

Article

Noether-Type First Integrals Associated with Autonomous Second-Order Lagrangians

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Abstract: In this paper, the analysis is centered on Noether-type first integrals in Lagrange-Hamilton dynamics based on autonomous second-order Lagrangians. More precisely, by using the classical Noether's theorem and a non-standard Legendrian duality, the single-time and multi-time versions of Noether's result are investigated for autonomous second-order Lagrangians. A correspondence is established between the invariances under flows and the first integrals for autonomous second-order Lagrangians. In this way, our results extend, unify and improve several existing theorems in the current literature.

Keywords: multi-time; conservation law; symmetry; first integral; autonomous second-order Lagrangian; Lagrange dynamics; Hamilton dynamics

MSC: 70H03; 70H05

1. Introduction

Over time, the multi-time version associated with Lagrange-Hamilton-Jacobi dynamics has been extensively studied by many researchers (see, for instance, Rochet [1], Motta and Rampazzo [2], Cardin and Viterbo [3], Udriște and Matei [4], Treanță [5–10]). The concept of *multi-time* has a long story and we make a dishonesty by mentioning only a part of the research works that contain it: Dirac et al. [11], Tomonaga [12], Friedman [13], Saunders [14], Udriște and Matei [4], Prepeliță [15], Mititelu and Treanță [16], Treanță [5–10].

In this paper, inspired and motivated by the ongoing research in this field, we consider *multi-time evolutions* and the notion *multi-time* is regarded as multiple parameter of evolution. For the multi-time case, the multi-index notation introduced by Saunders [14] is used. Throughout the paper, we develop our points of view, by developing new concepts and methods for a theory that involves single-time and multi-time second-order Lagrangians. More exactly, by using a non-standard Legendrian duality, the main aim of this work is to study the single-time and multi-time versions of Noether's result for autonomous second-order Lagrangians. We prove that there exists a correspondence between the invariances under flows and the first integrals for autonomous second-order Lagrangians. This actually reflects Noether type theorems between symmetries and conservation laws for dynamical systems. This work can be an important source for many research problems and it should be of interest to engineers and applied mathematicians. For other different but connected ideas to this subject, the reader is directed to Ma [17,18].

The present paper is structured as follows. Section 2 contains some auxiliary results including the classical Noether's theorem. Section 3 introduces the main results of this paper. More exactly, Noether-type

first integrals are investigated for autonomous second-order Lagrangians. Finally, Section 4 concludes the paper.

2. Auxiliary Results

In the classical (single-time) Lagrange-Hamilton dynamics it is well-known that if $L = L(x(t), \dot{x}(t))$ is an autonomous Lagrangian, then the associated Hamiltonian $H = H(x(t), p(t))$ is a first integral both for Euler-Lagrange and Hamilton equations (see, for instance, Udriște and Matei [4], Treanță [5–10]).

The next result formulates the classical Noether's theorem (for autonomous first-order Lagrangians).

Theorem 1. [Noether] Let $T(t, x)$ be the flow generated by the C^1 -class vector field $X(x) = (X^i(x))$, $i = \overline{1, n}$. If the Lagrangian $L(x(t), \dot{x}(t))$ is invariant under this flow, where $x : [t_0, t_1] \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$ and $\dot{x}(t) := \frac{d}{dt}x(t)$, then the function

$$I(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^i}(x, \dot{x}) X^i(x)$$

is a first integral associated with the movement generated by the Lagrangian L .

Proof. Denote $x_s(t) := T(s, x(t))$. The invariance of L means

$$\begin{aligned} 0 &= \frac{dL}{ds}(x_s(t), \dot{x}_s(t))|_{s=0} \\ &= \frac{\partial L}{\partial \dot{x}^i}(x(t), \dot{x}(t)) \frac{\partial X^i}{\partial x^j}(x(t)) \dot{x}^j(t) + \frac{\partial L}{\partial x^i}(x(t), \dot{x}(t)) X^i(x(t)). \end{aligned}$$

Consequently, using derivation formulas and Euler-Lagrange equations, we get

$$\begin{aligned} \frac{dI}{dt}(x, \dot{x}) &= \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i}(x, \dot{x}) \right) X^i(x) + \frac{\partial L}{\partial \dot{x}^i}(x, \dot{x}) \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j \\ &= \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i}(x, \dot{x}) - \frac{\partial L}{\partial x^i}(x, \dot{x}) \right) X^i(x) = 0 \end{aligned}$$

and the proof is complete. $\square$

Further, in order to formulate the multi-time version for Noether-type first integrals (associated with autonomous first-order Lagrangians), in accordance with Treanță [5–10] and following Udriște and Matei [4], consider $\Omega_{t_0, t_1} \subset \mathbb{R}^m$ a hyper-parallelepiped determined by diagonally opposite points t_0, t_1 from $\mathbb{R}^m$. If we define the *partial order product* on $\mathbb{R}^m$, then Ω_{t_0, t_1} is equivalent with the closed interval $[t_0, t_1]$. Also, consider the C^2 -class Lagrangian $L(x(t), x_\gamma(t), t)$, where

$$t = (t^1, \dots, t^m) = (t^\gamma) \in \Omega_{t_0, t_1}, \quad x = (x^1, \dots, x^n) = (x^i) : \Omega_{t_0, t_1} \rightarrow \mathbb{R}^n$$

$$x_\gamma(t) := \frac{\partial x}{\partial t^\gamma}(t), \quad \gamma \in \{1, 2, \dots, m\}, \quad i \in \{1, 2, \dots, n\}.$$

In the case of several variables of evolution (that is, the *multi-time* version), the Hamiltonian $H(x, p, t) = p_i^\alpha x_\alpha^i(x, p, t) - L(x, p, t)$ (see summation over the repeated indices and $p_i^\alpha(t) := \frac{\partial L}{\partial x_\alpha^i}(x(t), x_\gamma(t), t)$, $\alpha \in \{1, 2, \dots, m\}$, $t \in \Omega_{t_0, t_1}$) does not conserve, that is $D_\alpha H \neq 0$, where D_α is the total derivative operator, even in autonomous case.

In the following, let $L(x(t), x_\gamma(t))$ be an autonomous C^2 -class Lagrangian. Introducing the *multi-time anti-trace Euler-Lagrange PDEs* (for more details, see Udriște and Matei [4])

$$\frac{\partial L}{\partial x^i} \delta_\beta^\gamma - D_\beta \frac{\partial L}{\partial x_\gamma^i} = 0, \quad i = \overline{1, n}$$

and using *Legendrian duality*, we derive the *multi-time anti-trace Hamilton PDEs* (see δ_β^α as Kronecker's symbol)

$$\frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H}{\partial p_i^\beta}(x(t), p(t)), \quad \frac{\partial p_i^\alpha}{\partial t^\beta}(t) = -\delta_\beta^\alpha \frac{\partial H}{\partial x^i}(x(t), p(t)).$$

A direct computation gives us $D_\gamma H = 0$. Consequently, H is a first integral both for multi-time anti-trace Euler-Lagrange PDEs and multi-time anti-trace Hamilton PDEs.

On the other hand, if we introduce *energy-impulse tensor* T of components

$$T_\beta^\alpha(x, p, t) = p_i^\alpha x_\beta^i(x, p, t) - L(x, p, t) \delta_\beta^\alpha$$

and *Hamilton tensor*

$$H_\beta^\alpha(x, p) = p_i^\alpha x_\beta^i(x, p) - \frac{1}{m} L(x, p) \delta_\beta^\alpha,$$

we get the following results:

- (1) $\text{Div}(T) := D_\alpha T_\beta^\alpha = 0$ (conservation law);
- (2) the trace of Hamilton tensor is the Hamiltonian $H(x, p) = p_i^\alpha x_\alpha^i(x, p) - L(x, p)$.

Next, consider the *Hamiltonian tensor field* H_β^α defined by

$$H_\beta^\alpha(x, p) = p_i^\alpha x_\beta^i(x, p) - L_\beta^\alpha(x(t), x_\gamma(t)),$$

where x and p are the *canonical variables*. Here, $L_\beta^\alpha(x(t), x_\gamma(t))$ represents an extension of the Lagrangian $L(x(t), x_\gamma(t))$, satisfying:

- (a) the trace of tensor field L_β^α is the Lagrangian L ;
- (b) the Lagrangian 1-forms $L_\beta^\alpha(x(t), x_\gamma(t)) dt^\beta$ are completely integrable;
- (c) the functions $L_\beta^\alpha(x(t), x_\gamma(t))$ determine the multi-time anti-trace Euler-Lagrange PDEs

$$\frac{\partial L_\beta^\alpha}{\partial x^i} \delta_\lambda^\gamma - D_\lambda \frac{\partial L_\beta^\alpha}{\partial x_\gamma^i} = 0, \quad i = \overline{1, n}. \quad (1)$$

Further, we assume $p_i^\alpha(t) \delta_\beta^\gamma = \frac{\partial L_\beta^\alpha}{\partial x_\gamma^i}(x(t), x_\gamma(t))$ and that these nm^3 equations define the following m functions $x_\gamma = x_\gamma(x, p)$. If $L_\beta^\alpha = L \delta_\beta^\alpha$, then H_β^α is exactly the classical energy-impulse tensor.

The following result formulates the generalized Hamilton PDEs governed by first-order Lagrangians.

Theorem 2. (Generalized Hamilton PDEs, [4]) If $x(\cdot)$ is solution in (1) and $p = (p_i^\alpha(\cdot))$ is defined as above, then the pair $(x(\cdot), p(\cdot))$ is solution of the following generalized Hamilton PDEs:

$$\frac{\partial x^i}{\partial t^\beta}(t) \delta_\gamma^\alpha = \frac{\partial H_\beta^\alpha}{\partial p_i^\gamma}(x(t), p(t), t)$$

$$\frac{\partial p_i^\alpha}{\partial t^\beta}(t) = -\frac{\partial H_\beta^\alpha}{\partial x^i}(x(t), p(t), t).$$

Moreover, if the Lagrangian tensor field L_β^α is autonomous, then the divergence of the transposed Hamilton tensor field H_β^α is zero, that is $\sum_{\beta=1}^m D_\beta H_\beta^\alpha = 0$.

The next theorem formulates the multi-time version for Noether-type first integrals associated with autonomous first-order Lagrangians.

Theorem 3. ([4]) Let $T(t, x)$ be the m -flow generated by the C^1 -class vector fields $X_\alpha(x) = (X_\alpha^i(x))$, $\alpha = \overline{1, m}$, $i = \overline{1, n}$. If the Lagrangian $L(x(t), x_\gamma(t))$ is invariant under this flow, then the function

$$I(x, x_\gamma) = \frac{\partial L}{\partial x_\beta^i}(x, x_\gamma) X_\beta^i(x)$$

is a first integral of the movement generated by the Lagrangian L via multi-time anti-trace Euler-Lagrange PDEs.

Proof. The invariance of L means

$$0 = \frac{\partial L}{\partial x^i}(x(t), x_\gamma(t)) X_\alpha^i(x(t)) + \frac{\partial L}{\partial x_\beta^i}(x(t), x_\gamma(t)) \frac{\partial X_\beta^i}{\partial x^j}(x(t)) x_\alpha^j(t).$$

By direct computation and taking into account the multi-time anti-trace Euler-Lagrange PDEs, we get

$$\begin{aligned} D_\alpha I(x, x_\gamma) &= \left(D_\alpha \frac{\partial L}{\partial x_\beta^i}(x, x_\gamma) \right) X_\beta^i(x) + \frac{\partial L}{\partial x_\beta^i}(x, x_\gamma) \frac{\partial X_\beta^i}{\partial x^j}(x) x_\alpha^j \\ &= \left(D_\alpha \frac{\partial L}{\partial x_\beta^i}(x, x_\gamma) - \frac{\partial L}{\partial x^i}(x, x_\gamma) \delta_\alpha^\beta \right) X_\beta^i(x) = 0 \end{aligned}$$

and the proof is now complete. $\square$

3. Main Results

This section, taking into account the aforementioned auxiliary results, introduces the main results of this paper. More exactly, the single-time and multi-time versions of Noether's result are investigated for autonomous second-order Lagrangians.

Theorem 4. Let $T(t, x)$ be the flow generated by the C^2 -class vector field $X(x) = (X^i(x))$, $i = \overline{1, n}$. If the autonomous second-order Lagrangian $L(x(t), \dot{x}(t), \ddot{x}(t))$ is invariant under this flow, then the function

$$\begin{aligned} I(x, \dot{x}, \ddot{x}) &= \frac{\partial L}{\partial \ddot{x}^i}(x, \dot{x}, \ddot{x}) X^i(x) + \frac{\partial L}{\partial \ddot{x}^i}(x, \dot{x}, \ddot{x}) \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j \\ &\quad - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i}(x, \dot{x}, \ddot{x}) \right) X^i(x) \end{aligned}$$

is a first integral of the movement generated by the Lagrangian L .

Proof. The invariance of L means

$$0 = \frac{\partial L}{\partial x^i} (x(t), \dot{x}(t), \ddot{x}(t)) X^i(x(t)) + \frac{\partial L}{\partial \dot{x}^i} (x(t), \dot{x}(t), \ddot{x}(t)) \frac{\partial X^i}{\partial x^j} (x(t)) \dot{x}^j(t) \\ + \frac{\partial L}{\partial \ddot{x}^i} (x(t), \dot{x}(t), \ddot{x}(t)) \frac{d\ddot{x}^i}{dt}(t).$$

By using the associated Euler-Lagrange ODEs formulated as follows

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}^i} = 0, \quad i \in \{1, 2, \dots, n\},$$

it results

$$\begin{aligned} \frac{dI}{dt} &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} X^i(x) + \frac{\partial L}{\partial \ddot{x}^i} \frac{\partial X^i}{\partial x^j} (x) \dot{x}^j - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} X^i(x) + \frac{\partial L}{\partial \ddot{x}^i} \frac{\partial X^i}{\partial x^j} (x) \dot{x}^j \right) - \frac{d}{dt} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) X^i(x) + \frac{\partial L}{\partial \ddot{x}^i} \frac{\partial X^i}{\partial x^j} (x) \dot{x}^j + \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) \frac{\partial X^i}{\partial x^j} (x) \dot{x}^j \\ &\quad + \frac{\partial L}{\partial \ddot{x}^i} \frac{d\ddot{x}^i}{dt} - \frac{d}{dt} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) X^i(x) - \frac{\partial L}{\partial \dot{x}^i} X^i(x) - \frac{\partial L}{\partial \ddot{x}^i} \frac{d\ddot{x}^i}{dt} + \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) \frac{\partial X^i}{\partial x^j} (x) \dot{x}^j \\ &\quad + \frac{\partial L}{\partial \ddot{x}^i} \frac{d\ddot{x}^i}{dt} - \frac{d}{dt} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) \\ &= -\frac{\partial L}{\partial x^i} X^i(x) + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) X^i(x) - \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \\ &\quad + \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) \frac{\partial X^i}{\partial x^j} (x) \dot{x}^j - \frac{d}{dt} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) \\ &= \left(-\frac{\partial L}{\partial x^i} + \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \\ &\quad + \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) \frac{\partial X^i}{\partial x^j} (x) \dot{x}^j - \frac{d}{dt} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) \\ &= \left(-\frac{\partial L}{\partial x^i} + \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) + \frac{d}{dt} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) \\ &\quad - \frac{d}{dt} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) = \left(-\frac{\partial L}{\partial x^i} + \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) = 0. \end{aligned}$$

In consequence, the function $I(x, \dot{x}, \ddot{x})$ is a first integral of the movement generated by the autonomous second-order Lagrangian L . The proof is complete. $\square$

The following two corollaries establish some more restrictive results.

Corollary 1. Let $T(t, x)$ be the flow generated by the C^2 -class vector field $X(x) = (X^i(x))$, $i = \overline{1, n}$. If the autonomous second-order Lagrangian $L(x(t), \dot{x}(t), \ddot{x}(t))$ is invariant under this flow and $\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i}(x, \dot{x}, \ddot{x}) \right) X^i(x)$ is constant, then the function

$$I(x, \dot{x}, \ddot{x}) = \frac{\partial L}{\partial \dot{x}^i}(x, \dot{x}, \ddot{x}) X^i(x) + \frac{\partial L}{\partial \ddot{x}^i}(x, \dot{x}, \ddot{x}) \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j$$

is a first integral of the movement generated by the Lagrangian L .

Proof. The invariance of L means

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x^i}(x(t), \dot{x}(t), \ddot{x}(t)) X^i(x(t)) + \frac{\partial L}{\partial \dot{x}^i}(x(t), \dot{x}(t), \ddot{x}(t)) \frac{\partial X^i}{\partial x^j}(x(t)) \dot{x}^j(t) \\ &\quad + \frac{\partial L}{\partial \ddot{x}^i}(x(t), \dot{x}(t), \ddot{x}(t)) \frac{d\ddot{x}^i}{dt}(t). \end{aligned}$$

Consequently, we get

$$\begin{aligned} \frac{dI}{dt} &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} X^i(x) + \frac{\partial L}{\partial \ddot{x}^i} \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j \right) \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) X^i(x) + \frac{\partial L}{\partial \dot{x}^i} \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j + \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j + \frac{\partial L}{\partial \ddot{x}^i} \frac{d\ddot{x}^i}{dt} \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) X^i(x) - \frac{\partial L}{\partial x^i} X^i(x) - \frac{\partial L}{\partial \ddot{x}^i} \frac{d\ddot{x}^i}{dt} + \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j + \frac{\partial L}{\partial \ddot{x}^i} \frac{d\ddot{x}^i}{dt} \\ &= -\frac{\partial L}{\partial x^i} X^i(x) + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) X^i(x) - \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \\ &\quad + \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j \\ &= \left(-\frac{\partial L}{\partial x^i} + \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) + \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) \frac{\partial X^i}{\partial x^j}(x) \dot{x}^j \\ &= \left(-\frac{\partial L}{\partial x^i} + \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) + \frac{d}{dt} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i} \right) X^i(x) \right) = 0. \end{aligned}$$

Thus, the function $I(x, \dot{x}, \ddot{x})$ is a first integral and the proof is complete. $\square$

Corollary 2. For any autonomous regular second-order Lagrangian $L(\cdot) := L(x(t), \dot{x}(t), \ddot{x}(t))$, the Hamiltonian

$$H \left(x^i(t), \frac{\partial L}{\partial \dot{x}^i}(\cdot), \frac{\partial L}{\partial \ddot{x}^i}(\cdot) \right) := \dot{x}^i(t) \frac{\partial L}{\partial \dot{x}^i}(\cdot) + \ddot{x}^i(t) \frac{\partial L}{\partial \ddot{x}^i}(\cdot) - L(\cdot)$$

is conserved along to any extremal curve, $c(t) := (x^i(t))$, $i \in \{1, 2, \dots, n\}$, solution of the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}^i} = 0, \quad i \in \{1, 2, \dots, n\},$$

if $\dot{x}^i(t) \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{x}^i}(\cdot) \right) = ct$.

Remark 1. As with the first-order Lagrangians, in the case of multiple variables of evolution (that is, the multi-time version), the Hamiltonian

$$\begin{aligned} H(x, p, q, t) = & x_{\gamma}^i(x, p, q, t) \frac{\partial L}{\partial x_{\gamma}^i}(x, x_{\gamma}^i(x, p, q, t), x_{\alpha\beta}^i(x, p, q, t), t) \\ & + \frac{1}{n(\alpha, \beta)} x_{\alpha\beta}^i(x, p, q, t) \frac{\partial L}{\partial x_{\alpha\beta}^i}(x, x_{\gamma}^i(x, p, q, t), x_{\alpha\beta}^i(x, p, q, t), t) \\ & - L(x, x_{\gamma}^i(x, p, q, t), x_{\alpha\beta}^i(x, p, q, t), t), \end{aligned}$$

(multi-time second order non-standard Legendrian duality) or, shortly,

$$H = x_{\gamma}^i p_i^{\gamma} + x_{\alpha\beta}^i q_i^{\alpha\beta} - L,$$

(see summation over the repeated indices) with

$$\begin{aligned} p_i^{\gamma}(t) &:= \frac{\partial L}{\partial x_{\gamma}^i}(t, x(t), x_{\gamma}(t), x_{\alpha\beta}(t)), \quad t \in \Omega_{t_0, t_1}, \\ q_i^{\alpha\beta}(t) &:= \frac{1}{n(\alpha, \beta)} \frac{\partial L}{\partial x_{\alpha\beta}^i}(t, x(t), x_{\gamma}(t), x_{\alpha\beta}(t)), \quad t \in \Omega_{t_0, t_1}, \end{aligned}$$

does not conserve, even in autonomous case (for more details, see Treanță [8]).

4. Conclusions

In this paper, motivated and inspired by the ongoing research in this area, single-time and multi-time versions for Noether-type first integrals in Lagrange-Hamilton dynamics associated with autonomous second-order Lagrangians have been investigated. More exactly, by using a non-standard Legendrian duality, the results derived in this paper have extended and improved several existing theorems in the current literature.

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