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On Finite Quasi-Core-p p-Groups

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Abstract: Given a positive integer n, a finite group G is called quasi-core-n if $\langle x\rangle/\langle x\rangle_G$ has order at most n for any element x in G, where $\langle x\rangle_G$ is the normal core of $\langle x\rangle$ in G. In this paper, we investigate the structure of finite quasi-core-p p-groups. We prove that if the nilpotency class of a quasi-core-p p-group is $p+m$, then the exponent of its commutator subgroup cannot exceed p^{m+1}, where p is an odd prime and m is non-negative. If $p=3$, we prove that every quasi-core-3 3-group has nilpotency class at most 5 and its commutator subgroup is of exponent at most 9. We also show that the Frattini subgroup of a quasi-core-2 2-group is abelian.

Keywords: finite p-group; quasi-core-p p-group; commutator subgroup

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1. Introduction

Let G be a group and H is a subgroup of G. Then H_G is the normal core of H in G, where $H_G = \bigcap_{g\in G} g^{-1}Hg$ is the largest normal subgroup of G contained in H. A group G is called core-n if $|H/H_G| \leq n$ for every subgroup H of G, where n is a positive integer. Buckley, Lennox, Neumaan, Smith and Wiegold investigated the core-n groups in [1]. They show that every locally finite group G with H/H_G finite for all subgroups H is core-n for some n. Moreover, G has an abelian normal subgroup of index bounded in terms of n only. In [2], Lennox, Smith and Wiegold show that, for $p \neq 2$, a core-p p-group is nilpotent of class at most 3 and has an abelian normal subgroup of index at most p^5. Furthermore, Cutolo, Khukhro, Lennox, Wiegold, Rinauro and Smith [3] prove that a core-p p-group G has a normal abelian subgroup whose index in G is at most p^2 if $p \neq 2$. Furthermore, if $p=2$, Cutolo, Smith and Wiegold [4] prove that every core-2 2-group has an abelian subgroup of index at most 16. As a deepening of research in this area, it is interesting to study the following question.

How about the structure of a p-group G in which $|\langle x\rangle/\langle x\rangle_G| \leq p$, for any $x \in G$?

In this paper we hope to investigate the structure of a p-group G in which $|\langle x\rangle/\langle x\rangle_G| \leq p$, for any $x \in G$. For convenience, we call this kind of p-groups quasi-core-p p-groups.

2. Preliminaries

For convenience, we first recall some notations.

Let G be a p-group. We use $d(G)$ and $c(G)$ to denote the minimal number of generators and the nilpotency class of G respectively. We use C_{p^m} to denote the cyclic group of order p^m. Let $G_n = \langle [g_1, g_2, ..., g_n] \mid g_i \in G\rangle$. If H and K are groups, then $H \times K$ denotes a product of H and K. For other notations the reader is referred to [5].

Lemma 1. ([6], Section Appendix 1, Theorem A.1.4) *Let G be a p-group and $x, y \in G$.*

1. $(xy)^p \equiv x^p y^p$ *(mod $\mho_1(G')G_p$).*
2. $[x^p, y] \equiv [x, y]^p$ *(mod $\mho_1(N')N_p$), where $N = \langle x, [x, y]\rangle$.*

Lemma 2. ([7], Lemma 2.2) *Suppose that G is a finite non-abelian p-group. Then the following conditions are equivalent.*

1. *G is minimal non-abelian;*
2. *$d(G) = 2$ and $|G'| = p$;*
3. *$d(G) = 2$ and $\Phi(G) = Z(G)$.*

Lemma 3. ([8], Theorem) *Let p be a prime and d, e positive integers. A regular d-generator metabelian p-group G whose commutator subgroup has exponent p^e has nilpotency class at most $e(p-2)+1$ unless $e = 1, d > 2, p > 2$ when the class can be p. These bounds are best possible.*

Lemma 4. ([9], Theorem 2) *Let G be a metacyclic 2-group. Then G has one presentation of the following three kinds:*

1. *G has a cyclic maximal subgroup.*
2. *Ordinary metacyclic 2-groups $G = \langle a, b \mid a^{2^{r+s+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s}}, a^b = a^{1+2^r} \rangle$, where r, s, t, u are non-negative integers with $r \geq 2$ and $u \leq r$.*
3. *Exceptional metacyclic 2-groups $G = \langle a, b \mid a^{2^{r+s+v+t'+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s+v+t'}}, a^b = a^{-1+2^{r+v}} \rangle$, where r, s, v, t, t', u are non-negative integers with $r \geq 2, t' \leq r, u \leq 1, tt' = sv = tv = 0$, and if $t' \geq r-1$, then $u = 0$.*

Groups of different types or of the same type but with different values of parameters are not isomorphic to each other.

Lemma 5. ([5], Theorem 10.3) *Let G be a regular 3-group. Then G' is abelian.*

Lemma 6. *Let G be a quasi-core-p p-group. If H is a subgroup of G and N is a normal subgroup of G, then H and G/N are quasi-core-p p-groups.*

Proof. The proof of the lemma comes immediately from the definition of quasi-core-p p-groups. □

Lemma 7. *Let G be a p-group. Then G is quasi-core-p if and only if $\langle x^p \rangle \trianglelefteq G$, for any element x in G.*

Proof. Obviously, G is quasi-core-p if and only if $|\langle x \rangle_G / \langle x^p \rangle| \leq p$, for any $x \in G$, and this holds if and only if $\langle x^p \rangle \trianglelefteq G$, for any element x in G. □

Lemma 8. *Let G be a quasi-core-p p-group. Then $[G', \mho_1(G)] = 1$.*

Proof. For any $x \in G$, according to Lemma 7, we see $\langle x^p \rangle \trianglelefteq G$. Thus $G/C_G(x^p)$ is abelian and so $G' \leq C_G(x^p)$, which implies $[G', \mho_1(G)] = 1$. □

3. Quasi-Core-p p-Groups with $p > 2$

In this section we investigate the quasi-core-p p-groups for $p > 2$.

Theorem 1. *Let G be a quasi-core-p p-group and $p > 2$. If G' is cyclic, then $|G'| \leq p$.*

Proof. Suppose the result is not true and G is a counterexample of minimal order. Then there exist $a, b \in G$ such that $o([a, b]) \geq p^2$. Thus we may assume $G = \langle a, b \rangle$, $[a, b] = c$ and $L = \langle a, c \rangle$. Since G is regular, we may assume $\langle a \rangle \cap \langle b \rangle = 1$. By Lemma 1, we see $[a^p, b] = c^p x$, where $x \in \mho_1(L')L_p$. Since $L < G$, $\mho_1(L')L_p = 1$. So $x = 1$ and $[a^p, b] = c^p$. Similarly, $[a, b^p] = c^p$. It follows from Lemma 7 that $c^p \in \langle a \rangle \cap \langle b \rangle = 1$, in contradiction to the hypothesis. Thus the theorem is true. □

Corollary 1. *Let G be a quasi-core-p p-group with $p > 2$. Then $\mho_1(G)$ is abelian and $\mho_2(G) \leq Z(G)$.*

Proof. For any $a, b \in G$, we assume $H = \langle a^p, b \rangle$. By the hypotheses, we see $\langle a^p \rangle \trianglelefteq G$ and so H is metacyclic. By Theorem 1, $|H'| \leq p$ and so H is abelian or minimal non-abelian. Thus $\mho_1(H) \leq \Phi(H) \leq Z(H)$ by Lemma 2. It follows that $[a^{p^2}, b] = [a^p, b^p] = 1$, which implies $\mho_1(G)$ is abelian and $\mho_2(G) \leq Z(G)$. $\square$

Corollary 2. *Let G be a quasi-core-p p-group with $p > 2$. Then $G/C_G(a^p) \lesssim C_p$, for any $a \in G$.*

Proof. We may assume $a^p \notin Z(G)$ and $o(a) = p^n$. Then $n \geq 3$ and there exists an element $b \in G$ such that $b \notin C_G(a^p)$. By Theorem 1, we may assume $[a^p, b] = a^{p^{n-1}}$. Take $x \in G \setminus C_G(a^p)$. Assume $[a^p, x] = a^{ip^{n-1}}$, where $(i, p) = 1$. Then $[a^p, b^{-i}x] = 1$, which implies $x \in C_G(a^p)\langle b \rangle$ and so $G = C_G(a^p)\langle b \rangle$. It follows from $b^p \in C_G(a^p)$ that $G/C_G(a^p) \lesssim C_p$. $\square$

Corollary 3. *Let G be a quasi-core-p p-group with $p > 2$. If $c(G/\mho_1(G)) \leq n$, then $c(G) \leq n + 2$.*

Proof. Set $\overline{G} = G/\mho_1(G)$. Then $\overline{G}_{n+1} = \bar{1}$ and so $G_{n+1} \leq \mho_1(G)$. It follows from Theorem 1 that $[G_{n+1}, G] \leq [\mho_1(G), G] \leq Z(G)$, which implies $c(G) \leq n + 2$. $\square$

According to Lemma 3 and Corollary 3, we get the following theorem.

Theorem 2. *Suppose that G is a quasi-core-p p-group and G' is abelian with $p > 2$. If $d(G) = 2$, then $c(G) \leq p + 1$. If $d(G) > 2$, then $c(G) \leq p + 2$.*

If $p = 3$, then, according to Lemma 5 and Corollary 3, we get the theorem below.

Theorem 3. *Let G be a quasi-core-3 3-group. If $d(G) = 2$, then $c(G) \leq 4$. If $d(G) > 2$, then $c(G) \leq 5$.*

Theorem 4. *Let G be a quasi-core-3 3-group with $d(G) = 2$. Then $\Phi(G)$ is abelian.*

Proof. We may assume $G = \langle x, y \rangle$ and $[x, y] = z$. Then $G' = \langle z, [z, g] \mid g \in G \rangle$. For any $g_1, g_2 \in G$, it follows from Theorem 3 that $[z, [z, g]] \in [G_2, G_3] = 1$ and $[[z, g_1], [z, g_2]] = 1$, which implies G' is abelian. So, according to Lemma 8 and Corollary 1, $\Phi(G)$ is abelian. $\square$

Now, we investigate the exponent of commutator subgroups of the quasi-core-p p-groups.

Lemma 9. *Let G be a quasi-core-p p-group with $G_{p+1} = 1$ and $p > 2$. Then $\exp(G') \leq p$.*

Proof. Suppose the result is not true and G is a counterexample of minimal order. For any $g_1, g_2 \in G'$, let $H = \langle g_1, g_2 \rangle$. By Lemma 1, $(g_1 g_2)^p = g_1^p g_2^p x$, where $x \in \mho_1(H')H_p$. Since $c(H) < c(G)$, $H_p = 1$. By induction, $\exp(H') \leq p$ and so $\exp(\mho_1(H')) = 1$. Thus $x = 1$. It follows that there exist $a, b \in G$ such that $o([a, b]) > p$ and $\exp(G_3) \leq p$.

By induction, we may assume $G = \langle a, b \rangle$, $[a, b] = c$ and $L = \langle a, c \rangle$. Then, according to Lemma 1, we see $[a^p, b] = c^p y$, where $y \in \mho_1(L')L_p$. Since $c(L) < c(G)$, $L_p = 1$ and $\exp(L') \leq p$. Thus $y = 1$. Since G is a quasi-core-p p-group, $\langle a^p \rangle \trianglelefteq G$. So $c^p \in \langle a \rangle$. It follows from Theorem 1 that $o(c) = p^2$. Similarly, we see $c^p \in \langle b \rangle$.

Without loss of generality, we may assume $\langle a \rangle \cap \langle b \rangle = \langle a^{p^s} \rangle = \langle b^{p^t} \rangle$, $a^{p^s} = b^{p^t}$ and $s \geq t \geq 2$. If $s > t$, then, by letting $b_1 = a^{-p^{s-t}} b$, we see $[a, b_1^p] = c^p$ and $c^p \notin \langle b_1^p \rangle$, in contradiction to the hypothesis. So $s = t$. Let $b_2 = ab^{-1}$. Then, by Lemma 1, we see $b_2^p = a^p b^{-p} z$, where $z \in \mho_1(G')G_p$. Since $G' = \langle c, [c, g] \mid g \in G \rangle$, we see $\mho_1(G') = \langle c^p \rangle$. Then $\mho_1(G')G_p \leq Z(G)$ and $\exp(\mho_1(G')G_p) \leq p$. Thus $o(z) \leq p$ and $o(b_2) = p^s$. Noticing that $[a, b_2^p] = c^p$, we see $c^p \in \langle b_2^p \rangle$. If $s = 2$, then $\langle c^p \rangle = \langle b_2^p \rangle$, which implies $b_2^p = a^p b^{-p} z \in Z(G)$, a contradiction. If $s > 2$, then $\langle c^p \rangle = \langle b_2^{p^{s-1}} \rangle = \langle a^{p^{s-1}} b^{p^{s-1}} \rangle$. It follows that $\langle a \rangle \cap \langle b \rangle = \langle a^{p^{s-1}} \rangle$, another contradiction. $\square$

Corollary 4. *Let G be a quasi-core-p p-group and $\exp(G_{p+1}) = p^n$ with $p > 2$ and $n \geq 0$. Then $\exp(G') \leq p^{n+1}$.*

Proof. If $n = 0$, then the conclusion holds by Lemma 9. Thus we may assume $n \geq 1$. Set $\overline{G} = G/G_{p+1}$. Then $\overline{G}_{p+1} = \overline{G_{p+1}} = \bar{1}$. It follows from Lemma 9 that $\exp(\overline{G}') \leq p$, which implies $\exp(G') \leq p^{n+1}$. □

Corollary 5. *Let G be a quasi-core-p p-group and $c(G) = p + n$ with $p > 2$ and $n \geq 0$. Then $\exp(G') \leq p^{n+1}$.*

Proof. If $n = 0$, then the conclusion holds by Lemma 9. Thus we assume $n \geq 1$. Set $\overline{G} = G/G_{p+n}$. Then $c(\overline{G}) = p + n - 1$. By induction, we see $\exp(\overline{G}') \leq p^n$. Since $G_{p+n} = [G_{p+n-1}, G] \leq Z(G)$, by Lemma 9, we see $\exp(G_{p+n}) \leq p$. It follows that $\exp(G') \leq p^{n+1}$. □

Theorem 5. *Let G be a quasi-core-p p-group with $p > 2$. If G' is abelian, then $\exp(G') \leq p^2$ and $\exp(G_3) \leq p$.*

Proof. Suppose that the result is not true and G is a counterexample of minimal order. Then there exist $a, b \in G$ such that $o([a, b]) \geq p^3$. We may assume $G = \langle a, b\rangle$, $[a, b] = c$ and $L = \langle a, c\rangle$. By Lemma 1, $[a^p, b] = c^p x$, where $x \in \mho_1(L')L_p$. By induction, $\exp(L') \leq p^2$ and so $\exp(\mho_1(L')) \leq p$. On the other hand, since $[a, c]^p \in Z(G)$, it is easy to see that $\exp(L_3) \leq p$. So $o(x) \leq p$. According to Theorem 1, we see $o(c^p x) = p$, which implies $o(c) \leq p^2$, in contradiction to the hypothesis. So $\exp(G') \leq p^2$. Thus, for any $g \in G'$, we see $g^p \in Z(G)$. It follows that $\exp(G_3) \leq p$. □

Theorem 6. *Let G be a quasi-core-3 3-group. Then $\exp(G') \leq 9$ and $\exp(G_3) \leq 3$.*

Proof. Take $a, b \in G'$ with $o(a) \leq 9$ and $o(b) \leq 9$. Let $K = \langle a, b\rangle$. Then, by Lemma 1, $(ab)^3 = a^3b^3c$, where $c \in \mho_1(K')K_3$. Since $K' \leq G_4$, we see $c(K) \leq 3$ by Theorem 3. Thus $\exp(K') \leq 3$ by Corollary 5, which implies $o(c) \leq 3$. It follows that $(ab)^9 = a^9b^9 = 1$. So, we may assume $d(G) = 2$. According to Corollary 5 and Theorem 3, we see $\exp(G') \leq 9$.

Take $x \in G'$ and $y \in G$. Then $o(x) \leq 9$ and so $\langle x^3\rangle \leq Z(G)$. Assume $[x, y] = z$ and $L = \langle x, z\rangle$. Then, by Lemma 1, $1 = [x^3, y] = z^3w$, where $w \in \mho_1(L')L_3$. Since $L' \leq G_5 \leq Z(G)$, by Lemma 9, we see $\mho_1(L')L_3 = 1$. It follows that $z^3 = 1$. For any $g, h \in G_3$ with $o(g) \leq 3$ and $o(h) \leq 3$, then, by Theorem 3, we see $[g, h] \in G_6 = 1$. So $o(gh) \leq 3$, which implies $\exp(G_3) \leq 3$. □

4. Quasi-Core-2 2-Groups

In this section, we investigate the quasi-core-2 2-groups.

Lemma 10. *Let $G = \langle a, b\rangle$ be a non-abelian metacyclic quasi-core-2 2-group with $\langle a\rangle \trianglelefteq G$ and $o(a) = 2^n$. Then $[a, b] = a^{2^{n-1}}$, a^{-2} or $a^{-2+2^{n-1}}$.*

Proof. Since G is a non-abelian metacyclic 2-group , we see $n \geq 2$ and G is one of the groups listed in Lemma 4.

If G is a group listed in (1) in Lemma 4, then the conclusion holds by the classification of p-groups with a cyclic maximal subgroup.

If G is a group listed in (2) in Lemma 4, then $G = \langle a, b \mid a^{2^{r+s+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s}}, [a, b] = a^{2^r}\rangle$ with $r \geq 2$ and $u \leq r$. We may assume $s + u \geq 2$. By calculation, it is easy to see $\langle [a, b^2]\rangle = \langle a^{2^{r+1}}\rangle$. Since G is a quasi-core-2 2-group, we see $a^{2^{r+1}} \in \langle b^2\rangle$, which implies $s \leq 1$. Let $a_1 = ab^{-2^t}$. If $s = 0$, then $\langle a_1\rangle \cap \langle a\rangle$=1. It follows from G is quasi-core-2 that $a_1^2 \in Z(G)$, which implies $a^2 \in Z(G)$. However, it is impossible. If $s = 1$, then $o(a_1) = 2^{r+1}$ and $\langle [a_1^2, b]\rangle = \langle a^{2^{r+1}}\rangle \leq \langle a_1^2\rangle$. It follows that $\langle a^{2^{r+u}}\rangle = \langle a_1^{2^r}\rangle$, which implies $b^{2^{r+t}} \in \langle a\rangle$. It is also impossible. So $s + u = 1$ and therefore $[a, b] = a^{2^{n-1}}$.

If G is of type (3) in Lemma 4, then $G = \langle a, b \mid a^{2^{r+s+v+t'+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s+v+t'}}, [a,b] = a^{-2+2^{r+v}} \rangle$ with $r \geq 2$ and $u \leq 1$. It follows from $[a, b^2] \in \langle b \rangle$ that $s + t' \leq 1$ and so $s + t' + u \leq 2$. We may assume $s + t' + u = 2$ and so $u = s + t' = 1$. Then $b^{2^{r+s+t}} = a^{2^{n-1}}$ and $[a,b] = a^{-2+2^{n-2}}$. We assume $o(b) = 2^m$. If $r + s + t = 2$, then, since $(ba)^2 = b^2 a^{2^{n-2}}$, we see $o(ba) = 4$. On the other hand, $[a, (ba)^2] = a^{2^{n-1}}$. So, by the hypotheses, we see $a^{2^{n-1}} \in \langle (ba)^2 \rangle = \langle b^2 a^{2^{n-2}} \rangle$, a contradiction. If $r + s + t \geq 3$, then $o(b^{2^{m-3}} a^{2^{n-3}}) = 4$ and $[b, (b^{2^{m-3}} a^{2^{n-3}})^2] = a^{2^{n-1}}$. Thus $a^{2^{n-1}} \in \langle (b^{2^{m-3}} a^{2^{n-3}})^2 \rangle = \langle b^{2^{m-2}} a^{2^{n-2}} \rangle$, another contradiction. So the conclusion holds. □

Corollary 6. *Let G be a quasi-core-2 2-group. Then $\Phi(G)$ is abelian and $\mho_2(G) \leq G' Z(G)$.*

Proof. For any $a, b \in G$, we may assume $H = \langle a^2, b \rangle$ is not abelian and $o(a) = 2^n$. By the hypotheses, we see $\langle a^2 \rangle \trianglelefteq G$ and so H is metacyclic. It follows from Lemma 10 that $[a^2, b] = a^{2^{n-1}}, a^{-4}$ or $a^{-4+2^{n-1}}$. Then, it is easy to see that $[a^2, b^2] = 1$, which implies $\Phi(G)$ is abelian.

Take $g \in G$ with $g^4 \notin G'$. Then $[g^2, h] \in \Omega_1(\langle g \rangle)$ for any $h \in G$, which implies $[g^4, h] = 1$ and therefore $g^4 \in Z(G)$. So $\mho_2(G) \leq G' Z(G)$. □

Corollary 7. *Let G be a quasi-core-2 2-group. Then, for any $a \in G$, $G/C_G(a^2) \lesssim C_2 \times C_2$, $G/C_G(a^4) \lesssim C_2$ and if $G/C_G(a^4) \cong C_2$, then $a^4 \in G'$ and $\langle a \rangle \cap Z(G) = \Omega_1(\langle a \rangle)$.*

Proof. Without loss of generality, we may assume $a^2 \notin Z(G)$, $o(a) = 2^n$ and $n \geq 3$. By Corollary 6, we see $\Phi(G) \leq C_G(a^2)$, which implies $G/C_G(a^2)$ is elementary abelian. For any $g \in G \setminus C_G(a^2)$, according to Lemma 10, we see $[a^2, g] = a^{-4}, a^{2^{n-1}}$ or $a^{-4+2^{n-1}}$. It is easy to see that $G/C_G(a^2) \lesssim C_2 \times C_2$ and $G/C_G(a^4) \lesssim C_2$. If $G/C_G(a^4) \lesssim C_2$, then, there exists an element $b \in G \setminus C_G(a^4)$ such that $\langle [a^2, b] \rangle = \langle a^4 \rangle$. So $a^4 \in G'$ and $\langle a \rangle \cap Z(G) = \Omega_1(\langle a \rangle)$. □

Lemma 11. *Let G be a quasi-core-2 2-group with $c(G) = 2$. Then $\exp(G') \leq 4$.*

Proof. If not, then there exist $a, b \in G$ such that $o([a,b]) \geq 8$. We may assume $[a,b] = c$. Then $[a^2, b] = c^2$. By induction, $o(c^2) \leq 4$ and so $o(c) = 8$. It follows from Lemma 10 that $\langle c^2 \rangle = \langle a^4 \rangle$, which implies $a^4 \in Z(G)$. However, $[a^4, b] = c^4 \neq 1$, a contradiction. So the conclusion holds. □

Theorem 7. *Let G be a quasi-core-2 2-group with $c(G) = n$ and $n \geq 2$. Then $\exp(G') \leq 2^{2(n-1)}$.*

Proof. If $n = 2$, then the conclusion holds by Lemma 11. Thus we may assume $n \geq 3$. Set $\overline{G} = G/G_n$. Then $c(\overline{G}) = n - 1$. By induction, we see $\exp(\overline{G}') \leq 2^{2(n-2)}$. Since $G_n = [G_{n-1}, G] \leq Z(G)$, by Lemma 11, we see $\exp(G_n) \leq 4$. It follows that $\exp(G') \leq 2^{2(n-1)}$. □

Theorem 8. *Let G be a non-abelian quasi-core-2 2-group with $d(G) = 2$. Then $\mho_1(G')$, G_4 are cyclic, and either $G' \cap Z(G) \lesssim C_2 \times C_2 \times C_2$ or $G = \langle a, b \mid a^8 = 1, a^4 = b^4 = c^2, [a,b] = c, [c,a] = [c,b] = 1 \rangle$.*

Proof. If G is metacyclic, then the conclusion holds by Lemma 10. So we may assume $G = \langle a, b \rangle$ is non-metacyclic, $[a,b] = c$, $o(a) = 2^n$, $o(b) = 2^m$ and $o(c) = 2^t$ with $n \geq m$. Thus $G' = \langle c, [c,g] \mid g \in G \rangle$. By Corollary 6, $\Phi(G)$ is abelian. So $[c,g]^2 = [c^2, g] \in \langle c^2 \rangle$, which implies $\mho_1(G') \leq \langle c^2 \rangle$ and therefore $\mho_1(G')$ is cyclic. Now we consider the following two cases: $c(G) = 2$ and $c(G) > 2$.

Case 1. $c(G) = 2$.

By Lemma 11, we see $\exp(G') \leq 4$. We may assume $\exp(G') = 4$. Then $o(c) = 4$ and $[a^2, b] = [a, b^2] = c^2$. Thus $n \geq m \geq 3$ and $c^2 \in \langle a \rangle \cap \langle b \rangle$. Without loss of generality, we may assume $\langle a \rangle \cap \langle b \rangle = \langle a^{2^u} \rangle = \langle b^{2^v} \rangle$, $a^{2^u} = b^{2^v}$ and $u \geq v \geq 2$. Let $b_1 = a^{-2^{u-v}} b$. Then $[a, b_1^2] = c^2$. If $u > v$ or $v \geq 3$, then $o(b_1) = 2^v$. Thus $\langle c^2 \rangle = \langle b_1^{2^{v-1}} \rangle$, which implies $a^{2^{u-1}} \in \langle b \rangle$, a contradiction. So $u = v = 2$ and $a^4 = b^4$. Noticing that $G = \langle a, b_1 \rangle$ and

$[a,b_1]=c$, we see $a^4=b_1^4$ by the above. It follows from $o(b_1)=8$ that $o(a)=8$. So, we see $G=\langle a,b \mid a^8=1, a^4=b^4=c^2, [a,b]=c, [c,a]=[c,b]=1\rangle$.

Case 2. $c(G)>2$.

In this case, we consider the following two subcases: G' is cyclic and G' is not cyclic.

Subcase 1. G' is cyclic.

If $o(c)\le 4$, then $c^2\in Z(G)$ and $G'\cap Z(G)\lesssim C_2$. So we may assume $t\ge 3$. By Lemma 10, we see $[c,a]=1, c^{-2}, c^{-2+2^{t-1}}$ or $c^{2^{t-1}}$. If $\langle [c,a]\rangle=\langle c^2\rangle$, then $\exp(G'\cap Z(G))=2$. Thus we may assume $[c,a]=c^{2^{t-1}}$ and $[c,b]=1$. It follows that $[a^2,b]=c^{2+2^{t-1}}$. According to Lemma 10, it is easy to see $\langle c^2\rangle=\langle a^4\rangle$. So $[a^4,b]=1$ and therefore $o(c)\le 4$, in contradiction to the hypothesis.

Subcase 2. G' is not cyclic.

Since $[a,b]=c$, $[a^2,b]=c^2[c,a]$. By Lemma 10, we see $[c,a]=c^{-2}a^{-4}, c^{-2}a^{-4+2^{n-1}}, c^{-2}$ or $c^{-2}a^{2^{n-1}}$. Similarly, $[c,b]=c^{-2}b^{-4}, c^{-2}b^{-4+2^{m-1}}, c^{-2}$ or $c^{-2}b^{2^{m-1}}$. It follows that $G'\le\langle c,a^4,b^4\rangle$, $[\langle [c,a]\rangle, G]\le \mho_1(\langle [c,a]\rangle)$ and $[\langle [c,b]\rangle, G]\le \mho_1(\langle [c,b]\rangle)$. Then $[G_3,G]\le\mho_1(G_3)\le\mho_1(G')$. So G_4 is cyclic.

Now we prove $\exp(G'\cap Z(G))=2$. Assume $[c,a]=c^{-2}a^{-4}$ or $c^{-2}a^{-4+2^{n-1}}$, and $n\ge 4$.

If $[c,b]=c^{-2}$, then $G'=\langle c,a^4\rangle$. Since G' is not cyclic, we see $[c,a]\ne 1$. Take $g\in G'\cap Z(G)$ and assume $g=c^{2i}a^{4j}$. It follows from $[g,b]=1$ that $o(g)\le 2$. So $\exp(G'\cap Z(G))=2$.

If $[c,b]=c^{-2}b^{2^{m-1}}$, then $G'=\langle c,a^4,b^{2^{m-1}}\rangle$. If $[c,a]=1$, then $a^4\in\langle c\rangle$ and $G'=\langle c,b^{2^{m-1}}\rangle$. It is easy to see that $\exp(G'\cap Z(G))=2$. Assume $[c,a]\ne 1$. Take $h\in G'\cap Z(G)$ and assume $h=c^{2k}a^{4l}$. It follows from $[h,b]=1$ that $o(h)\le 2$ and so $\exp(G'\cap Z(G))=2$.

If $[c,b]=c^{-2}b^{-4}$ or $c^{-2}b^{-4+2^{m-1}}$, we may assume $m\ge 4$ by the above. It is easy to see that $\langle a^8,b^8\rangle\le\langle c\rangle$. Thus $[b^8,a]=1$, which implies $o(b)=16$ and $b^8=a^{2^{n-1}}$. On the other hand, we see $[(a^{2^{n-3}}b^2)^2,a]=b^8$ and therefore $b^8=a^{2^{n-2}}b^4$. It follows that $[a,b^4]=1$. However, it is impossible.

Assume $[c,a]=c^{-2}$ or $c^{-2}a^{2^{n-1}}$. Without loss of generality, we may assume $[c,b]=c^{-2}$ or $c^{-2}b^{2^{m-1}}$. Then $G'\le\langle c,a^{2^{n-1}},b^{2^{m-1}}\rangle$. It is clear that $\exp(G'\cap Z(G))=2$. □

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