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Results on Functions on Dedekind Multisets

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Abstract: Many real-life problems are well represented only by sets which allow repetition(s), such as the multiset. Although not limited to the following, such cases may arise in a database query, chemical structures and computer programming. The set of roots of a polynomial, say $f(x)$, has been found to correspond to a multiset, say F. If $f(x)$ and $g(x)$ are polynomials whose sets of roots respectively correspond to the multisets $F(x)$ and $G(x)$, the set of roots of their product, $f(x)g(x)$, corresponds to the multiset $F \uplus G$, which is the sum of multisets F and G. In this paper, some properties of the algebraic sum of multisets $\uplus$ and some results on selection are established. Also, the count function of the image of any function on Dedekind multisets is defined and some of its properties are established. Some applications of these multisets are also given.

Keywords: multisets; functions on multiset; selection operation; submultiset

MSC: 03B70

1. Introduction

Let us start with the words of Prof. Irina Cristea that motivated us for this study; in many cases *Symmetry plays a fundamental role in our daily lives and in the study of the structure of different objects in physics, chemistry, biology, mathematics, architecture, arts, sociology, linguistics, etc. For example, the structure of molecules is well explained by their symmetry properties, described by symmetry elements and symmetry operations. A symmetry operation is a change, a transformation after which certain objects remain invariant, such as rotations, reflections, inversions, or permutation operations. Until now, the most efficient way to better describe symmetry, is using mathematical tools offered by group theory.*

The notion of multiset can be traced to as far back as 1888 where Dedekind in [1] said that an element of a set may belongs to it more than once. In 1989, Blizard in his paper [2] developed a first-order two-sorted multi set theory for multisets that "contains" classical set theory. Later on, in 1993 [3], he identified a kind of multiset which is based on the function of the root set. This idea was primarily from the work of Dedekind in [1].

However, Syropoulos in [4] also studied various operations on multisets and extended his work to category of multisets. Wildberger in [5] considered the use of multisets in data structure and also related it to tropical mathematics and gave some applications of it responding to meet catalogue of orders emanating from various customers from a set of inventory of a sales company. Knuth [6] related multisets to various aspects of computer programming.

Multisets are furthermore studied in the form of, and substituted with, numerous concepts such as bag, fireset (**f**initely **r**epeated **e**lement **s**et), heap, bunches, etc. These concepts have all been studied by various mathematicians with different applications.

To be specific, Nazmul et al. [7] extended the study of multisets to multigroup and other related algebraic properties as in the classical group. Concepts such as multicosets, [8,9] symmetric multigroup, and many others have all been studied [10–12]. Yohanna and Simon studied symmetric groups under multiset in [13]. Congruences of Multialgebra were studie by Ameri and Rosenberg in [14]. Even if the list of related papers is long we would like to draw the readers attention only to some of them, e.g., [4,5,15,16].

In this paper, we present some of the operations on multisets and some applications to real-life problems, number theory and management [17].

2. Preliminaries

In this paper, we shall use X to denote a non-empty set.

Definition 1 ([7,10])**.** *A multiset M drawn from a set X is denoted by the count function $C_M\colon X \longrightarrow N$ defined by $C_M(x) = n \in N$, the multiplicity or number of occurrence of x in M, where N is the set of non-negative integers.*

Definition 2 ([7])**.** *Let multisets A and B be drawn from X. A is said to be a* submultiset *of B and is denoted $A \subseteq B$ if $C_A(x) \leq C_B(x)\ \forall x \in X$.*

Definition 3 ([7])**.** *The* root set *or* support *of a multiset M, which is denoted by M^*, is the set which contains the distinct elements in the multiset. Hence, M^* is the set of $x \in M$ such that $C_M(x) > 0$.*

A multiset M is called a *regular multiset* if $C_M(x) = C_M(y)\ \forall x, y \in M$. The count function of the intersection of two multisets A and B both drawn from X is denoted by $C_A(x) \cap C_B(x) = \min\{C_A(x), C_B(x)\}$ and that of their union is denoted $C_A(x) \cup C_B(x) = \max\{C_A(x), C_B(x)\}$.

Multisets A and B are said to be equal if and only if $C_A(x) = C_B(x)$. Denote by $[X]^{\alpha}$, all the multisets whose elements have the multiplicity not more than α and $MS(X)$ the *set of all multisets drawn from* X. An empty multiset ϕ is such that $C_{\phi}(x) = 0,\ \forall x \in X$. *Cardinality* of a multiset M is denoted by $|M| = \sum C_M(x),\ \forall x \in M$. The *peak element* $x \in M$ is such that $C_M(x) \geq C_M(y),\ \forall y \in M$.

Definition 4 ([18])**.** *i.* *Consider $A \in MS(X)$. The insertion of x into A results into a multiset denoted by $\mathcal{C} = x \biguplus A$ which has the count function*

$$C_{\mathcal{C}}(y) = \begin{cases} C_A(y), & y \neq x \\ C_A(x) + 1, & y = x. \end{cases}$$

ii. *Consider $A, B \in MS(X)$. The insertion of A into B or of B into A results into a multiset $\mathcal{C}$ which has the count function denoted by $C_{\mathcal{C}}(x) = C_A(x) + C_B(x)$.*

It should be noted that the operation of insertion ($\biguplus$) on the set of all multisets drawn from X, that is $MS(X)$, is commutative and associative.

Definition 5 ([18])**.** *i.* *Consider $A \in MS(X)$. The removal of x from A results into a multiset denoted by $\mathcal{D} = A \ominus x$ which has the count function*

$$C_{\mathcal{D}}(y) = \begin{cases} \max\{C_A(y) - 1, 0\}, & y = x \\ C_A(y), & y \neq x. \end{cases}$$

ii. *Consider $A, B \in MS(X)$. The removal of B from A results into a multiset which has the count function denoted by $C_{\mathcal{D}}(x) = \max\{C_A(x) - C_B(x), 0\}$.*

It should be noted that the removal operation is neither commutative nor associative. Besides, it is also possible to make some kind of selection in multisets using the following operations.

Definition 6 ([10,18])**.** *Consider $A \in MS(X)$ and $B \subseteq X$.*

i. *The multiset $\mathcal{E} = A \otimes B$ is such that $\mathcal{E}$ only contains elements of A which also occur in B. The count function of $\mathcal{E}$ is denoted by*

$$C_{\mathcal{E}}(x) = \begin{cases} C_A(x), \ x \in \ B \\ \\ 0, \qquad x \notin B. \end{cases}$$

ii. *The multiset $\mathcal{F} = A \bigodot B$ is such that $\mathcal{F}$ only contains elements of A which do not occur in B. The count function of $\mathcal{F}$ is denoted by*

$$C_{\mathcal{F}}(x) = \begin{cases} C_A(x), \ x \notin \ B \\ \\ 0, \qquad x \in B. \end{cases}$$

Operations defined above "$\otimes$" or "$\bigodot$" are called selection operations.

Definition 7 ([7])**.** *Let X be a group and $e \in X$ its identity. Then, $\forall x, y \in X$, a multiset M drawn from X is called a* multigroup *if*

i. $C_M(xy) \geq C_M(x) \wedge C_M(y)$,
ii. $C_M(x^{-1}) \geq C_M(x)$.

Remark 1. *The immediate consequenceof this is that $C_M(e) \geq C_M(x)$ and $C_M(x^{-1}) = C_M(x)$, for any $x \in X$. We shall call $MG(X)$ the* set of all multigroups drawn from X.

Example 1. *Let $X = \{1,2,3,4,5\}$. Let the multiset $M = \{1,1,1,1,2,2,3,3,3,4,4, 5,5\}$, $N = \{1,1,2,2,3,4,4,4,5\}$ and $W = \{1,1,1,2,3,3\}$ be drawn from X. It is also justifiable to say that X is the support of M and N. Furthermore, the root set of W is $W^* = \{1,2,3\}$.*

i. $C_N(1) = 2$, $C_N(2) = 2$, $C_N(3) = 1$, $C_N(4) = 3$ *and* $C_N(5) = 1$
ii. $M \cap N = \{1,1,2,2,3,4,4,5\}$. *When two or more multisets are intersected, the minimum multiplicity of the common elements is taken.*
iii. $M \cup N = \{1,1,1,1,2,2,3,3,3,4,4,4,5,5\}$. *When the union of two or more multisets is taken, the maximum multiplicity of the common elements is taken.*
iv. *Neither is $M \subseteq N$, since $C_M(x) \leq C_N(x)$ for all $x \in X$, nor $M \supseteq N$, since $C_M(x) \geq C_N(x)$ for all $x \in X$, but $W \subseteq M$, since $C_W(x) \leq C_M(x)$ for all $x \in X$.*

Example 2. *i.* *Let $G = \{1,-1,i,-i\}$ be a group with the usual multiplication. $M = \{1,1,1,-1,-1,-1,i,i,-i,-i\}$ is a multigroup since it satisfies Definition 7 and Remark 1.*
ii. *Consider the group $Z_3 = \{0,1,2\}$ with the modulo addition. $M = \{0,0,0,1,1,2,2\}$ is a multigroup.*

Definition 8 ([10])**.** *Let $A \in MS(X)$, where X is a group.*

i. $A_n = \{x : C_A(x) \geq n\}$;
ii. *We denote a multiset containing only one element x with multiplicity n as $[n]_x$–a* simple multiset;
iii. *The* complement of the multiset *$M \in [X]^\alpha$ denoted by M' is such that $C_{M'}(x) = \alpha - C_M(x)$;*
iv. $nA = \{x^n, \ \forall x \in A \ , n$ *is the multiplicity of each element that appears in* $nA\}$.

Example 3. *Let $X = \{0,1,2,3\}$ which is a group with respect to addition modulo 4 and $A = \{0,0,0,1,1,2,2\}$ a multiset.*

i. $2A = \{0,0,0,0,0,0,1,1,1,1,2,2,2,2\}$, *the multiplicities of each element in A is doubled;*

ii. $[2]_0 = \{0, 0\}$, *this multiset comprises only 0 and its multiplicity is 2;*
iii. $A_2 = A_1 = A_0 = \{0, 1, 2\}$ *and* $A_3 = \{0\}$. *These are the sets of elements of A respectively with multiplicities 2, 1, 0 and 3.*

Remark 2. *For a multigroup A drawn from a group X,* A_n *is a group, indeed the subgroup of X* [7].

Proposition 1 ([7], p. 645). *Let* $A, B \in MS(X)$ *and* $m, n \in \mathbb{N}$.

i. If $A \subseteq B$, then $A_n \subseteq B_n$;
ii. If $m \leq n$, then $A_m \supseteq A_n$;
iii. $(A \cap B)_n = A_n \cap B_n$;
iv. $(A \cup B)_n = A_n \cup B_n$;
v. $A = B$ if and only if $A_n = B_n$, $\forall n \in \mathbb{N}$.

Definition 9 ([7]). *Let X and Y be two nonempty sets such that* $f\colon X \longrightarrow Y$ *is a mapping. Consider the multisets* $M \in [X]^\alpha$ *and* $N \in [Y]^\alpha$. *Then,*

i. *the image of M under f denoted* $f(M)$ *has the count function*

$$C_{f(M)}(y) = \begin{cases} \bigvee_{f(x)=y} C_M(x), \text{ if } f^{-1}(y) \neq \emptyset \\ 0, \quad\quad \text{otherwise;} \end{cases}$$

ii. *the inverse image of N under f denoted* $f^{-1}(N)$ *has the count function* $C_{f^{-1}(N)}(x) = C_N[f(x)]$.

The following Propositions were proved in [7]. But we shall later show that the items (iv), (v) and (vii) are not true and that the Proposition 2 needs to be restated.

Proposition 2 ([7]). *Let X, Y and Z be three nonempty sets such that* $f\colon X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$ *are mappings. If* $M_i \in [X]^\alpha$, $N_i \in [Y]^\alpha$, $i \in I$ *then*

i. $M_1 \subseteq M_2 \Rightarrow f(M_1) \subseteq f(M_2)$;
ii. $f(\cup_{i \in I} M_i) = \cup_{i \in I} f(M_i)$;
iii. $N_1 \subseteq N_2 \Rightarrow f^{-1}(N_1) \subseteq f^{-1}(N_2)$;
iv. $f^{-1}(\cup_{i \in I} M_i) = \cup_{i \in I} f^{-1}(M_i)$;
v. $f^{-1}(\cap_{i \in I} M_i) = \cap_{i \in I} f^{-1}(M_i)$;
vi. $f(M_i) \subseteq N_j \Rightarrow M_j \subseteq f^{-1}(N_j)$;
vii. $g[f(M_i)] = [gf](M_i)$ *and* $f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j)$.

3. Some Illustrations of Properties of Operations on Multisets

In the following section we now introduce some new results and properties of defined operations.

Proposition 3. *The operation* $\uplus$ *in Definition 4(ii) is such that:*

i. *Let A be a multiset drawn a nonempty set X. The n insertion of A into itself denoted* $\biguplus_n A = nA$;
ii. $A \uplus A = A \Leftrightarrow A = \phi$.

Proof. i. By Definition 4(ii),

$$C_{\biguplus_n A}(x) = \underbrace{C_A(x) + C_A(x) + \cdots + C_A(x)}_{n \text{ times}} = nC_A(x) \ \forall x \in A;$$

ii. Assume that $A \uplus A = A$ but $A \neq \emptyset$. By (i), $A \uplus A = 2A$. Then, $2A = A$. This is not possible by Definition 4(ii). Thus, $A = \emptyset$.

Conversely, assume that $A = \emptyset$ but $A \uplus A \neq A = \emptyset$. Then $A \uplus A = 2A$. But $C_A(x) = 0;\ \forall x \in X$. Then, $C_{A \uplus A}(x) = C_A(x) + C_A(x) = 0$. This implies that $A \uplus A = \emptyset$. This is a contradiction. Hence, $A \uplus A = A$. □

Remark 3. *Some properties of the selection operation $\bigotimes$ in Definition 6(i) and $\bigodot$ in Definition 6(ii) will be illustrated here. Let X be a nonempty set, $A \in MS(X)$, $B \subseteq X$ and $E = A \bigodot B$.*

i. Let $D = A \bigotimes B$. If $B \subset A^*$ then $D \subseteq A$. If $A^* \subseteq B$ then $D = A$.
ii. If $B \subset A^*$ then $E \subseteq A$. If $A^* \subseteq B$ then $E = \emptyset$.
iii. $D^* \cup E^* = A^*$.

This will be illustrated by the following Examples.

Example 4. *Let $X = \{1,2,3,4,5\}$, $B = \{2,3,5\}$.*

i. If $A = \{1,1,1,2,3,3,3,4,4,5,5\}$. Then, $D = \{2,3,3,3,5,5\}$, $E = \{1,1,1,4,4\}$ and $A^* = \{1,2,3,4,5\}$. Obviously, $B \subseteq A^*$, $D \subseteq A$ and $E \subseteq A$.
ii. If, on the other hand, $A = \{3,3,3,5,5\}$, $A^* = \{3,5\}$. Then, $A^* \subseteq B$, $D = \{3,3,3,5,5\} = A$ and $E = \emptyset$.
iii. In [i], $D^* = \{2,3,5\}$ and $E^* = \{1,4\}$. Hence, $D^* \cup E^* = A^*$. Also, in [ii] $D^* = \{3,5\}$ and $E^* = \emptyset$. Hence, $D^* \cup E^* = A^*$.

Proposition 4. *Let A be a multiset drawn from X and $B \subseteq X$. Then, $A^* \cap B = (A \bigotimes B)^*$.*

Proof. Let $x \in (A^* \cap B)$. Then $x \in A^*$ and $x \in B$. Thus, $C_A(x) \neq 0$. Hence, $x \in A \bigotimes B$. It can be concluded that $x \in (A \bigotimes B)^*$, in which case $(A^* \cap B) \subseteq (A \bigotimes B)^*$.

Also, let $x \in (A \bigotimes B)^*$, then $C_{(A \bigotimes B)}(x) \neq 0$. This implies that $x \in B$ and $C_A(x) \neq 0$ in which case $x \in A^*$.Then $x \in (A^* \cap B)$. Thus, $(A \bigotimes B)^* \subseteq (A^* \cap B)$. □

Proposition 5. *Let A and B be multisets drawn from X.*

i. $A \cup B \subseteq A \uplus B$;
ii. $A \cup B = A \uplus B$ if $A \cap B = \emptyset$.

Proof. Note that for any non-negative integers n and m, $\max(n,m) \leq n+m$ and that $\max(n,m) = n+m$ if either m or n is 0. Let $C_A(x) = n$ and $C_B(x) = m$.

i. $C_{A \cup B}(x) = C_A(x) \vee C_B(x) = \max(n,m) \leq n+m = C_A(x) + C_B(x) = C_{A \uplus B}(x)$;
ii. Since $A \cap B = \emptyset$, if $x \in A$, $C_A(x) = n$ and $C_B(x) = 0$. On the other hand, if $x \in B$, $C_B(x) = m$ and $C_A(x) = 0$. $C_{A \cup B}(x) = C_A(x) \vee C_B(x) = \max(n,m) = n+m = C_A(x) + C_B(x) = C_{A \uplus B}(x)$. □

Recall the Definition of A_n and the complement of a multiset M denoted M' by [7]. For a non-empty set Y, define a *characteristic function*

$$\mu_Y(y) = \begin{cases} 1, \text{ if } y \in Y; \\ \\ 0, \text{ if } y \notin Y. \end{cases}$$

We now introduce the complement of A_n denoted by A'_n.

Definition 10. *Let A be a multiset drawn from X and A_n as defined in Definition 8(iii). $A'_n = \{x \in X : \mu_{A_n}(x) = 0$ and $C_M(x) < n\}$.*

The following results shows that A'_n is well-defined.

Proposition 6. *i.* *If $m \leq n$, then $A'_m \subseteq A'_n$*
ii. $A \subseteq B \Rightarrow \ B'_n \subseteq A'_n$.
iii. $A'_n \cup B'_n = (A \cap B)'_n$;
iv. $A'_n \cap B'_n = (A \cup B)'_n$.

Proof. ii. Since $A \subseteq B, C_B(x) \geq C_A(x)$. Let $x \in A_n, C_A(x) \geq n$. But $C_B(x) \geq C_A(x) \geq n$. Then $x \in B_n$ and that implies that $A_n \subseteq B_n$. From elementary set theory, $B'_n \subseteq A'_n$.

iii. Let $x \in (A \cap B)'_n$ then $C_{(A \cap B)}(x) < n$. Thus, $\min\{C_A(x), C_B(x)\} < n$ in which case, $C_A(x) < n$ or $C_B(x) < n$. We conclude that $x \in A'_n$ or $x \in B'_n$. Hence, $x \in (A'_n \cup B'_n)$ and $(A \cap B)'_n \subseteq (A'_n \cup B'_n)$. Now let $x \in (A'_n \cup B'_n)$. $x \in A'_n$ or $x \in B'_n$ or both. Then, $x \notin A_n$ or $x \notin B_n$ or not in both. $C_A(x) < n$ and $C_B(x) < n$. Therefore, $C_A(x) \wedge C_B(x) = C_{(A \cap B)}(x) < n$. We conclude that $x \in (A \cap B)'_n$ and $(A'_n \cup B'_n) \subseteq (A \cap B)'_n$.

iv. Let $x \in (A'_n \cap B'_n)$. Then, $x \in A'_n$ and $x \in B'_n$, which implies that $x \notin A_n$ and $x \notin B_n$. Thus, $C_A(x) < n$ and $C_B(x) < n$. As a result, $C_A(x) \vee C_B(x) = C_{A \cup B}(x) < n$. Consequently, $x \in (A \cup B)'_n \Rightarrow A'_n \cap B'_n \subseteq (A \cup B)'_n$.
On the other hand, let $x \in (A \cup B)'_n$. Then, $x \notin (A \cup B)_n$ and $C_{A \cup B}(x) < n$. Furthermore, if $C_{A \cup B}(x) = C_A(x) \vee C_B(x) < n$, then $C_A(x) < n$ and $C_B(x) < n$. The consequence is that $x \in A'_n$ and $x \in B'_n$. Hence, $x \in (A'_n \cap B'_n) \Rightarrow (A \cup B)'_n \subseteq (A'_n \cap B'_n)$. □

Then the complement of A_n denoted A'_n is well-defined.

4. Results on Function on Multisets

Dedekind, in his paper "Was sind und was sollen die Zahlen?" had said that "the frequency-number of an image is the number of its preimages" [1]. Hence, if there are n elements in a domain X (of a function f mapping X to Y) which are mapped to an element $y \in Y$, then y has frequency n so that it is an n-fold element of Y. This defines a kind of multiset (Dedekind's multiset) denoted by M_f and $|dom(f)| = |M_f|$. This fact will be illustrated by the following Example.

Definition 11. *Let $f : X \to Y$ be a mapping on two non-empty sets and $f : M \to N$ be a mapping on multisets M and N respectively drawn from X and Y. If $C_N(y) = C_M(x)$ for all $y \in Y$ such that $f(x) = y$, M and N are Dedekind's multisets.*

Example 5. *Let $X = \{-1, -1, -1, 1, 1, 2, 2\}$ and $f(x) = x^2$. Then, the Dedekind's multiset $M_f = \{1, 1, 1, 1, 1, 4, 4\}$.*

But, following the Definition by Nazmul et al. [7] which is stated in this article as Definition 2(i), if a multiset $M = \{-1, -1, -1, 1, 1, 2, 2\}$ is mapped by $f(x) = x^2, f(M) = \{1, 1, 1, 4, 4\}$. This is because $\emptyset \neq f^{-1}(1) = \pm 1$ and $\emptyset \neq f^{-1}(4) = \pm 2$; $\max\{C_M(1), C_M(-1)\} = 3$ and $\max\{C_M(2), C_M(-2)\} = 2$; thus $|M| \neq |f(M)|$. Hence, Nazmul et al.'s Definition of function on multisets fails for Dedekind's multisets.

Moreover, in Proposition 2, since $M_i \in [X]^\alpha$ and $N_i \in [Y]^\alpha$, and f maps X to Y, $f^{-1}(M_i)$ is undefined but $f^{-1}(N_i)$ is. Also, $f^{-1}g^{-1}[N_j]$ is not defined. There should be a multiset say $W_i \in [Z]^\alpha$ so that $f^{-1}g^{-1}[W_i]$ is defined. Hence, properties (iv), (v) and (vii) are not true. Against these backgrounds, there is the need to redefine Definition 2(i) and state some properties of this new Definition.

Definition 12. *Let X and Y be two non-empty sets and $f : X \longrightarrow Y$ a mapping such that $M \in [X]^\alpha$. Then, $C_{f(M)}(y) = \sum_{f^{-1}(y) \neq \emptyset f(x) = y} C_M(x)$.*

Proposition 7. *Let X, Y and Z be three nonempty sets such that $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are mappings. If $M_i \in [X]^\alpha$, $N_i \in [Y]^\alpha$ and $W_i \in [Z]^\alpha$ with $i \in I$, then*

i. $M_1 \subseteq M_2 \Rightarrow f(M_1) \subseteq f(M_2)$;
ii. $f(\cup_{i\in I} M_i) \supseteq \cup_{i\in I} f(M_i)$;
iii. $N_1 \subseteq N_2 \Rightarrow f^{-1}(N_1) \subseteq f^{-1}(N_2)$;
iv. $f^{-1}(\cup_{i\in I} N_i) = \cup_{i\in I} f^{-1}(N_i)$;
v. $f^{-1}(\cap_{i\in I} N_i) = \cap_{i\in I} f^{-1}(N_i)$;
vi. $f(M_i) \subseteq N_j \Rightarrow M_i \subseteq f^{-1}(N_j)$;
vii. $g[f(M_i)] = [gf](M_i)$ *and* $f^{-1}[g^{-1}(W_j)] = [gf]^{-1}(W_j)$.

Proof. i. Suppose $M_1 \subseteq M_2$ and let $y \in f(M_1)$. $C_{f(M_1)}(y) = \sum C_{M_1}(x) \leq \sum C_{M_2}(x) = C_{f(M_2)}(y)$;
ii. Note that $C_{\cup M_i}(x) \geq C_{M_i}(x) \Rightarrow \sum C_{\cup M_i}(x) \geq \sum C_{M_i}(x)$. Then, $C_{f(\cup M_i)}(y) = \sum C_{\cup M_i}(x) = \sum \vee C_{M_i}(x) \geq \vee \sum C_{M_i}(x) = \vee C_{f(M_i)}(y) = C_{\cup f(M_i)}(y)$;
iii. Let $M_1 \subseteq M_2$. Then, $f(M_1) = N_1 \subseteq N_2 = f(M_2)$. Let $x \in f^{-1}(N_1)$. $C_{f^{-1}(N_1)}(x) = C_{N_1}(f(x)) = \sum C_{M_1}(x) \leq \sum C_{M_2}(x) = C_{N_2}(f(x)) = C_{f^{-1}(N_2)}(x)$.
iv. $C_{f^{-1}\cup N_i}(x) = C_{\cup N_i}(f(x)) = \vee C_{N_i}(f(x)) = \vee C_{f^{-1}(N_i)}(x) = C_{\cup f^{-1}(N_i)}(x)$.
v. $C_{f^{-1}\cap N_i}(x) = C_{\cap N_i}(f(x)) = \wedge C_{N_i}(f(x)) = \wedge C_{f^{-1}(N_i)}(x) = C_{\cap f^{-1}(N_i)}(x)$.
vi. Let $f(M_i) \subseteq N_j$ and $x \in M_i$ such that $f(x) = y$. $C_{M_i}(x) = C_{f(M_i)}(y) \leq C_{N_j}(y) = C_{f^{-1}(N_j)}(x)$.
vii. $C_{g[f(M_i)]}(z) = \sum_{g^{-1}(z)\neq\emptyset g(y)=z} C_{f(M_i)}(y) = \sum_{g^{-1}(z)\neq\emptyset g(y)=z} \sum_{f^{-1}(y)\neq\emptyset f(x)=y} C_{M_i}(x) =$
$= \sum_{gf(x)=z} C_{M_i}(x) = C_{gf(M_i)}(z)$. □

The following Examples will illustrate some of the properties in Proposition 7.

Example 6. *Let* $X = \{-1, 1, 2\}$ *and* $f(x) = x^2$. *Let* $M_1 = \{-1, -1, 1, 12, 2\}$, $M_2 = \{1, 1, -2, -2, -2\}$, $f(M_i) = N_i; i = 1, 2$. *Then,*
$f(M_1 \cup M_2) = \{1, 1, 1, 1, 4, 4, 4, 4\}$ *and* $f(M_1) \cup f(M_2) = \{1, 1, 1, 1, 4, 4, 4\}$.

5. Some Applications of Operations on Multiset

This chapter offers some possible applications: we are going to draw our attention to applying "removal and selection" operations defined in Definitions 5 and 6, respectively. Those may be and are used for mathematical sorting and blacklisting.

Informatics understands the expressions blacklist or blocklist as lists containing something forbidden; conversely, the so-called whitelist is used to create a list of entities, which are to be allowed. Server lists (more precisely, their IP addresses) are among of the most common uses; it is unsuitable to receive e-mails from them because they send spams. The blacklist is also used in programs for instant messaging to create a list of users from which the information sent is not to be received; they can also serve for the same purpose on social networks. Similarly, a list of e-mail addresses from which messages are not to be received can be created in an e-mail client or at an e-mail provider.

Both blacklists and whitelists are sometimes used at the same time. A typical case is a situation when a program uses its algorithm; an antispam filter can be used as an example of such a program. It decides according to the e-mail content, whether it is spam. If a user needs to inform the filter that only spams and no useful e-mails are received from a particular e-mail address, he/she places this address on the blacklist. Then, the filter will automatically classify e-mails sent from that address as spams. On the other hand, if a user needs to inform the filter that he/she receives no spams from a particular address or he/she does not want to miss any important e-mail, he/she places this address on the whitelist. Then, the filter will treat all messages from that particular address as useful and will not classify them as spam.

We also used a so-called yellow list containing for instance the IP address list of e-mail servers, which predominantly send non-spam e-mails, however, sometimes there are some spam as well, e.g., yahoo, Hotmail and Gmail. The yellow list comprises servers, which should never appear on the blacklist (for Example because of mistakes or misprints). The yellow list is checked as the first one and if a server is listed there, blacklist tests are ignored afterwards.

Operations such as *removal* " $\ominus$" and *selection* " $\bigodot$" and " $\bigotimes$" are useful for the above mentioned applications: see previous sections of this paper. See also [6]. Let us give several Examples:

Example 7. *Consider the polynomial* $f(x) = (x-1)^2(x-2)(x-3)^3$. *The associated multiset of roots is* $F = \{1,1,2,3,3,3\}$. *If it is intended to get a polynomial* $g(x) = \dfrac{f(x)}{x-2}$, *the associated multiset of roots is* $G = F \ominus 2 = \{1,1,3,3,3\}$. *If it is intended to find a factor of the polynomial* $f(x)$ *which does not contain the linear factor* $x-3$, *the multiset* $B = \{3\}_{C_F(3)} = \{3,3,3\}$ *is defined and the removal operation* $F \ominus B$ *is performed.*

Example 8. *Consider a unique factorisation of a positive integer* N *into its primes with the associated multiset of primes*

$$M = \{x_1^{m_1}, x_2^{m_2}, x_3^{m_3}, \cdots, x_n^{m_n}\},$$

where m_i'*s are the multiplicities of* x_i'*s for* $1 \leq i \leq n$. *To have a number which is not a multiple of* x_i, *construct a multiset* $B = \{x_i\}_{m_i}$ *and perform* $M \ominus B$; *to have a number which is just* $\frac{N}{x_i}$, *perform* $M \ominus x_i$.

Example 9. *Again, consider the polynomial* $f(x) = (x-1)^2(x-2)(x-3)^3$ *with the associated multiset of roots* $F = \{1,1,2,3,3,3\}$. *If it is intended to get a polynomial in which all linear factors* $(x-3)$ *is eliminated, define a subset* $B = \{1,2\}$ *of* $X = \{1,2,3\}$ *and perform* $F \bigotimes B$ *and the new polynomial has the associated multiset of roots as* $G = F \bigotimes B = \{1,1,2\}$. *It is another way of getting a polynomial whose only factors are* $x-1$ *and* $x-2$.

Example 10. *If tag numbers were to be given to a set of 20 workers in a manufacturing company from the set of numbers* $X = \{1,2,3,4\}$ *with which they are allocated into four different work stations, and if the allocation of these personnel is the multiset* $A = \{1^6, 2^4, 3^4, 4^6\}$, *the code to blacklist any of the group would consist the selection operation* $A \bigodot B$, *where* B *is the subset of* X *consisting the tag number of the group to be blacklisted. For instance, if the company wishes to blacklist or remove from their payroll everyone carrying the tag number 1 and 2,* $B = \{1,2\}$ *and* $A \bigodot B = \{3^4, 4^6\}$.

6. Conclusions

The theory of multisets is an important generalization of the classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set just once. It can be used in many applications, e.g., data encryption, data mining, coding theory, decision making or to write a mathematical programme which could do some sorting of data. The algebraic structure of such data could be studied by applying group theory to a multiset.

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