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New Hermite–Hadamard Type Inequalities Involving Non-Conformable Integral Operators

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Abstract: At present, inequalities have reached an outstanding theoretical and applied development and they are the methodological base of many mathematical processes. In particular, Hermite–Hadamard inequality has received considerable attention. In this paper, we prove some new results related to Hermite–Hadamard inequality via symmetric non-conformable integral operators.

Keywords: Hermite–Hadamard type inequalities; non-conformable derivatives and integrals; fractional integral inequalities; convex functions

1. Introduction

The significant role of inequalities in the development and evolution of Mathematics is well known. Some basic notions related to them were already in use by the ancient Greeks, such as triangle and isoperimetric inequalities. However, inequalities were not employed either in arithmetic or any other kind of number manipulation [1]. The formalization of the Mathematical Theory of Inequalities essentially begins in the 18th century with the studies carried out by Gauss. It was continued by Cauchy, and Chebyshov, who had the idea to apply some inequalities to Mathematical Analysis. Later, the Russian mathematician Bunyakovsky, proved in 1859 the well-known Cauchy–Schwarz inequality for the case of infinite dimensions.

Likewise, the research conducted by Hardy on this subject should be recognized as particularly significant, since it went beyond particular inequalities. Hardy succeeded in gathering together the best mathematicians of the moment to solve problems related to inequalities. Furthermore, he founded the Journal of the London Mathematical Society, a magazine especially suitable to publish papers on inequalities. Together with renowned mathematicians such as Littlewood and Polya, he developed the famous volume entitled "Inequalities" [2], which was the first monograph on this subject.

The book became a milestone in the field of inequalities, and it achieved the goal of giving structure, systematization and formalization to an apparently isolated set of results, and, by doing so, it changed them into a theory. At present, inequalities have reached an outstanding theoretical and applied development and they are the methodological base of processes of approximation, estimation, boundedness, interpolation, etc. In general, they are fundamental in every modeling problem.

As usual, a function $f:\mathcal{I}\subseteq\mathbb{R}\to\mathbb{R}$ is said to be *convex* on the interval $\mathcal{I}$, if the inequality

$$f\big(tx+(1-t)y\big)\le tf(x)+(1-t)f(y)$$

holds for all $x, y \in \mathcal{I}$ and $t \in [0,1]$. We say that f is *concave* if $-f$ is convex. It is well known that every convex function is continuous and thus integrable on any compact interval.

Among many important inequalities involving convex functions, we will focus here on the following ones. If $f : I \to \mathbb{R}$ is a convex function on the interval I, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}$$

for every $a, b \in I$ with $a < b$.

The converse inequalities hold if the function f is concave on the interval I. This seminal result was proved in [3] and it is known as Hermite–Hadamard inequality (see [4,5] for more details). Since its discovery, this inequality has received considerable attention.

In recent years, this inequality has been generalized to conformable integrals in [6–14]. In addition, there are many works generalizing other classical inequalities from the fractional calculus viewpoint (see, e.g., [15,16]). The aim of this paper is to show some new results related to Hermite–Hadamard inequalities via non-conformable integrals.

The authors in [17] introduced a useful conformable derivative; in addition, a non-conformable derivative is introduced in [18]. These derivatives are interesting from a theoretical viewpoint and useful in many applications [19–21].

Next, we give the definition of the non-conformable derivative related to our results.

Definition 1. *Given an interval $I \subseteq [0,\infty)$, a function $f : I \to \mathbb{R}$, $\alpha \in (0,1)$ and $t \in I$, the* non-conformable derivative of f of order α at t *is defined by*

$$N_3^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^\alpha) - f(t)}{\varepsilon}\,.$$

We say that f is α-differentiable at t *if there exists $N_3^\alpha(f)(t)$ and it is finite.*

Note that if f is differentiable at t, then

$$N_3^\alpha(f)(t) = t^\alpha f'(t),$$

where $f'(t)$ denotes the usual derivative.

Following the ideas in [18], we can easily prove the next result.

Theorem 1. *Let $\alpha \in (0,1)$, $t > 0$ and f, g α-differentiable functions at t. Then:*
(1) $N_3^\alpha(af+bg)(t) = aN_3^\alpha(f)(t) + bN_3^\alpha(g)(t)$ *for all $a, b \in \mathbb{R}$,*
(2) $N_3^\alpha(fg)(t) = g(t)N_3^\alpha(f)(t) + f(t)N_3^\alpha(g)(t)$,
(3) $N_3^\alpha\left(\frac{f}{g}\right)(t) = \frac{g(t)N_3^\alpha(f)(t) - f(t)N_3^\alpha(g)(t)}{g(t)^2}$,
(4) $N_3^\alpha(c) = 0$ *for every constant function $c \in \mathbb{R}$,*
(5) $N_3^\alpha\left(\frac{1}{1-\alpha}t^{1-\alpha}\right) = 1$.

Definition 2. *Let $\alpha \in \mathbb{R}$ and $a < b$. We define the following linear spaces:*

$$L_{\alpha,0}[a,b] = \left\{ f : [a,b] \to \mathbb{R} \;\middle|\; |t-u|^{-\alpha} f(t) \in L^1[a,b] \text{ for every } u \in [a,b] \right\},$$
$$L_\alpha[a,b] = \left\{ f : [a,b] \to \mathbb{R} \;\middle|\; (t-a)^{-\alpha} f(t), (b-t)^{-\alpha} f(t) \in L^1[a,b] \right\}.$$

Note that, if $\alpha \le 0$, then $L_\alpha[a,b] = L^1[a,b]$.

Motivated by this non-conformable derivative, we define the non-conformable integrals that appear in the inequalities of this paper.

Definition 3. *Let $\alpha \in \mathbb{R}$ and $0 < a < b$. For each function $f \in L^1[a,b]$, we define*

$$ {}_{N_3}J_u^\alpha f(x) = \int_u^x t^{-\alpha} f(t)\, dt $$

for every $x, u \in [a,b]$.

Definition 4. *Let $\alpha \in \mathbb{R}$ and $a < b$. For each function $f \in L_{\alpha,0}[a,b]$, let us define the fractional integrals*

$$ \begin{aligned} {}_{N_3}J_{a^+}^\alpha f(x) &= \int_a^x (x-t)^{-\alpha} f(t)\, dt, \\ {}_{N_3}J_{b^-}^\alpha f(x) &= \int_x^b (t-x)^{-\alpha} f(t)\, dt, \end{aligned} $$

for every $x \in [a,b]$.

The symmetry of these non-conformable integral operators will allow for obtaining new results related to Hermite–Hadamard inequality.

2. Main Results

We start with an equality that will be useful.

Lemma 1. *Let $\alpha < 1$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function. If $f' \in L_{\alpha-1}[a,b]$, then*

$$ \frac{\alpha-1}{(b-a)^{2-\alpha}} \left[{}_{N_3}J_{b^-}^\alpha f(a) + {}_{N_3}J_{a^+}^\alpha f(b) \right] + \frac{f(b)+f(a)}{b-a} = I_0, $$

with

$$ I_0 = \int_0^1 \left[(1-t)^{1-\alpha} - t^{1-\alpha} \right] f'\big(at + (1-t)b\big)\, dt. $$

Proof. First of all, note that Hardy's inequalities

$$ \begin{aligned} \int_a^b (t-a)^{-\alpha} |f(t)|\, dt &\le K \int_a^b (t-a)^{1-\alpha} |f'(t)|\, dt, \\ \int_a^b (b-t)^{-\alpha} |f(t)|\, dt &\le K \int_a^b (b-t)^{1-\alpha} |f'(t)|\, dt \end{aligned} $$

give that $f \in L_\alpha[a,b]$ since $f' \in L_{\alpha-1}[a,b]$ and $\alpha < 1$.

We can write I_0 as follows:

$$ I_0 = \int_0^1 (1-t)^{1-\alpha} f'\big(at + (1-t)b\big)\, dt - \int_0^1 t^{1-\alpha} f'\big(at + (1-t)b\big)\, dt. $$

Integration by parts gives that the first integral is equal to

$$ \begin{aligned} \int_0^1 (1-t)^{1-\alpha} f'\big(at + (1-t)b\big)\, dt &= \frac{1}{b-a} f(b) - \frac{1-\alpha}{b-a} \int_0^1 (1-t)^{-\alpha} f\big(at + (1-t)b\big)\, dt \\ &= \frac{1}{b-a} f(b) - \frac{1-\alpha}{b-a} \int_a^b \left(\frac{x-a}{b-a} \right)^{-\alpha} \frac{f(x)}{b-a}\, dx \\ &= \frac{1}{b-a} f(b) + \frac{\alpha-1}{(b-a)^{2-\alpha}}\, {}_{N_3}J_{b^-}^\alpha f(a). \end{aligned} $$

We obtain, in a similar way,

$$\int_0^1 t^{1-\alpha} f'\big(at+(1-t)b\big)\,dt = \frac{-1}{b-a}\,f(a) - \frac{\alpha-1}{(b-a)^{2-\alpha}}\,{}_{N_3}J^{\alpha}_{a+}f(b).$$

These equalities give the desired result. □

Lemma 1 allows for proving several inequalities.

Proposition 1. *Let $\alpha<1$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function. If $f'\in L_{\alpha-1}[a,b]$ and $f'(s)\ge f'(a+b-s)$ for every $s\in[(a+b)/2,b]$, then*

$$\frac{1-\alpha}{(b-a)^{2-\alpha}}\left[{}_{N_3}J^{\alpha}_{b-}f(a)+{}_{N_3}J^{\alpha}_{a+}f(b)\right]\le\frac{f(b)+f(a)}{b-a}.$$

Proof. We have

$$\begin{aligned}
I_0 &= \int_0^{1/2}\left[(1-t)^{1-\alpha}-t^{1-\alpha}\right]f'\big(at+(1-t)b\big)\,dt \\
&\quad+\int_{1/2}^{1}\left[(1-s)^{1-\alpha}-s^{1-\alpha}\right]f'\big(as+(1-s)b\big)\,ds \\
&=\int_0^{1/2}\left[(1-t)^{1-\alpha}-t^{1-\alpha}\right]f'\big(at+(1-t)b\big)\,dt \\
&\quad+\int_0^{1/2}\left[t^{1-\alpha}-(1-t)^{1-\alpha}\right]f'\big(a(1-t)+bt\big)\,dt \\
&=\int_0^{1/2}\left[(1-t)^{1-\alpha}-t^{1-\alpha}\right]\left[f'\big(at+(1-t)b\big)-f'\big(a+b-at-(1-t)b\big)\right]dt\ge 0,
\end{aligned}$$

since the integrand is the product of two non-negative functions. Thus, Lemma 1 gives the inequality. □

Corollary 1. *Let $\alpha<1$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function. If $f'\in L_{\alpha-1}[a,b]$, and f is decreasing on $[a,(a+b)/2]$ and increasing on $[(a+b)/2,b]$, then*

$$\frac{1-\alpha}{(b-a)^{2-\alpha}}\left[{}_{N_3}J^{\alpha}_{b-}f(a)+{}_{N_3}J^{\alpha}_{a+}f(b)\right]\le\frac{f(b)+f(a)}{b-a}.$$

Proof. Since f is decreasing on $[a,(a+b)/2]$ and increasing on $[(a+b)/2,b]$, we have $f'(s)\ge 0\ge f'(a+b-s)$ for every $s\in[(a+b)/2,b]$, and Proposition 1 gives the inequality. □

Theorem 2. *Let $\alpha\in(0,1)$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function. If $f'\in L_{\alpha-1}[a,b]$ and $|f'|$ is a convex function, then*

$$\begin{aligned}
&\left|\frac{1-\alpha}{(b-a)^{2-\alpha}}\left[{}_{N_3}J^{\alpha}_{b-}f(a)+{}_{N_3}J^{\alpha}_{a+}f(b)\right]-\frac{f(a)+f(b)}{b-a}\right| \\
&\le\left(\frac{(1-\alpha)2^{\alpha-2}}{(2-\alpha)(3-\alpha)}+\frac{5}{24}\right)|f'(a)|+\left(\frac{(5-\alpha)2^{\alpha-2}}{(2-\alpha)(3-\alpha)}+\frac{1}{24}\right)|f'(b)|.
\end{aligned}$$

Proof. Since $\alpha\in(0,1)$, the function $u(x)=x^{1-\alpha}$ is concave, and thus $u(x)\le u(x_0)+u'(x_0)(x-x_0)$ for every $x\ge 0$ and $x_0>0$. Hence, $(1-t)^{1-\alpha}-t^{1-\alpha}\le(1-\alpha)(1-2t)t^{-\alpha}$ for every $t\in(0,1]$, and thus $\left|(1-t)^{1-\alpha}-t^{1-\alpha}\right|\le(1-\alpha)(1-2t)t^{-\alpha}$ for every $t\in(0,1/2]$, since $(1-t)^{1-\alpha}-t^{1-\alpha}\ge 0$ on $(0,1/2]$.

If we define $g(t)=t^{1-\alpha}-(1-t)^{1-\alpha}$, then

$$g''(t)=\alpha(1-\alpha)\big((1-t)^{-\alpha-1}-t^{-\alpha-1}\big)\ge 0$$

for every $t \in [1/2, 1)$, and, thus, since $h(t) = 2t - 1$ satisfies $h(1/2) = 0 = g(1/2)$ and $h(1) = 1 = g(1)$, we conclude $t^{1-\alpha} - (1-t)^{1-\alpha} \le 2t - 1$ for every $t \in [1/2, 1]$. Therefore, $\left|(1-t)^{1-\alpha} - t^{1-\alpha}\right| \le 2t - 1$ for every $t \in [1/2, 1]$.

Then, we have

$$\begin{aligned} I_0 &= \int_0^1 \left[(1-t)^{1-\alpha} - t^{1-\alpha}\right] f'\big(at + (1-t)b\big)\, dt \\ |I_0| &\le \int_0^{1/2} (1-\alpha)(1-2t)t^{-\alpha} \left|f'\big(at + (1-t)b\big)\right| dt \\ &\quad + \int_{1/2}^1 (2t-1) \left|f'\big(at + (1-t)b\big)\right| dt. \end{aligned}$$

Since $|f'|$ is a convex function, we obtain

$$\begin{aligned} &\int_0^{1/2} (1-2t)t^{-\alpha} \left|f'\big(at + (1-t)b\big)\right| dt \\ &\le \int_0^{1/2} (1-2t)t^{-\alpha} t \left|f'(a)\right| dt + \int_0^{1/2} (1-2t)t^{-\alpha}(1-t) \left|f'(b)\right| dt \\ &= \left|f'(a)\right| \left[\frac{t^{2-\alpha}}{2-\alpha} - 2\,\frac{t^{3-\alpha}}{3-\alpha}\right]_0^{1/2} + \left|f'(b)\right| \left[\frac{t^{1-\alpha}}{1-\alpha} - 3\,\frac{t^{2-\alpha}}{2-\alpha} + 2\,\frac{t^{3-\alpha}}{3-\alpha}\right]_0^{1/2} \\ &= \frac{2^{\alpha-2}}{(2-\alpha)(3-\alpha)} \left|f'(a)\right| + \frac{(5-\alpha)2^{\alpha-2}}{(1-\alpha)(2-\alpha)(3-\alpha)} \left|f'(b)\right|, \end{aligned}$$

and

$$\begin{aligned} &\int_{1/2}^1 (2t-1) \left|f'\big(at + (1-t)b\big)\right| dt \\ &\le \int_{1/2}^1 (2t-1)t \left|f'(a)\right| dt + \int_{1/2}^1 (2t-1)(1-t) \left|f'(b)\right| dt \\ &= \left|f'(a)\right| \left[\frac{2t^3}{3} - \frac{t^2}{2}\right]_{1/2}^1 + \left|f'(b)\right| \left[\frac{-2t^3}{3} + \frac{3t^2}{2} - t\right]_{1/2}^1 \\ &= \frac{5}{24} \left|f'(a)\right| + \frac{1}{24} \left|f'(b)\right|. \end{aligned}$$

Hence,

$$|I_0| \le \left(\frac{(1-\alpha)2^{\alpha-2}}{(2-\alpha)(3-\alpha)} + \frac{5}{24}\right) \left|f'(a)\right| + \left(\frac{(5-\alpha)2^{\alpha-2}}{(2-\alpha)(3-\alpha)} + \frac{1}{24}\right) \left|f'(b)\right|,$$

and Lemma 1 gives the inequality. □

The argument in the proof of Theorem 2 also allows for dealing with the case $\alpha \le 0$.

Theorem 3. *Let $\alpha < 0$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function. Assume that $f' \in L_{\alpha-1}[a, b]$ and $|f'|$ is a convex function.*

(1) *If $\alpha \in (-1, 0)$, then*

$$\begin{aligned} &\left| \frac{1-\alpha}{(b-a)^{2-\alpha}} \left[{}_{N_3}J^{\alpha}_{b-} f(a) + {}_{N_3}J^{\alpha}_{a+} f(b) \right] - \frac{f(a) + f(b)}{b-a} \right| \\ &\le \left((1-\alpha) \frac{2^{\alpha-2} + 1 - \alpha}{(2-\alpha)(3-\alpha)} + \frac{2+\alpha}{24} \right) \left|f'(a)\right| + \left(\frac{(5-\alpha)2^{\alpha-2} - 1 - \alpha}{(2-\alpha)(3-\alpha)} + \frac{7+2\alpha}{24} \right) \left|f'(b)\right|. \end{aligned}$$

(2) *If* $\alpha \leq -1$, *then*

$$\begin{aligned}&\left| \frac{1-\alpha}{(b-a)^{2-\alpha}} \left[{}_{N_3}J^{\alpha}_{b-} f(a) + {}_{N_3}J^{\alpha}_{a+} f(b) \right] - \frac{f(a)+f(b)}{b-a} \right| \\ &\leq \Big((1-\alpha) \frac{2^{\alpha-2}+1-\alpha}{(2-\alpha)(3-\alpha)} + \frac{1}{24} \Big) \left| f'(a) \right| + \Big(\frac{(5-\alpha)2^{\alpha-2}-1-\alpha}{(2-\alpha)(3-\alpha)} + \frac{5}{24} \Big) \left| f'(b) \right| .\end{aligned}$$

Proof. Since $\alpha < 0$, the function $u(x) = x^{1-\alpha}$ is convex, and thus $u(x) \geq u(x_0) + u'(x_0)(x - x_0)$ for every $x, x_0 \geq 0$. Hence, $(1-t)^{1-\alpha} - t^{1-\alpha} \geq (1-\alpha)(1-2t)t^{-\alpha}$ for every $t \in [0,1]$, and thus $\left|(1-t)^{1-\alpha} - t^{1-\alpha}\right| \leq (1-\alpha)(2t-1)t^{-\alpha}$ for every $t \in [1/2,1]$, since $(1-t)^{1-\alpha} - t^{1-\alpha} \leq 0$ on $[1/2,1]$.

If we define $G(t) = (1-t)^{1-\alpha} - t^{1-\alpha}$, then $G(t) \geq 0$ for every $t \in [0,1/2]$ and

$$G''(t) = -\alpha(1-\alpha)\big((1-t)^{-\alpha-1} - t^{-\alpha-1}\big).$$

If $\alpha \leq -1$, then $G''(t) \geq 0$ for every $t \in (0,1/2]$, and, thus, since $H(t) = 1-2t$ satisfies $H(0) = 1 = G(0)$ and $H(1/2) = 0 = G(1/2)$, we conclude $(1-t)^{1-\alpha} - t^{1-\alpha} \leq 1-2t$ for every $t \in [0,1/2]$. Therefore, $\left|(1-t)^{1-\alpha} - t^{1-\alpha}\right| \leq 1-2t$ for every $t \in [0,1/2]$. Note that this inequality also holds for $\alpha = 0$.

If $\alpha \in (-1,0)$, then $G''(t) \leq 0$ for every $t \in (0,1/2]$, and so, $G(t) \leq G(0) + G'(0)t = 1 + (\alpha-1)t$ for every $t \in [0,1/2]$. Therefore, $\left|(1-t)^{1-\alpha} - t^{1-\alpha}\right| \leq 1 - (1-\alpha)t$ for every $t \in [0,1/2]$.

Hence, we have

$$\begin{aligned}|I_0| \leq \int_0^{1/2} &\left|(1-t)^{1-\alpha} - t^{1-\alpha}\right| \left|f'\big(at+(1-t)b\big)\right| dt \\ &+ \int_{1/2}^{1} (1-\alpha)(2t-1)t^{-\alpha} \left|f'\big(at+(1-t)b\big)\right| dt.\end{aligned}$$

Since $|f'|$ is a convex function, we obtain

$$\begin{aligned}&\int_{1/2}^{1} (2t-1)t^{-\alpha} \left|f'\big(at+(1-t)b\big)\right| dt \\ &\leq \int_{1/2}^{1} (2t-1)t^{-\alpha} t \left|f'(a)\right| dt + \int_{1/2}^{1} (2t-1)t^{-\alpha}(1-t) \left|f'(b)\right| dt \\ &= \left|f'(a)\right| \Big[\frac{-t^{2-\alpha}}{2-\alpha} + 2\,\frac{t^{3-\alpha}}{3-\alpha} \Big]_{1/2}^{1} + \left|f'(b)\right| \Big[\frac{-t^{1-\alpha}}{1-\alpha} + 3\,\frac{t^{2-\alpha}}{2-\alpha} - 2\,\frac{t^{3-\alpha}}{3-\alpha} \Big]_{1/2}^{1} \\ &= \frac{2^{\alpha-2}+1-\alpha}{(2-\alpha)(3-\alpha)} \left|f'(a)\right| + \frac{(5-\alpha)2^{\alpha-2}-1-\alpha}{(1-\alpha)(2-\alpha)(3-\alpha)} \left|f'(b)\right| .\end{aligned}$$

If $\alpha \in (-1,0)$, then

$$\begin{aligned}&\int_0^{1/2} \left|(1-t)^{1-\alpha} - t^{1-\alpha}\right| \left|f'\big(at+(1-t)b\big)\right| dt \\ &\leq \int_0^{1/2} \big(1-(1-\alpha)t\big) t \left|f'(a)\right| dt + \int_0^{1/2} \big(1-(1-\alpha)t\big)(1-t) \left|f'(b)\right| dt \\ &= \left|f'(a)\right| \Big[\frac{t^2}{2} - (1-\alpha)\frac{t^3}{3} \Big]_0^{1/2} + \left|f'(b)\right| \Big[t - (2-\alpha)\frac{t^2}{2} + (1-\alpha)\frac{t^3}{3} \Big]_0^{1/2} \\ &= \frac{2+\alpha}{24} \left|f'(a)\right| + \frac{7+2\alpha}{24} \left|f'(b)\right| .\end{aligned}$$

If $\alpha \leq -1$, then

$$\begin{aligned}
&\int_0^{1/2} \left|(1-t)^{1-\alpha} - t^{1-\alpha}\right| \left|f'\big(at+(1-t)b\big)\right| dt \\
&\quad \leq \int_0^{1/2} (1-2t)t\left|f'(a)\right| dt + \int_0^{1/2} (1-2t)(1-t)\left|f'(b)\right| dt \\
&\quad = \frac{1}{24}\left|f'(a)\right| + \frac{5}{24}\left|f'(b)\right|.
\end{aligned}$$

Hence,

$$|I_0| \leq \left((1-\alpha)\frac{2^{\alpha-2}+1-\alpha}{(2-\alpha)(3-\alpha)} + \frac{2-\alpha}{24}\right)\left|f'(a)\right| + \left(\frac{(5-\alpha)2^{\alpha-2}-1-\alpha}{(2-\alpha)(3-\alpha)} + \frac{7+2\alpha}{24}\right)\left|f'(b)\right|,$$

if $\alpha \in (-1,0)$, and

$$|I_0| \leq \left((1-\alpha)\frac{2^{\alpha-2}+1-\alpha}{(2-\alpha)(3-\alpha)} + \frac{1}{8}\right)\left|f'(a)\right| + \left(\frac{(5-\alpha)2^{\alpha-2}-1-\alpha}{(2-\alpha)(3-\alpha)} + \frac{5}{24}\right)\left|f'(b)\right|,$$

if $\alpha \leq -1$. Thus, Lemma 1 gives the inequalities. □

We deal now with the case $\alpha = 0$.

Proposition 2. *Let $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function. If $f' \in L^1[a,b]$ and $|f'|$ is a convex function, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Proof. Let us define $\alpha = 0$.

First of all, note that $L_{\alpha-1}[a,b] = L_{-1}[a,b] = L^1[a,b]$.

In addition, we have $\left|(1-t)^{1-\alpha} - t^{1-\alpha}\right| = \left|1-2t\right|$ for every $t \in [0,1]$. Thus, the argument in the proof of Theorem 3 allows for concluding

$$\left|\frac{1}{(b-a)^2}\, 2\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{b-a}\right| \leq \frac{1}{4}\left(\left|f'(a)\right| + \left|f'(b)\right|\right),$$

since

$${}_{N_3}J^{0}_{b^-}f(a) = {}_{N_3}J^{0}_{a^+}f(b) = \int_a^b f(t)\,dt.$$

□

Theorem 4. *Let $\alpha < 1$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function. If $f' \in L_{\alpha-1}[a,b]$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{1-\alpha}{(b-a)^{2-\alpha}}\left[{}_{N_3}J^{\alpha}_{b^-}f(a) + {}_{N_3}J^{\alpha}_{a^+}f(b)\right] - \frac{f(a)+f(b)}{b-a}\right| \leq \frac{1-2^{\alpha-1}}{2-\alpha}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Proof. By Lemma 1, we have

$$\frac{\alpha-1}{(b-a)^{2-\alpha}}\left[{}_{N_3}J^{\alpha}_{b^-}f(a) + {}_{N_3}J^{\alpha}_{a^+}f(b)\right] + \frac{f(a)+f(b)}{b-a} = I_0. \tag{1}$$

Since $|f'|$ is convex on $[a,b]$, we have $|I_0| \leq J_1 + J_2$, with

$$J_1 = \int_0^{1/2} \left((1-t)^{1-\alpha} - t^{1-\alpha}\right)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right)dt,$$
$$J_2 = \int_{1/2}^{1} \left(t^{1-\alpha} - (1-t)^{1-\alpha}\right)\left(t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right)dt.$$

A simple computation gives

$$J_1 = \left|f'(a)\right| \frac{1-(3-\alpha)2^{\alpha-2}}{(2-\alpha)(3-\alpha)} + \left|f'(b)\right| \left(\frac{1}{3-\alpha} - \frac{2^{\alpha-2}}{2-\alpha}\right),$$
$$J_2 = \left|f'(a)\right| \left(\frac{1}{3-\alpha} - \frac{2^{\alpha-2}}{2-\alpha}\right) + \left|f'(b)\right| \frac{1-(3-\alpha)2^{\alpha-2}}{(2-\alpha)(3-\alpha)}$$

and we obtain the inequality by adding these expressions of J_1 and J_2. □

Let us state a result relating the three integral operators.

Proposition 3. *Let $\alpha > 0$, $0 < a < b$ and $f : [a,b] \to [0,\infty)$ be a convex function. Then,*

$$_{N_3}J_a^{\alpha} f(b) \leq \min\left\{ \left(\frac{b-a}{a}\right)^{\alpha} {}_{N_3}J_{a+}^{\alpha} f(b),\ \left(\frac{b-a}{b}\right)^{\alpha} {}_{N_3}J_{b-}^{\alpha} f(a) \right\}.$$

Proof. Since

$$\frac{1}{t^{\alpha}} = \left(\frac{t-a}{t}\right)^{\alpha} \left(\frac{1}{t-a}\right)^{\alpha},$$
$$\frac{1}{t^{\alpha}} = \left(\frac{b-t}{t}\right)^{\alpha} \left(\frac{1}{b-t}\right)^{\alpha},$$

for every $t \in (a,b)$, we obtain

$$t^{-\alpha} \leq \left(\frac{b-a}{b}\right)^{\alpha} (t-a)^{-\alpha}, \tag{2}$$

$$t^{-\alpha} \leq \left(\frac{b-a}{a}\right)^{\alpha} (b-t)^{-\alpha}, \tag{3}$$

for every $t \in (a,b)$.

Since $f \geq 0$, Equation (2) gives

$$\int_a^b f(t)\, t^{-\alpha} dt \leq \left(\frac{b-a}{b}\right)^{\alpha} \int_a^b f(t)(t-a)^{-\alpha} dt,$$
$$_{N_3}J_a^{\alpha} f(b) \leq \left(\frac{b-a}{b}\right)^{\alpha} {}_{N_3}J_{b-}^{\alpha} f(a).$$

By using Equation (3), we obtain

$$_{N_3}J_a^{\alpha} f(b) \leq \left(\frac{b-a}{a}\right)^{\alpha} {}_{N_3}J_{a+}^{\alpha} f(b).$$

□

Theorem 5. *Let $\alpha < 1$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be a convex function. Then,*

$$
\begin{aligned}
{}_{N_3}J^{\alpha}_{a^+} f(b) &\leq (b-a)^{1-\alpha}\left(\frac{f(b)}{(1-\alpha)(2-\alpha)} + \frac{f(a)}{2-\alpha}\right),\\
{}_{N_3}J^{\alpha}_{b^-} f(a) &\leq (b-a)^{1-\alpha}\left(\frac{f(a)}{(1-\alpha)(2-\alpha)} + \frac{f(b)}{2-\alpha}\right).
\end{aligned}
$$

Proof. By using the convexity of f, we obtain

$$
\begin{aligned}
{}_{N_3}J^{\alpha}_{b^-} f(a) = \int_a^b f(t)(t-a)^{-\alpha}dt &= (b-a)^{1-\alpha}\int_a^b f(t)\left(\frac{t-a}{b-a}\right)^{-\alpha}\frac{dt}{b-a}\\
&= (b-a)^{1-\alpha}\int_0^1 f(sb+(1-s)a)s^{-\alpha}ds\\
&\leq (b-a)^{1-\alpha}\int_0^1 (sf(b)+(1-s)f(a))s^{-\alpha}ds\\
&= (b-a)^{1-\alpha}\left(\frac{f(a)}{(1-\alpha)(2-\alpha)} + \frac{f(b)}{2-\alpha}\right).
\end{aligned}
$$

The other inequality follows from a similar argument. □

Theorem 6. *Let $\alpha > 0$, $0 < a < b$ and $f : [a,b] \to \mathbb{R}$ be a convex function. Then,*

$$
\frac{1}{b^\alpha} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\, {}_{N_3}J^{\alpha}_{a} f(b) \leq \frac{1}{a^\alpha}\,\frac{f(b)+f(a)}{2}.
$$

Furthermore, if $0 < \alpha < 1$ and $f \geq 0$, then

$$
\begin{aligned}
&\frac{1}{b^\alpha} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\, {}_{N_3}J^{\alpha}_{a} f(b)\\
&\leq \min\left\{\frac{1}{a^\alpha}\,\frac{f(b)+f(a)}{2},\ \frac{1}{b^\alpha}\left(\frac{f(a)}{(1-\alpha)(2-\alpha)} + \frac{f(b)}{2-\alpha}\right)\right\}.
\end{aligned}
$$

Proof. Since $b^{-\alpha} \leq t^{-\alpha} \leq a^{-\alpha}$, the classical Hermite–Hadamard inequality gives

$$
\frac{1}{b^\alpha} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\, {}_{N_3}J^{\alpha}_{a} f(b) \leq \frac{1}{a^\alpha}\,\frac{f(b)+f(a)}{2}.
$$

If $0 < \alpha < 1$ and $f \geq 0$, then Proposition 3 and Theorem 5 give

$$
\frac{1}{b-a}\, {}_{N_3}J^{\alpha}_{a} f(b) \leq \min\left\{\frac{1}{a^\alpha}\left(\frac{f(b)}{(1-\alpha)(2-\alpha)} + \frac{f(a)}{2-\alpha}\right), \frac{1}{b^\alpha}\left(\frac{f(a)}{(1-\alpha)(2-\alpha)} + \frac{f(b)}{2-\alpha}\right)\right\}.
$$

Since $0 < (1-\alpha)(2-\alpha) < (2-\alpha) < 2$, we obtain

$$
\frac{1}{a^\alpha}\,\frac{f(b)+f(a)}{2} < \frac{1}{a^\alpha}\left(\frac{f(b)}{(1-\alpha)(2-\alpha)} + \frac{f(a)}{2-\alpha}\right),
$$

and thus we have

$$
\frac{1}{b-a}\, {}_{N_3}J^{\alpha}_{a} f(b) \leq \min\left\{\frac{1}{a^\alpha}\,\frac{f(b)+f(a)}{2},\ \frac{1}{b^\alpha}\left(\frac{f(a)}{(1-\alpha)(2-\alpha)} + \frac{f(b)}{2-\alpha}\right)\right\}.
$$

□

Theorem 7. *Let $\alpha \neq 1, 2$, $0 < a < b$ and $f : [a,b] \to \mathbb{R}$ be a convex function. Then,*

$$
{}_{N_3}J_a^{\alpha} f(b) \leq f(b) \left(\frac{b^{1-\alpha}}{1-\alpha} - \frac{b^{2-\alpha} - a^{2-\alpha}}{(b-a)(1-\alpha)(2-\alpha)} \right) + f(a) \left(\frac{b^{2-\alpha} - a^{2-\alpha}}{(b-a)(1-\alpha)(2-\alpha)} - \frac{a^{1-\alpha}}{1-\alpha} \right).
$$

Proof. The change of variables $t = bs + a(1-s)$ and the convexity of f give

$$
\begin{aligned}
\frac{1}{b-a}\, {}_{N_3}J_a^{\alpha} f(b) &= \frac{1}{b-a} \int_a^b f(t)\, t^{-\alpha} dt \\
&= \int_0^1 f\big(bs + a(1-s)\big) \big(bs + a(1-s)\big)^{-\alpha} ds \\
&\leq f(b) \int_0^1 s \big(bs + a(1-s)\big)^{-\alpha} ds + f(a) \int_0^1 (1-s) \big(bs + a(1-s)\big)^{-\alpha} ds.
\end{aligned}
$$

Integration by parts gives

$$
\begin{aligned}
\int_0^1 s \big(bs + a(1-s)\big)^{-\alpha} ds &= \frac{b^{1-\alpha}}{(b-a)(1-\alpha)} - \frac{b^{2-\alpha} - a^{2-\alpha}}{(b-a)^2(1-\alpha)(2-\alpha)}, \\
\int_0^1 (1-s) \big(bs + a(1-s)\big)^{-\alpha} ds &= \frac{-a^{1-\alpha}}{(b-a)(1-\alpha)} + \frac{b^{2-\alpha} - a^{2-\alpha}}{(b-a)^2(1-\alpha)(2-\alpha)}
\end{aligned}
$$

and this finishes the proof. □

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