## Article

# A New Identity Involving Balancing Polynomials and Balancing Numbers 

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#### Abstract

In this paper, a second-order nonlinear recursive sequence $M(h, i)$ is studied. By using this sequence, the properties of the power series, and the combinatorial methods, some interesting symmetry identities of the structural properties of balancing numbers and balancing polynomials are deduced.


Keywords: balancing numbers; balancing polynomials; combinatorial methods; symmetry sums

## 1. Introduction

For any positive integer $n \geq 2$, we denote the balancing number by $B_{n}$ and the balancer corresponding to it by $r(n)$ if

$$
1+2+\cdots+\left(B_{n}-1\right)=\left(B_{n}+1\right)+\left(B_{n}+2\right)+\cdots+\left(B_{n}+r(n)\right)
$$

holds for some positive integer $r(n)$ and $B_{n}$. It is clear that $r(n)=\frac{B_{n}-B_{n-1}-1}{2}$, for example, $r(2)=2$, $r(3)=14, r(4)=84, r(5)=492 \ldots$

It is found that the balancing numbers satisfy the second order linear recursive sequence $B_{n+1}=$ $6 B_{n}-B_{n-1}(n \geq 1)$, providing $B_{0}=0$ and $B_{1}=1$ [1].

The balancing polynomials $B_{n}(x)$ are defined by $B_{0}(x)=1, B_{1}(x)=6 x, B_{2}(x)=36 x^{2}-1$, $B_{3}(x)=216 x^{3}-12 x, B_{4}(x)=1296 x^{4}-108 x^{2}+1$, and the second-order linear difference equation:

$$
B_{n+1}(x)=6 x B_{n}(x)-B_{n-1}(x), n \geq 1
$$

where $x$ is any real number. While $n \geq 1$, we get $B_{n+1}=6 B_{n}-B_{n-1}$ with $B_{n}(1)=B_{n+1}$. Such balancing numbers have been widely studied in recent years. G. K. Panda and T. Komatsu [2] studied the reciprocal sums of the balancing numbers and proved the following inequation holds for any positive integer $n$ :

$$
\frac{1}{B_{n}-B_{n-1}}<\sum_{k=n}^{\infty} \frac{1}{B_{k}}<\frac{1}{B_{n}-B_{n-1}-1} .
$$

G. K. Panda [3] studied some fascinating properties of balancing numbers and gave the following result for any natural numbers $m>n$ :

$$
\left(B_{m}+B_{n}\right)\left(B_{m}-B_{n}\right)=B_{m+n} \cdot B_{m-n} .
$$

Other achievements related to balancing numbers can be found in [4-7].
It is found that the balancing polynomials $B_{n}(x)$ can be generally expressed as

$$
B_{n}(x)=\frac{1}{2 \sqrt{9 x^{2}-1}}\left[\left(3 x+\sqrt{9 x^{2}-1}\right)^{n+1}-\left(3 x-\sqrt{9 x^{2}-1}\right)^{n+1}\right],
$$

and the generating function of the balancing polynomials $B_{n}(x)$ is given by

$$
\begin{equation*}
\frac{1}{1-6 x t+t^{2}}=\sum_{n=0}^{\infty} B_{n}(x) \cdot t^{n} \tag{1}
\end{equation*}
$$

Recently, our attention was drawn to the sums of polynomials calculating problem [8-11], which is important in mathematical application. We are going to study the computational problem of the symmetry summation:

$$
\sum_{a_{1}+a_{2}+\cdots+a_{h+1}=n} B_{a_{1}}(x) B_{a_{2}}(x) \cdots B_{a_{h+1}}(x)
$$

where $h$ is any positive integer. We shall prove the following theorem holds.
Theorem 1. For any specific positive integer $h$ and any integer $n \geq 0$, the following identity stands:

$$
\begin{aligned}
& \sum_{a_{1}+a_{2}+\cdots+a_{h+1}=n} B_{a_{1}}(x) B_{a_{2}}(x) \cdots B_{a_{h+1}}(x) \\
= & \frac{1}{2^{h} \cdot h!} \cdot \sum_{j=1}^{h} \frac{M(h, j)}{(3 x)^{2 h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \frac{B_{n-i+j}(x)}{(3 x)^{i}} \cdot\binom{2 h+i-j-1}{i},
\end{aligned}
$$

where $M(h, i)$ is defined by $M(h, 0)=0, M(h, i)=\frac{(2 h-i-1)!}{2^{h-i} \cdot(h-i)!\cdot(i-1)!}$ for all positive integers $1 \leq i \leq h$.
In particular, for $n=0$, the following corollary can be deduced.
Corollary 1. For any positive integer $h \geq 1$, the following formula holds:

$$
\sum_{j=1}^{h} M(h, j) \cdot j!\cdot(3 x)^{j} \cdot B_{j}(x)=2^{h} \cdot h!\cdot(3 x)^{2 h}
$$

The formula in Corollary 1 shows the close relationship among the balancing polynomials. For $h=2$, the following corollary can be inferred by Theorem 1 .

Corollary 2. For any integer $n \geq 0$, we obtain

$$
\begin{aligned}
\sum_{a+b+c=n} B_{a}(x) \cdot B_{b}(x) \cdot B_{c}(x) & =\frac{1}{216 x^{3}} \sum_{i=0}^{n}(n-i+1)(i+1)(i+2) \cdot \frac{B_{n-i+2}}{(3 x)^{i}} \\
& +\frac{1}{72 x^{2}} \sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1) \cdot \frac{B_{n-i+3}}{(3 x)^{i}} .
\end{aligned}
$$

For $x=1, h=2$ and 3 , according to Theorem 1 we can also infer the following corollaries:
Corollary 3. For any integer $n \geq 0$, we obtain

$$
\begin{aligned}
\sum_{a+b+c=n} B_{a+1} \cdot B_{b+1} \cdot B_{c+1} & =\frac{1}{216} \sum_{i=0}^{n}(n-i+1)(i+1)(i+2) \cdot \frac{B_{n-i+2}}{3^{i}} \\
& +\frac{1}{72} \sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1) \cdot \frac{B_{n-i+3}}{3^{i}} .
\end{aligned}
$$

Corollary 4. For any integer $n \geq 0$, we obtain:

$$
\begin{aligned}
& \sum_{a+b+c+d=n} B_{a+1} \cdot B_{b+1} \cdot B_{c+1} \cdot B_{d+1} \\
= & \frac{1}{3888} \sum_{i=0}^{n}(n-i+1)(i+1)(i+2)(i+3)(i+4) \cdot \frac{B_{n-i+2}}{3^{i}} \\
+ & \frac{1}{1296} \sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)(i+2)(i+3) \cdot \frac{B_{n-i+3}}{3^{i}} \\
+ & \frac{1}{1296} \sum_{i=0}^{n}(n-i+1)(n-i+2)(n-i+3)(i+1)(i+2) \cdot \frac{B_{n-i+4}}{3^{i}} .
\end{aligned}
$$

Corollary 5. For any odd prime $p$, we have the congruence $M(p, i) \equiv 0(\bmod p), 0 \leq i \leq p-1$.
Corollary 6. The balancing polynomials are essentially Chebyshev polynomials of the second kind, specifically $B_{n}(x)=U_{n}(3 x)$. Taking $x=\frac{1}{3} x$ in Theorem 1, we can get the following:

$$
\begin{aligned}
& \sum_{a_{1}+a_{2}+\cdots+a_{h+1}=n} U_{a_{1}}(x) U_{a_{2}}(x) \cdots U_{a_{h+1}}(x) \\
= & \frac{1}{2^{h} \cdot h!} \cdot \sum_{j=1}^{h} \frac{(2 h-j-1)!}{2^{h-j} \cdot(h-j)!\cdot(j-1)!\cdot x^{2 h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \frac{U_{n-i+j}(x)}{x^{i}} \cdot\binom{2 h+i-j-1}{i} .
\end{aligned}
$$

Compared with [8], we give a more precise result for $\sum_{a_{1}+a_{2}+\cdots+a_{h+1}=n} U_{a_{1}}(x) U_{a_{2}}(x) \cdots U_{a_{h+1}}(x)$ with the specific expressions of $M(h, i)$. This shows our novelty.

Here, we list the first several terms of $M(h, i)$ in Table 1 in order to demonstrate the properties of the sequence $M(h, i)$ clearly.

Table 1. Values of $M(h, i)$.

| $\boldsymbol{M}(\boldsymbol{h}, \boldsymbol{i})$ | $\boldsymbol{i = 1}$ | $\boldsymbol{i = 2}$ | $\boldsymbol{i = 3}$ | $\boldsymbol{i = 4}$ | $\boldsymbol{i = 5}$ | $\boldsymbol{i = 6}$ | $\boldsymbol{i = 7}$ | $\boldsymbol{i = 8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $h=1$ | 1 |  |  |  |  |  |  |  |
| $h=2$ | 1 | 1 |  |  |  |  |  |  |
| $h=3$ | 3 | 3 | 1 |  |  |  |  |  |
| $h=4$ | 15 | 15 | 6 | 1 |  |  |  |  |
| $h=5$ | 105 | 105 | 45 | 10 | 1 |  |  |  |
| $h=6$ | 945 | 945 | 420 | 105 | 15 | 1 |  |  |
| $h=7$ | 10,395 | 10,395 | 4725 | 1260 | 210 | 21 | 1 |  |
| $h=8$ | 135,135 | 135,135 | 62,370 | 17,325 | 3150 | 378 | 28 | 1 |

## 2. Several Lemmas

For the sake of clarity, several lemmas that are necessary for proving our theorem will be given in this section.

Lemma 1. For the sequence $M(n, i)$, the following identity holds for all $1 \leq i \leq n$ :

$$
M(n, i)=\frac{(2 n-i-1)!}{2^{n-i} \cdot(n-i)!\cdot(i-1)!}
$$

Proof. We present a straightforward proof of this lemma by using mathematical introduction. It is obvious that

$$
M(1,1)=\frac{0!}{1 \cdot 0!\cdot 0!}=1
$$

This means Lemma 1 is valid for $n=1$. Without loss of generality, we assume that Lemma 1 holds for $1 \leq n=h$ and all $1 \leq i \leq h$. Then, we have

$$
\begin{gathered}
M(h, i)=\frac{(2 h-i-1)!}{2^{h-i} \cdot(h-i)!\cdot(i-1)!}, \\
M(h, i+1)=\frac{(2 h-i-2)!}{2^{h-i-1} \cdot(h-i-1)!\cdot i!} .
\end{gathered}
$$

According to the definitions of $M(n, i)$, it is easy to find that

$$
\begin{aligned}
M(h+1, i+1) & =(2 h-1-i) \cdot M(h, i+1)+M(h, i) \\
& =(2 h-1-i) \cdot \frac{2(h-i)}{(2 h-i-1) i} \cdot M(h, i)+M(h, i) \\
& =\frac{2 h-i}{i} M(h, i)=\frac{(2 h-i)!}{2^{h-i} \cdot(h-i)!\cdot i!} \\
& =\frac{(2(h+1)-(i+1)-1)!}{2^{h-i} \cdot(h-i)!\cdot i!}
\end{aligned}
$$

Thus, Lemma 1 is also valid for $n=h+1$. From now on, Lemma 1 has been proved.
Lemma 2. If we have a function $f(t)=\frac{1}{1-6 x t+t^{2}}$, then for any positive integer $n$, real numbers $x$ and $t$ with $|t|<|3 x|$, the following identity holds:

$$
2^{n} \cdot n!\cdot f^{n+1}(t)=\sum_{i=1}^{n} M(n, i) \cdot \frac{f^{(i)}(t)}{(3 x-t)^{2 n-i}}
$$

where $f^{(i)}(t)$ denotes the $i$-th order derivative of $f(t)$, with respect to variable $t$ and $M(n, i)$, which is defined in the theorem.

Proof. Similarly, Lemma 2 will be proved by mathematical induction. We start by showing that Lemma 2 is valid for $n=1$. Using the properties of the derivative, we have:

$$
f^{\prime}(t)=(6 x-2 t) \cdot f^{2}(t)
$$

or

$$
2 f^{2}(t)=\frac{f^{\prime}(t)}{3 x-t}=M(1,1) \cdot \frac{f^{\prime}(t)}{3 x-t}
$$

This is in fact true and provides the main idea to show the following steps. Without loss of generality, we assume that Lemma 2 holds for $1 \leq n=h$. Then, we have

$$
\begin{equation*}
2^{h} \cdot h!\cdot f^{h+1}(t)=\sum_{i=1}^{h} M(h, i) \cdot \frac{f^{(i)}(t)}{(3 x-t)^{2 h-i}} \tag{2}
\end{equation*}
$$

As an immediate consequence, we can tell by (2), the properties of $M(n, i)$, and the derivative, we get

$$
\begin{aligned}
& 2^{h} \cdot(h+1)!\cdot f^{h}(t) \cdot f^{\prime}(t)=2^{h+1} \cdot(h+1)!\cdot(3 x-t) \cdot f^{h+2}(t) \\
= & \sum_{i=1}^{h} \frac{M(h, i)}{(3 x-t)^{2 h-i}} \cdot f^{(i+1)}(t)+\sum_{i=1}^{h} \frac{(2 h-i) M(h, i)}{(3 x-t)^{2 h-i+1}} \cdot f^{(i)}(t)
\end{aligned}
$$

$$
\begin{align*}
= & \frac{M(h, h)}{(3 x-t)^{h}} \cdot f^{(h+1)}(t)+\sum_{i=1}^{h-1} \frac{M(h, i)}{(3 x-t)^{2 h-i}} \cdot f^{(i+1)}(t)+\frac{(2 h-1) M(h, 1)}{(3 x-t)^{2 h}} \cdot f^{\prime}(t) \\
& +\sum_{i=1}^{h-1} \frac{(2 h-i-1) M(h, i+1)}{(3 x-t)^{2 h-i}} \cdot f^{(i+1)}(t) \\
= & \frac{M(h+1, h+1)}{(3 x-t)^{h}} \cdot f^{(h+1)}(t)+\frac{M(h+1,1)}{(3 x-t)^{2 h}} \cdot f^{\prime}(t)+\sum_{i=1}^{h-1} \frac{M(h+1, i+1)}{(3 x-t)^{2 h-i}} \cdot f^{(i+1)}(t) \\
= & \frac{M(h+1, h+1)}{(3 x-t)^{h}} \cdot f^{(h+1)}(t)+\frac{M(h+1,1)}{(3 x-t)^{2 h}} \cdot f^{\prime}(t)+\sum_{i=2}^{h} \frac{M(h+1, i)}{(3 x-t)^{2 h+1-i}} \cdot f^{(i)}(t) \\
= & \sum_{i=1}^{h+1} M(h+1, i) \cdot \frac{f^{(i)}(t)}{(3 x-t)^{2 h+1-i}} . \tag{3}
\end{align*}
$$

Then, it is deduced that

$$
2^{h+1} \cdot(h+1)!\cdot(3 x-t) \cdot f^{h+2}(t)=\sum_{i=1}^{h+1} M(h+1, i) \cdot \frac{f^{(i)}(t)}{(3 x-t)^{2 h+1-i}}
$$

or

$$
2^{h+1} \cdot(h+1)!\cdot f^{h+2}(t)=\sum_{i=1}^{h+1} M(h+1, i) \cdot \frac{f^{(i)}(t)}{(3 x-t)^{2 h+2-i}} .
$$

Thus, Lemma 2 is also valid for $n=h+1$. From now on, Lemma 2 has been proved.
Lemma 3. The following power series expansion holds for arbitrary positive integers $h$ and $k$ :

$$
\frac{f^{(h)}(t)}{(3 x-t)^{k}}=\frac{1}{(3 x)^{k}} \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} \frac{(n-i+h)!}{(n-i)!} \cdot \frac{B_{n-i+h}(x)}{(3 x)^{i}} \cdot\binom{i+k-1}{i}\right) t^{n}
$$

where $t$ and $x$ are any real numbers with $|t|<|3 x|$.
Proof. According to the definition of the balancing polynomials $B_{n}(x)$, we have:

$$
f(t)=\frac{1}{1-6 x t+t^{2}}=\sum_{n=0}^{\infty} B_{n}(x) \cdot t^{n} .
$$

For any positive integer $h$, from the properties of the power series, we can obtain

$$
\begin{align*}
& f^{(h)}(t)=\sum_{n=0}^{\infty}(n+h)(n+h-1) \cdots(n+1) \cdot B_{n+h}(x) \cdot t^{n} \\
= & \sum_{n=0}^{\infty} \frac{(n+h)!}{n!} \cdot B_{n+h}(x) \cdot t^{n} . \tag{4}
\end{align*}
$$

For all real $t$ and $x$ with $|t|<|3 x|$, we have the following power series expansion:

$$
\frac{1}{3 x-t}=\frac{1}{3 x} \cdot \sum_{n=0}^{\infty} \frac{t^{n}}{(3 x)^{n}}
$$

and

$$
\begin{equation*}
\frac{1}{(3 x-t)^{k}}=\frac{1}{(3 x)^{k}} \cdot \sum_{n=0}^{\infty}\binom{n+k-1}{n} \cdot \frac{t^{n}}{(3 x)^{n}}, \tag{5}
\end{equation*}
$$

with any positive integer $k$. Then, it is found that

$$
\begin{aligned}
& \frac{f^{(h)}(t)}{(3 x-t)^{k}} \\
= & \frac{1}{(3 x)^{k}} \cdot\left(\sum_{n=0}^{\infty} \frac{(n+h)!}{n!} \cdot B_{n+h}(x) \cdot t^{n}\right)\left(\sum_{n=0}^{\infty}\binom{n+k-1}{n} \cdot \frac{t^{n}}{(3 x)^{n}}\right) \\
= & \frac{1}{(3 x)^{k}} \sum_{n=0}^{\infty}\left(\sum_{i+j=n} \frac{(j+h)!}{j!} \cdot B_{j+h}(x) \cdot\binom{i+k-1}{i} \cdot \frac{1}{(3 x)^{i}}\right) t^{n} \\
= & \frac{1}{(3 x)^{k}} \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} \frac{(n-i+h)!}{(n-i)!} \cdot B_{n-i+h}(x) \cdot\binom{i+k-1}{i} \cdot \frac{1}{(3 x)^{i}}\right) t^{n},
\end{aligned}
$$

where we have used the multiplicative of the power series. Lemma 3 has been proved.

## 3. Proof of Theorem

Based on the lemmas in the above section, it is easy to deduce the proof of Theorem 1. For any positive integer $h$, we can derive

$$
\begin{align*}
& 2^{h} \cdot h!\cdot f^{h+1}(t)=2^{h} \cdot h!\cdot\left(\sum_{n=0}^{\infty} B_{n}(x) \cdot t^{n}\right)^{h+1} \\
= & 2^{h} \cdot h!\cdot \sum_{n=0}^{\infty}\left(\sum_{a_{1}+a_{2}+\cdots+a_{h+1}=n} B_{a_{1}}(x) B_{a_{2}}(x) \cdots B_{a_{h+1}}(x)\right) \cdot t^{n} . \tag{6}
\end{align*}
$$

On the other hand, by the observation made in Lemma 3, it is deduced that

$$
\begin{align*}
& 2^{h} \cdot h!\cdot f^{h+1}(t)=\sum_{j=1}^{h} M(h, j) \cdot \frac{f^{(j)}(t)}{(3 x-t)^{2 h-j}} \\
= & \sum_{j=1}^{h} \frac{M(h, j)}{(3 x)^{2 h-j}} \cdot\left(\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot B_{n-i+j}(x) \cdot\binom{2 h+i-j-1}{i} \cdot \frac{1}{(3 x)^{i}}\right) t^{n}\right) \\
= & \sum_{n=0}^{\infty}\left(\sum_{j=1}^{h} \frac{M(h, j)}{(3 x)^{2 h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \frac{B_{n-i+j}(x)}{(3 x)^{i}} \cdot\binom{2 h+i-j-1}{i}\right) \cdot t^{n} . \tag{7}
\end{align*}
$$

Altogether, we obtain the identity:

$$
\begin{aligned}
& 2^{h} \cdot h!\sum_{a_{1}+a_{2}+\cdots+a_{h+1}=n} B_{a_{1}}(x) B_{a_{2}}(x) \cdots B_{a_{h+1}}(x) \\
= & \sum_{j=1}^{h} \frac{M(h, j)}{(3 x)^{2 h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \frac{B_{n-i+j}(x)}{(3 x)^{i}} \cdot\binom{2 h+i-j-1}{i} .
\end{aligned}
$$

This proves Theorem 1.

## 4. Conclusions

In this paper, a representation of a linear combination of balancing polynomials $B_{i}(x)$ (see Theorem 1) is obtained. Moreover, the specific expressions of $M(h, i)$ is given by using mathematical induction (see Lemma 1).

Theorem 1 can be reduced to various studies for the specific values of $x, n$, and $h$ in the literature. For example, if $n=0$, our results reduce to Corollary 1. Taking $h=2$, our results reduce to Corollary 2. Taking $x=1, h=2,3$, our results reduce to Corollary 3 and Corollary 4, respectively.

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