



Article A New Identity Involving Balancing Polynomials and Balancing Numbers

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Abstract: In this paper, a second-order nonlinear recursive sequence M(h, i) is studied. By using this sequence, the properties of the power series, and the combinatorial methods, some interesting symmetry identities of the structural properties of balancing numbers and balancing polynomials are deduced.

Keywords: balancing numbers; balancing polynomials; combinatorial methods; symmetry sums

1. Introduction

For any positive integer $n \ge 2$, we denote the balancing number by B_n and the balancer corresponding to it by r(n) if

$$1 + 2 + \dots + (B_n - 1) = (B_n + 1) + (B_n + 2) + \dots + (B_n + r(n))$$

holds for some positive integer r(n) and B_n . It is clear that $r(n) = \frac{B_n - B_{n-1} - 1}{2}$, for example, r(2) = 2, r(3) = 14, r(4) = 84, r(5) = 492...

It is found that the balancing numbers satisfy the second order linear recursive sequence $B_{n+1} = 6B_n - B_{n-1}$ ($n \ge 1$), providing $B_0 = 0$ and $B_1 = 1$ [1].

The balancing polynomials $B_n(x)$ are defined by $B_0(x) = 1$, $B_1(x) = 6x$, $B_2(x) = 36x^2 - 1$, $B_3(x) = 216x^3 - 12x$, $B_4(x) = 1296x^4 - 108x^2 + 1$, and the second-order linear difference equation:

$$B_{n+1}(x) = 6xB_n(x) - B_{n-1}(x), n \ge 1,$$

where *x* is any real number. While $n \ge 1$, we get $B_{n+1} = 6B_n - B_{n-1}$ with $B_n(1) = B_{n+1}$. Such balancing numbers have been widely studied in recent years. G. K. Panda and T. Komatsu [2] studied the reciprocal sums of the balancing numbers and proved the following inequation holds for any positive integer *n*:

$$rac{1}{B_n - B_{n-1}} < \sum_{k=n}^{\infty} rac{1}{B_k} < rac{1}{B_n - B_{n-1} - 1}.$$

G. K. Panda [3] studied some fascinating properties of balancing numbers and gave the following result for any natural numbers m > n:

$$(B_m + B_n)(B_m - B_n) = B_{m+n} \cdot B_{m-n}.$$

Other achievements related to balancing numbers can be found in [4–7].

It is found that the balancing polynomials $B_n(x)$ can be generally expressed as

$$B_n(x) = \frac{1}{2\sqrt{9x^2 - 1}} \left[\left(3x + \sqrt{9x^2 - 1} \right)^{n+1} - \left(3x - \sqrt{9x^2 - 1} \right)^{n+1} \right],$$

and the generating function of the balancing polynomials $B_n(x)$ is given by

$$\frac{1}{1 - 6xt + t^2} = \sum_{n=0}^{\infty} B_n(x) \cdot t^n.$$
 (1)

Recently, our attention was drawn to the sums of polynomials calculating problem [8–11], which is important in mathematical application. We are going to study the computational problem of the symmetry summation:

$$\sum_{a_2+\cdots+a_{h+1}=n} B_{a_1}(x) B_{a_2}(x) \cdots B_{a_{h+1}}(x),$$

where h is any positive integer. We shall prove the following theorem holds.

 a_1

Theorem 1. For any specific positive integer h and any integer $n \ge 0$, the following identity stands:

$$\sum_{a_1+a_2+\dots+a_{h+1}=n} B_{a_1}(x)B_{a_2}(x)\cdots B_{a_{h+1}}(x)$$

$$= \frac{1}{2^h \cdot h!} \cdot \sum_{j=1}^h \frac{M(h,j)}{(3x)^{2h-j}} \sum_{i=0}^n \frac{(n-i+j)!}{(n-i)!} \cdot \frac{B_{n-i+j}(x)}{(3x)^i} \cdot \begin{pmatrix} 2h+i-j-1\\i \end{pmatrix},$$

where M(h,i) is defined by M(h,0) = 0, $M(h,i) = \frac{(2h-i-1)!}{2^{h-i}\cdot(h-i)!\cdot(i-1)!}$ for all positive integers $1 \le i \le h$.

In particular, for n = 0, the following corollary can be deduced.

Corollary 1. For any positive integer $h \ge 1$, the following formula holds:

$$\sum_{j=1}^{h} M(h,j) \cdot j! \cdot (3x)^{j} \cdot B_{j}(x) = 2^{h} \cdot h! \cdot (3x)^{2h}.$$

The formula in Corollary 1 shows the close relationship among the balancing polynomials. For h = 2, the following corollary can be inferred by Theorem 1.

Corollary 2. *For any integer* $n \ge 0$ *, we obtain*

$$\sum_{a+b+c=n} B_a(x) \cdot B_b(x) \cdot B_c(x) = \frac{1}{216x^3} \sum_{i=0}^n (n-i+1)(i+1)(i+2) \cdot \frac{B_{n-i+2}}{(3x)^i} \\ + \frac{1}{72x^2} \sum_{i=0}^n (n-i+1)(n-i+2)(i+1) \cdot \frac{B_{n-i+3}}{(3x)^i}.$$

For x = 1, h = 2 and 3, according to Theorem 1 we can also infer the following corollaries:

Corollary 3. *For any integer* $n \ge 0$ *, we obtain*

$$\sum_{a+b+c=n} B_{a+1} \cdot B_{b+1} \cdot B_{c+1} = \frac{1}{216} \sum_{i=0}^{n} (n-i+1)(i+1)(i+2) \cdot \frac{B_{n-i+2}}{3^{i}} + \frac{1}{72} \sum_{i=0}^{n} (n-i+1)(n-i+2)(i+1) \cdot \frac{B_{n-i+3}}{3^{i}}.$$

Corollary 4. *For any integer* $n \ge 0$ *, we obtain:*

$$\sum_{a+b+c+d=n} B_{a+1} \cdot B_{b+1} \cdot B_{c+1} \cdot B_{d+1}$$

$$= \frac{1}{3888} \sum_{i=0}^{n} (n-i+1)(i+1)(i+2)(i+3)(i+4) \cdot \frac{B_{n-i+2}}{3^{i}}$$

$$+ \frac{1}{1296} \sum_{i=0}^{n} (n-i+1)(n-i+2)(i+1)(i+2)(i+3) \cdot \frac{B_{n-i+3}}{3^{i}}$$

$$+ \frac{1}{1296} \sum_{i=0}^{n} (n-i+1)(n-i+2)(n-i+3)(i+1)(i+2) \cdot \frac{B_{n-i+4}}{3^{i}}$$

Corollary 5. For any odd prime p, we have the congruence $M(p,i) \equiv 0 \pmod{p}, 0 \le i \le p-1$.

Corollary 6. The balancing polynomials are essentially Chebyshev polynomials of the second kind, specifically $B_n(x) = U_n(3x)$. Taking $x = \frac{1}{3}x$ in Theorem 1, we can get the following:

$$\sum_{a_1+a_2+\dots+a_{h+1}=n} U_{a_1}(x)U_{a_2}(x)\cdots U_{a_{h+1}}(x)$$

$$= \frac{1}{2^h \cdot h!} \cdot \sum_{j=1}^h \frac{(2h-j-1)!}{2^{h-j} \cdot (h-j)! \cdot (j-1)! \cdot x^{2h-j}} \sum_{i=0}^n \frac{(n-i+j)!}{(n-i)!} \cdot \frac{U_{n-i+j}(x)}{x^i} \cdot \begin{pmatrix} 2h+i-j-1\\i \end{pmatrix}.$$

Compared with [8], we give a more precise result for $\sum_{a_1+a_2+\cdots+a_{h+1}=n} U_{a_1}(x)U_{a_2}(x)\cdots U_{a_{h+1}}(x)$ with the specific expressions of M(h, i). This shows our novelty.

Here, we list the first several terms of M(h, i) in Table 1 in order to demonstrate the properties of the sequence M(h, i) clearly.

M(h,i)	<i>i</i> =1	<i>i</i> =2	<i>i</i> =3	<i>i</i> =4	i=5	<i>i</i> =6	<i>i</i> =7	<i>i</i> =8
h=1	1							
h=2	1	1						
h = 3	3	3	1					
$h\!=\!4$	15	15	6	1				
h = 5	105	105	45	10	1			
h = 6	945	945	420	105	15	1		
h = 7	10,395	10,395	4725	1260	210	21	1	
h = 8	135,135	135,135	62,370	17,325	3150	378	28	1

Table 1. Values of M(h, i).

2. Several Lemmas

For the sake of clarity, several lemmas that are necessary for proving our theorem will be given in this section.

Lemma 1. For the sequence M(n, i), the following identity holds for all $1 \le i \le n$:

$$M(n,i) = \frac{(2n-i-1)!}{2^{n-i} \cdot (n-i)! \cdot (i-1)!}$$

Proof. We present a straightforward proof of this lemma by using mathematical introduction. It is obvious that

$$M(1,1) = \frac{0!}{1 \cdot 0! \cdot 0!} = 1.$$

This means Lemma 1 is valid for n = 1. Without loss of generality, we assume that Lemma 1 holds for $1 \le n = h$ and all $1 \le i \le h$. Then, we have

$$M(h,i) = \frac{(2h-i-1)!}{2^{h-i} \cdot (h-i)! \cdot (i-1)!},$$
$$M(h,i+1) = \frac{(2h-i-2)!}{2^{h-i-1} \cdot (h-i-1)! \cdot i!}$$

According to the definitions of M(n, i), it is easy to find that

$$\begin{split} M(h+1,i+1) &= (2h-1-i) \cdot M(h,i+1) + M(h,i) \\ &= (2h-1-i) \cdot \frac{2(h-i)}{(2h-i-1)i} \cdot M(h,i) + M(h,i) \\ &= \frac{2h-i}{i} M(h,i) = \frac{(2h-i)!}{2^{h-i} \cdot (h-i)! \cdot i!} \\ &= \frac{(2(h+1)-(i+1)-1)!}{2^{h-i} \cdot (h-i)! \cdot i!}. \end{split}$$

Thus, Lemma 1 is also valid for n = h + 1. From now on, Lemma 1 has been proved. \Box

Lemma 2. If we have a function $f(t) = \frac{1}{1-6xt+t^2}$, then for any positive integer *n*, real numbers *x* and *t* with |t| < |3x|, the following identity holds:

$$2^{n} \cdot n! \cdot f^{n+1}(t) = \sum_{i=1}^{n} M(n,i) \cdot \frac{f^{(i)}(t)}{(3x-t)^{2n-i}},$$

where $f^{(i)}(t)$ denotes the *i*-th order derivative of f(t), with respect to variable t and M(n, i), which is defined in the theorem.

Proof. Similarly, Lemma 2 will be proved by mathematical induction. We start by showing that Lemma 2 is valid for n = 1. Using the properties of the derivative, we have:

$$f'(t) = (6x - 2t) \cdot f^2(t),$$

or

$$2f^{2}(t) = \frac{f'(t)}{3x - t} = M(1, 1) \cdot \frac{f'(t)}{3x - t}.$$

This is in fact true and provides the main idea to show the following steps. Without loss of generality, we assume that Lemma 2 holds for $1 \le n = h$. Then, we have

$$2^{h} \cdot h! \cdot f^{h+1}(t) = \sum_{i=1}^{h} M(h,i) \cdot \frac{f^{(i)}(t)}{(3x-t)^{2h-i}}.$$
(2)

As an immediate consequence, we can tell by (2), the properties of M(n, i), and the derivative, we get

$$2^{h} \cdot (h+1)! \cdot f^{h}(t) \cdot f'(t) = 2^{h+1} \cdot (h+1)! \cdot (3x-t) \cdot f^{h+2}(t)$$
$$= \sum_{i=1}^{h} \frac{M(h,i)}{(3x-t)^{2h-i}} \cdot f^{(i+1)}(t) + \sum_{i=1}^{h} \frac{(2h-i)M(h,i)}{(3x-t)^{2h-i+1}} \cdot f^{(i)}(t)$$

$$= \frac{M(h,h)}{(3x-t)^{h}} \cdot f^{(h+1)}(t) + \sum_{i=1}^{h-1} \frac{M(h,i)}{(3x-t)^{2h-i}} \cdot f^{(i+1)}(t) + \frac{(2h-1)M(h,1)}{(3x-t)^{2h}} \cdot f'(t) \\ + \sum_{i=1}^{h-1} \frac{(2h-i-1)M(h,i+1)}{(3x-t)^{2h-i}} \cdot f^{(i+1)}(t) \\ = \frac{M(h+1,h+1)}{(3x-t)^{h}} \cdot f^{(h+1)}(t) + \frac{M(h+1,1)}{(3x-t)^{2h}} \cdot f'(t) + \sum_{i=1}^{h-1} \frac{M(h+1,i+1)}{(3x-t)^{2h-i}} \cdot f^{(i+1)}(t) \\ = \frac{M(h+1,h+1)}{(3x-t)^{h}} \cdot f^{(h+1)}(t) + \frac{M(h+1,1)}{(3x-t)^{2h}} \cdot f'(t) + \sum_{i=2}^{h-1} \frac{M(h+1,i)}{(3x-t)^{2h-i}} \cdot f^{(i)}(t) \\ = \sum_{i=1}^{h+1} M(h+1,i) \cdot \frac{f^{(i)}(t)}{(3x-t)^{2h+1-i}}.$$
(3)

Then, it is deduced that

$$2^{h+1} \cdot (h+1)! \cdot (3x-t) \cdot f^{h+2}(t) = \sum_{i=1}^{h+1} M(h+1,i) \cdot \frac{f^{(i)}(t)}{(3x-t)^{2h+1-i}},$$

or

$$2^{h+1} \cdot (h+1)! \cdot f^{h+2}(t) = \sum_{i=1}^{h+1} M(h+1,i) \cdot \frac{f^{(i)}(t)}{(3x-t)^{2h+2-i}}.$$

Thus, Lemma 2 is also valid for n = h + 1. From now on, Lemma 2 has been proved. \Box

Lemma 3. The following power series expansion holds for arbitrary positive integers h and k:

$$\frac{f^{(h)}(t)}{(3x-t)^k} = \frac{1}{(3x)^k} \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \frac{(n-i+h)!}{(n-i)!} \cdot \frac{B_{n-i+h}(x)}{(3x)^i} \cdot \begin{pmatrix} i+k-1\\i \end{pmatrix} \right) t^n,$$

where t and x are any real numbers with |t| < |3x|.

Proof. According to the definition of the balancing polynomials $B_n(x)$, we have:

$$f(t) = \frac{1}{1 - 6xt + t^2} = \sum_{n=0}^{\infty} B_n(x) \cdot t^n.$$

For any positive integer *h*, from the properties of the power series, we can obtain

$$f^{(h)}(t) = \sum_{n=0}^{\infty} (n+h)(n+h-1)\cdots(n+1) \cdot B_{n+h}(x) \cdot t^{n}$$

=
$$\sum_{n=0}^{\infty} \frac{(n+h)!}{n!} \cdot B_{n+h}(x) \cdot t^{n}.$$
 (4)

For all real *t* and *x* with |t| < |3x|, we have the following power series expansion:

$$\frac{1}{3x-t} = \frac{1}{3x} \cdot \sum_{n=0}^{\infty} \frac{t^n}{(3x)^n},$$

and

$$\frac{1}{(3x-t)^k} = \frac{1}{(3x)^k} \cdot \sum_{n=0}^{\infty} \left(\begin{array}{c} n+k-1\\ n \end{array} \right) \cdot \frac{t^n}{(3x)^n},$$
(5)

with any positive integer *k*. Then, it is found that

$$\begin{aligned} & \frac{f^{(h)}(t)}{(3x-t)^k} \\ &= \frac{1}{(3x)^k} \cdot \left(\sum_{n=0}^{\infty} \frac{(n+h)!}{n!} \cdot B_{n+h}(x) \cdot t^n\right) \left(\sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot \frac{t^n}{(3x)^n}\right) \\ &= \frac{1}{(3x)^k} \sum_{n=0}^{\infty} \left(\sum_{i+j=n} \frac{(j+h)!}{j!} \cdot B_{j+h}(x) \cdot \binom{i+k-1}{i} \cdot \frac{1}{(3x)^i}\right) t^n \\ &= \frac{1}{(3x)^k} \sum_{n=0}^{\infty} \left(\sum_{i=0}^n \frac{(n-i+h)!}{(n-i)!} \cdot B_{n-i+h}(x) \cdot \binom{i+k-1}{i} \cdot \frac{1}{(3x)^i}\right) t^n, \end{aligned}$$

where we have used the multiplicative of the power series. Lemma 3 has been proved. \Box

3. Proof of Theorem

Based on the lemmas in the above section, it is easy to deduce the proof of Theorem 1. For any positive integer *h*, we can derive

$$2^{h} \cdot h! \cdot f^{h+1}(t) = 2^{h} \cdot h! \cdot \left(\sum_{n=0}^{\infty} B_{n}(x) \cdot t^{n}\right)^{h+1}$$

= $2^{h} \cdot h! \cdot \sum_{n=0}^{\infty} \left(\sum_{a_{1}+a_{2}+\dots+a_{h+1}=n} B_{a_{1}}(x) B_{a_{2}}(x) \cdots B_{a_{h+1}}(x)\right) \cdot t^{n}.$ (6)

On the other hand, by the observation made in Lemma 3, it is deduced that

$$2^{h} \cdot h! \cdot f^{h+1}(t) = \sum_{j=1}^{h} M(h,j) \cdot \frac{f^{(j)}(t)}{(3x-t)^{2h-j}}$$

$$= \sum_{j=1}^{h} \frac{M(h,j)}{(3x)^{2h-j}} \cdot \left(\sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot B_{n-i+j}(x) \cdot \left(\begin{array}{c} 2h+i-j-1\\i\end{array}\right) \cdot \frac{1}{(3x)^{i}}\right) t^{n}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=1}^{h} \frac{M(h,j)}{(3x)^{2h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \frac{B_{n-i+j}(x)}{(3x)^{i}} \cdot \left(\begin{array}{c} 2h+i-j-1\\i\end{array}\right)\right) \cdot t^{n}.$$
(7)

Altogether, we obtain the identity:

$$2^{h} \cdot h! \sum_{a_{1}+a_{2}+\dots+a_{h+1}=n} B_{a_{1}}(x) B_{a_{2}}(x) \cdots B_{a_{h+1}}(x)$$

= $\sum_{j=1}^{h} \frac{M(h,j)}{(3x)^{2h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \frac{B_{n-i+j}(x)}{(3x)^{i}} \cdot \begin{pmatrix} 2h+i-j-1\\i \end{pmatrix}.$

This proves Theorem 1.

4. Conclusions

In this paper, a representation of a linear combination of balancing polynomials $B_i(x)$ (see Theorem 1) is obtained. Moreover, the specific expressions of M(h, i) is given by using mathematical induction (see Lemma 1).

Theorem 1 can be reduced to various studies for the specific values of x, n, and h in the literature. For example, if n = 0, our results reduce to Corollary 1. Taking h = 2, our results reduce to Corollary 2. Taking x = 1, h = 2, 3, our results reduce to Corollary 3 and Corollary 4, respectively. Funding: This work is supported by the N. S. F. (11771351) of China.

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References

- 1. Behera, A.; Panda, G.K. On the square roots of triangular numbers. *Fibonacci Quart.* 1999, 37, 98–105.
- 2. Panda, G.K.; Komatsu, T.; Davala, R.K. Reciprocal sums of sequences involving balancing and lucas-balancing numbers. *Math. Rep.* **2018**, *20*, 201–214.
- 3. Panda, G.K. Some fascinating properties of balancing numbers. *Congr. Numer.* 2009, 194, 185–189.
- Patel, B.K.; Ray, P.K. The period, rank and order of the sequence of balancing numbers modulo m. *Math. Rep.* 2016, 18, 395–401.
- 5. Komatsu, T.; Szalay, L. Balancing with binomial coefficients. *Int. J. Number. Theory* **2014**, *10*, 1729–1742. [CrossRef]
- 6. Finkelstein, R. The house problem. Am. Math. Mon. 1965, 72, 1082–1088. [CrossRef]
- 7. Ray, P.K. Balancing and Cobalancing Numbers. Ph.D. Thesis, National Institute of Technology, Rourkela, India, 2009.
- 8. Zhang, Y.X.; Chen, Z.Y. A new identity involving the Chebyshev polynomials. *Mathematics* **2018**, *6*, 244. [CrossRef]
- 9. Zhao, J.H.; Chen, Z.Y. Some symmetric identities involving Fubini polynomials and Euler numbers. *Symmetry* **2018**, *10*, 303.
- Ma, Y.K.; Zhang, W.P. Some identities involving Fibonacci polynomials and Fibonacci numbers. *Mathematics* 2018, 6, 334. [CrossRef]
- 11. Kim, D.S.; Kim, T. A generalization of power and alternating power sums to any Appell polynomials. *Filomat* **2017**, *1*, 141–157. [CrossRef]



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