplus division value.

Symmetry says that the players, whose marginal contributions are equal, should get the same payoff. This axiom is used to characterize the ES value by van den Brink and Funaki [9]. However, asymmetry arises when the weight coefficient is given to the participant. If players still wish to get equitable treatment when removing the impact of the weights, we consider the following property,

- ω -symmetry for zero-normalized game: For any zero-normalized game $v \in \mathcal{G}^N$ and weight vector $\omega \in \Delta^n$, the value $\phi : \mathcal{G}^N \rightarrow \mathbb{R}^N$ verifies that

$$\frac{\phi_i(v)}{\omega_i} = \frac{\phi_j(v)}{\omega_j}, \text{ for any } i, j \in N.$$

The ω -symmetry for zero-normalized game says that discarding the effects of the weights, the players will get the same in the zero-normalized games. It is trivial that the weighted surplus division value verifies this axiom. Moreover, the weighted surplus division value can be axiomatized by this property as follows.

Theorem 4. *Given any weight vector $\omega \in \Delta^n$, the weighted surplus division value is the unique value that possesses efficiency, additivity, the inessential game property and the ω -symmetry for zero-normalized game.*

Proof. Given any weight vector $\omega \in \Delta^n$. It is straightforward that the weighted surplus division value satisfies the axioms listed in the theorem.

The proof of the uniqueness. Let $\phi : \mathcal{G}^N \rightarrow \mathbb{R}^N$ be a solution satisfies the above properties. For any game $v \in \mathcal{G}^N$, define

$$v^0(S) := v(S) - \sum_{j \in S} v(\{j\}), S \subseteq N.$$

For $i \in N$, $v^0(\{i\}) = 0$, so v^0 is a zero-normalized game. Because of the ω -symmetry for zero-normalized game, we get

$$\frac{\phi_i(v^0)}{\omega_i} = \frac{\phi_j(v^0)}{\omega_j}, \text{ for any } i, j \in N.$$

Combined with the efficiency, we can obtain

$$\phi_i(v^0) = \omega_i v^0(N) = \omega_i [v(N) - \sum_{j \in N} v(\{j\})].$$

Let $v^* := v - v^0$, then v^* is an inessential game. We conclude from the inessential game axioms that $\phi_i(v^*) = v^*(\{i\}) = v(\{i\})$.

Because $v = v^* + v^0$, involved with additivity we have

$$\begin{aligned} \phi_i(v) &= \phi_i(v^*) + \phi_i(v^0) \\ &= v(\{i\}) + \omega_i [v(N) - \sum_{j \in N} v(\{j\})] \\ &= SD_i^\omega(v). \end{aligned}$$

□

Remark 3. *According to the above proof, linearity can be used to replace additivity in the Theorem 4.*

Individual rationality recommends that a player gets at least his individual worth in weakly essential games. We can substitute the inessential game property in Theorem 4 with this axiom.

Corollary 3. Given any weight vector $\omega \in \Delta^n$, the weighted surplus division value is the unique value that possesses efficiency, linearity, the individual rationality and the ω -symmetry for zero-normalized game.

Proof. Given any weight vector $\omega \in \Delta^n$, we can easily check the weighted surplus division value satisfies efficiency, linearity, the individual rationality and the ω -symmetry for zero-normalized game. The proof of uniqueness. Let $\phi : \mathcal{G}^N \rightarrow \mathbb{R}^N$ be a solution possesses these four axioms.

- (i) For any unanimity game u_T , $T \subseteq N$ and $t = 1$, it follows that $\sum_{j \in N} u_T(\{j\}) \leq u_T(N)$. According to the individual rationality,

$$\begin{cases} i \notin T, & \phi_i(u_T) \geq u_T(\{i\}) = 0, \\ i \in T, & \phi_i(u_T) \geq u_T(\{i\}) = 1. \end{cases}$$

With efficiency, we obtain that

$$\phi_i(N, u_T) = \begin{cases} 1, & i \in T, \\ 0, & i \notin T. \end{cases}$$

- (ii) Given a standard game b_T , $T \subseteq N$ and $1 < t < n$, it holds that $\sum_{j \in N} b_T(\{j\}) \leq b_T(N)$. According to the individual rationality, for any $i \in N$, $\phi_i(b_T) \geq b_T(\{i\}) = 0$. With efficiency, $\phi_i(b_T) = 0, i \in N$.
- (iii) For $T = N$, the standard game b_N is a zero-normalized game. According to the ω -symmetry for zero-normalized game, we have, for any $i, j \in N$, $\frac{\phi_i(b_N)}{\omega_i} = \frac{\phi_j(b_N)}{\omega_j}$. With efficiency, $\phi_i(b_N) = \omega_i, i \in N$.

The unanimity games $\{u_T\}_{T \subseteq N, t=1}$ and standard games $\{b_T\}_{T \subseteq N, t \geq 2}$ make up a basis of $\mathcal{G}^N$. Together with linearity for ϕ , we have proved $\phi(v)$ is unique for $v \in \mathcal{G}^N$. $\square$

Remark 4. If a value satisfies efficiency and individual rationality, it also satisfies the inessential game property. Therefore, Corollary 3 can be deduced from Theorem 4. Here we exhibit a new method to prove the above Corollary.

- Individual equity: For any game $v \in \mathcal{G}^N$, if $\sum_{j \in N} v(\{j\}) = v(N)$, then $\phi_i(v) = v(\{i\})$, for all $i \in N$.

The individual equity demands that a player merely gets his individual worth if the sum of all players' individual worth equals to $v(N)$, which means the cooperation cannot bring any extra worth. Evidently, the individual equity implies the inessential game property. Therefore, it upholds the following Corollary.

Corollary 4. Given any weight vector $\omega \in \Delta^n$, the weighted surplus division value is the unique value that possesses efficiency, linearity, the individual equity and ω -symmetry for zero-normalized game.

In contrast to additivity and linearity, covariance describes that the value can keep the same affine transformation with the game.

- Covariance: For any game $v \in \mathcal{G}^N$, $p \in \mathbb{R}$ and $q \in \mathbb{R}^N$, $\phi_i(pv + q) = p\phi_i(v) + q_i$, for $i \in N$, where $(pv + q)(S) = pv(S) + \sum_{j \in S} q_j, S \subseteq N$.

Corollary 5. Given any weight vector $\omega \in \Delta^n$, the weighted surplus division value is the unique value that possesses efficiency, linearity, covariance and ω -symmetry for zero-normalized game.

Proof. Given any weight vector $\omega \in \Delta^n$. It is trivial that the weighted surplus division value satisfies efficiency, linearity, covariance and the ω -symmetry for zero-normalized game. The uniqueness will be proved in the following paragraphs.

Let $\phi : \mathcal{G}^N \rightarrow \mathbb{R}^N$ be a solution satisfies the above axioms.

- (i) For any unanimity game $u_{\{k\}}$, $k \in N$, let

$$u^0(S) := u_{\{k\}}(S) - \sum_{j \in S} u_{\{k\}}(\{j\}), S \subseteq N.$$

For $i \in N$, $u^0(\{i\}) = 0$, so u^0 is a zero-normalized game. Because of the ω -symmetry for zero-normalized game,

$$\frac{\phi_i(u^0)}{\omega_i} = \frac{\phi_j(u^0)}{\omega_j}, \text{ for any } i, j \in N.$$

Combined with the efficiency, we can obtain

$$\phi_i(u^0) = \omega_i u^0(N) = 0, \text{ for } i \in N.$$

Because $u_{\{k\}}(S) = u^0(S) + \sum_{j \in S} u_{\{k\}}(\{j\})$, $S \subseteq N$, with covariance, it holds that

$$\phi_i(u_{\{k\}}) = \phi_i(u^0) + u_{\{k\}}(\{i\}) = u_{\{k\}}(\{i\}), \text{ for } i \in N.$$

- (ii) Given any standard game b_T , $T \subseteq N$ and $t \geq 2$, they are all zero-normalized games. According to the ω -symmetry for zero-normalized game, we have, for any $i, j \in N$, $\frac{\phi_i(b_T)}{\omega_i} = \frac{\phi_j(b_T)}{\omega_j}$. With efficiency, $\phi_i(b_T) = \omega_i b_T(N)$, $i \in N$.

The unanimity games $\{u_T\}_{T \subseteq N, t=1}$ and standard games $\{b_T\}_{T \subseteq N, t \geq 2}$ constitute a basis of $\mathcal{G}^N$. Combining with linearity for ϕ , we conclude that $\phi(v)$ is unique for $v \in \mathcal{G}^N$. $\square$

Example 1. Three persons set up a new company with certain amount of money. They create alliance profits through collaboration of their funds and labor. Deduction of funds for company expansion, how the overall payoff be divided among all participants?

This situation can be condensed into a cooperative game. The player set is $N = \{1, 2, 3\}$. The overall payoff can be written as $v(N)$. Each person first receives his salary according to their labor contribution, denoted by $v(\{i\})$, ($i = 1, 2, 3$).

- (i) If we divide the surplus division value $v(N) - \sum_{i \in N} v(\{i\})$ equally, then the payoff coincides with the equal surplus division value $ES(v)$.
- (ii) If we allocate the surplus part according to a weight w_i , ($i = 1, 2, 3$), the payoff coincides with the weighted surplus division value $SD^\omega(v)$. Naturally, the weight can be set to be the proportion of their investment.

The $SD^\omega(v)$ is more feasible than $ED(v)$ and the $ES(v)$ in this economic example. In the wireless network, this value can be applied to distribute joint payoff through the cooperation of each node and the weight can be adjust flexible according to the performance of the node periodically.

6. Conclusions

This article defines a new value called the weighted surplus division value in cooperative games. This value considers the exogenous weights when distributing the surplus portion after every player takes his individual worth. Through modifying the classical procedural values, we determine that

the result coincides with the weighted surplus division value. Moreover, the weighted surplus division value is also the only solution minimizing the disweighted variance of players' complaints. Additionally, several new axioms are proposed to depict the weighted surplus division value. For further research, some more general properties are expected to characterize the weighted surplus division value. How to apply this new value to the economic model is also deserving to research.

Author Contributions: H.Y. and W.W. conceived and designed this work; Z.D. carried out feasibility analysis; H.Y. wrote the paper.

Funding: This work is funded by the National Natural Science Foundation of China (No. 71571143), the Science and Technology Research and Development Program in Shaanxi Province (Nos. 2017GY-095, 2017JM5147).

Acknowledgments: Our deepest gratitude goes to the anonymous reviewers for their careful work and thoughtful suggestions that have helped improve this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Myerson, R.B. *Game Theory: Analysis of Conflict*; The President and Fellows of Harvard College: Cambridge, MA, USA, 1991.
2. Binmore, K. Egalitarianism versus Utilitarianism. *Utilitas* **1998**, *3*, 353–367. [\[CrossRef\]](#)
3. Rachmilevitch, S. Egalitarianism, utilitarianism, and the Nash bargaining solution. *Soc. Choice Welf.* **2019**, *4*, 741–751. [\[CrossRef\]](#)
4. Shapley, L.S. A Value for n-Person Games. In *Contributions to the Theory of Games II*; Kuhn, H.W., Tucker, A.W., Eds.; Annals of Mathematics Studies; Princeton University Press: Princeton, NJ, USA, 1953; pp. 307–317.
5. Driessen, T.S.H.; Funaki, F. Coincidence of and collinearity between game theoretic solutions. *OR Spectr.* **1991**, *13*, 15–30. [\[CrossRef\]](#)
6. Moulin, H. The separability axiom and equal sharing method. *J. Econ. Theory* **1985**, *36*, 120–148. [\[CrossRef\]](#)
7. Malawski, M. “Procedural” values for cooperative games. *Int. J. Game Theory* **2013**, *42*, 305–324. [\[CrossRef\]](#)
8. Wang, W.; Sun, H.; Xu, G.; Hou, D. Procedural interpretation and associated consistency for the egalitarian Shapley values. *OR Lett.* **2017**, *45*, 164–169. [\[CrossRef\]](#)
9. van den Brink, R.; Funaki, Y. Axiomatizations of a class of equal surplus sharing solutions for TU-games. *Theory Decis.* **2009**, *67*, 303–340. [\[CrossRef\]](#)
10. Béal, S.; Casajus, A.; Huettner, F.; Rémila, E.; Solal, P. Characterizations of weighted and equal division values. *Theory Decis.* **2016**, *80*, 649–667. [\[CrossRef\]](#)
11. Shapley, L.S. Additive and Non-Additive Set Functions. Ph.D. Thesis, Princeton University, Princeton, NJ, USA, 1953.
12. Kalai, E.; Samet, D. On weighted Shapley values. *Int. J. Game Theory* **1987**, *16*, 205–222. [\[CrossRef\]](#)
13. Ruiz, L.M.; Valenciano, F.; Zarzuelo, J.M. The least square prenucleolus and the least square nucleolus. Two values for TU games based on the excess vector. *Int. J. Game Theory* **1996**, *25*, 113–134. [\[CrossRef\]](#)
14. Wang, W.; Sun, H.; van den Brink, R.; Xu, G. The family of ideal values for cooperative games. *J. Optim. Theory Appl.* **2019**, *180*, 1065–1086. [\[CrossRef\]](#)
15. Xu, G.; Dai, H.; Shi, H. Axiomatizations and a noncooperative interpretation of the α -CIS Value. *Asia Pac. J. Oper. Res.* **2015**, *32*, 1550031. [\[CrossRef\]](#)



© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).