

The Standard Model of Particle Physics with Diracian Neutrino Sector

Theodorus Maria Nieuwenhuizen 

Institute for Theoretical Physics, University of Amsterdam, Science Park 904,
1098 XH Amsterdam, The Netherlands; t.m.nieuwenhuizen@uva.nl; Tel.: +31-20-525-6332

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Abstract: The minimally extended standard model of particle physics contains three right handed or sterile neutrinos, coupled to the active ones by a Dirac mass matrix and mutually by a Majorana mass matrix. In the pseudo-Dirac case, the Majorana terms are small and maximal mixing of active and sterile states occurs, which is generally excluded for solar neutrinos. In a “Diracian” limit, the physical masses become pairwise degenerate and the neutrinos attain a Dirac signature. Members of a pair do not oscillate mutually so that their mixing can be undone, and the standard neutrino model follows as a limit. While two Majorana phases become physical Dirac phases and three extra mass parameters occur, a better description of data is offered. Oscillation problems are worked out in vacuum and in matter. With lepton number -1 assigned to the sterile neutrinos, the model still violates lepton number conservation and allows very feeble neutrinoless double beta decay. It supports a sterile neutrino interpretation of Earth-traversing ultra high energy events detected by ANITA.

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1. Introduction

Thus far the Large Hadron Collider (LHC) has not produced evidence for physics beyond the standard model (BSM). But the neutrino sector must involve BSM because neutrinos have mass. Indeed, the 2015 Noble prize in physics was awarded to T. Kajita and A. B. McDonald “for the discovery of neutrino oscillations which show that neutrinos have mass” [1].

The standard neutrino model (S ν M) with its three Majorana neutrinos has measured values for the mass-squared differences, the mixing angles θ_{12} , θ_{23} and θ_{13} and the weak Dirac phase δ . But the absolute mass scale, the order of the hierarchy, normal or inverted, and the Majorana phases are unknown. There is stress in the fit to the standard solar model [2]; there is a reactor neutrino anomaly [3,4]; MiniBooNE finds 4.5σ evidence for a sterile neutrino [5], while MINOS/MINOS+ does not [6]. At present, there is no definite conclusion about the existence of an eV sterile neutrino [7].

There is also input from cosmology. From the lensing of background galaxies by the large, reasonably relaxed galaxy clusters Abell 1689 [8–10] and Abell 1835 [11] there is indication for three active and three sterile neutrinos with common mass of 1.5–1.9 eV, which act as the cluster dark matter. We shall not dwell here into the many questions this raised and counter-evidence to that possibility, but refer to the discussion and cited articles in these references. Be it as it may, the $3 + 3$ case puts forward the minimal extension of the standard model (SM) in the neutrino sector for consideration. By default, this accepts all SM physics without extension in the Higgs, gauge, quark and charged lepton sectors. Gauge invariance then forbids the presence of a ‘left handed’ Majorana mass matrix between the left handed active neutrinos, so that there *must* be a Dirac mass matrix to give them mass. As

such a term mixes left and right handed fields, this presupposes the existence of three right handed neutrinos, also called sterile, i.e., not involved in elementary particle processes [12]. For that reason, they are allowed to have a mutual ‘right handed’ Majorana mass matrix. In order to make up for half of the cluster dark matter, sterile neutrinos have to be generated in the early cosmos by oscillation of active ones. This is only possible when the Dirac mass matrix is accompanied by a non-trivial right handed Majorana mass matrix.

In the pseudo-Dirac limit, the right handed Majorana masses are much smaller than the eigenvalues of the Dirac mass matrix. The maximal mixing of the resulting pseudo-Dirac neutrinos implies that in principle half of the emitted solar neutrinos has become sterile here on Earth, and thus unobservable (see Section 2.4 for details); this is ruled out by the standard solar model [2]. Hence the pseudo-Dirac case is often considered to be ruled out. We intend to show, however, that there is a way out of this conundrum, so as to faithfully include neutrino mass in the SM without changing its high energy sector.

While excellent studies such as [12–14] discuss the theory for general number N_s of sterile neutrinos, we shall work out the case $N_s = 3$ in a nontrivial limit where the 6 Majorana neutrinos combine into three Dirac neutrinos so that the maximal mixing is harmless and can be circumvented. We call them Diracian neutrinos, i.e., Dirac neutrinos in a model with both Dirac and Majorana masses. In Section 2 we treat the theory and in Section 3 we consider various applications. We close with a summary.

2. The Lagrangian for Active Plus Sterile Neutrinos

In this section we concentrate on the neutrino sector of the SM. For completeness we present the full Lagrangian in Appendix B.

2.1. Active Neutrinos Only

We start from the SM Lagrangian where the e , μ and τ fields are diagonal in the mass basis. Left handed neutrinos and right handed antineutrinos exist, and are called “active neutrinos” since they participate in the weak interactions (left and right handedness refers to the chirality; see Appendix A). Additional neutrinos are not involved in them, and are called sterile. If only active ones exist, they are Majorana particles. Their mass term involves the quantized left handed fermionic flavor fields $\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$,

$$\mathcal{L}_{mL}^M = \frac{1}{2} \sum_{\alpha, \beta=e, \mu, \tau} \nu_{\alpha L}^T \mathcal{C}^\dagger (M_L^M)_{\alpha\beta} \nu_{\beta L} + h.c. \quad (1)$$

where \mathcal{C} is the charge conjugation matrix, T denotes transposition, \dagger Hermitian conjugation, and $h.c.$ Hermitian conjugated terms. M_L^M is called the left handed Majorana mass matrix. In the SM gauge invariance forces M_L^M to vanish [12]; if it is present, it must originate from high energy BSM, such as Weinberg’s dimension-five operator. Considering new physics only in the neutrino sector, we neglect M_L^M .

2.2. The Dirac and Majorana Mass Matrices

In absence of M_L^M , the only possibility to give mass to the active neutrinos is by a Dirac mass matrix. Since that involves products of left and right handed fields, this presupposes the existence of $N_s \geq 3$ sterile neutrinos, that must be right handed and represented by quantized fermionic fields ν_{iR} ,

$$\mathcal{L}_m^D = - \sum_{\alpha=e, \mu, \tau} \sum_{i=1}^{N_s} \left(\overline{\nu_{\alpha L}} M_{\alpha i}^D \nu_{iR} + \overline{\nu_{iR}} M_{i\alpha}^{D\dagger} \nu_{\alpha L} \right), \quad (2)$$

where the Dirac mass matrix M^D is a complex $3 \times N_s$ matrix. The sterile fields do not enter the weak interactions; they are singlets under the $U(1)_Y \times SU(2)_L \times SU(3)_C$ gauge groups of the SM and affect neither gauge invariance, anomalies nor renormalization. Hence they preserve its full functioning while accounting for neutrino masses. Moreover, the sterile fields may have a mutual mass term like Equation (1),

$$\mathcal{L}_{mR}^M = \frac{1}{2} \sum_{i,j=1}^{N_s} \left(v_{iR}^T C^\dagger M_{R,ij}^{M\dagger} v_{jR} + (v_{iR}^c)^T C^\dagger M_{R,ij}^M v_{jR}^c \right), \quad (3)$$

where the right handed Majorana mass matrix M_R^M is symmetric and complex valued, and where v_{iR}^c is the charge conjugate of v_{iR} ,

$$v_{iR}^c = C \overline{v_{iR}}^T = C (\gamma^0)^T (v_{iR}^\dagger)^T = -\gamma^0 C (v_{iR}^\dagger)^T. \quad (4)$$

While v_{iR} is a right handed field, v_{iR}^c is left handed (see Appendix A for properties of γ and C matrices).

The kinetic term has a common form for all species [12],

$$\mathcal{L}_k = \sum_{\alpha=e,\mu,\tau} \overline{v_{\alpha L}} i \overleftrightarrow{\partial} v_{\alpha L} + \sum_{i=1}^{N_s} \overline{v_{iR}} i \overleftrightarrow{\partial} v_{iR}, \quad (5)$$

where the slash denotes contraction with γ matrices, and the partial derivatives acting as

$$\overleftrightarrow{\partial} = \sum_{\mu=0}^3 \gamma^\mu \overrightarrow{\partial}_\mu, \quad \overrightarrow{\partial}_\mu = \frac{\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu}{2}, \quad \overline{a} \not{b} \equiv \frac{1}{2} \sum_{\mu=0}^3 \left(\overline{a} \gamma^\mu \frac{\partial b}{\partial x^\mu} - \frac{\partial \overline{a}}{\partial x^\mu} \gamma^\mu b \right). \quad (6)$$

2.3. The General Mass Matrix for Three Sterile Neutrinos

Though the number of right handed neutrinos is not fixed in principle, the case $N_s = 3$ has, if not a practical value [8–11], at least an esthetic one: for each left handed neutrino there is a right handed one, in the way it occurs for charged leptons and quarks. The three families of active left and sterile right handed neutrinos have the flavor three vectors (in our case $N_s = 3$ one may be tempted to denote (v_{1R}, v_{2R}, v_{3R}) as $(v_{eR}, v_{\mu R}, v_{\tau R})$)

$$v_{fL} \equiv v_{aL} = (v_{eL}, v_{\mu L}, v_{\tau L})^T, \quad v_{fR} \equiv v_{sR} = (v_{1R}, v_{2R}, v_{3R})^T. \quad (7)$$

With the combined left handed flavor vector

$$\mathbf{N}_{fL} = (v_{fL}^T, v_{fR}^{cT})^T = (v_{aL}^T, v_{sR}^{cT})^T, \quad (8)$$

the above mass Lagrangians combine into

$$\mathcal{L}_m = \frac{1}{2} \mathbf{N}_{fL}^T C^\dagger M^{DM} \mathbf{N}_{fL} + h.c. \quad (9)$$

In general, the mass matrix consists of four 3×3 blocks,

$$M^{DM} = \begin{pmatrix} M_L^M & M^{DT} \\ M^D & M_R^M \end{pmatrix}. \quad (10)$$

As stated, we take $M_L^M = 0$. In the (“standard”, “pure” or “trivial”) Dirac limit also $M_R^M = 0$. For pseudo-Dirac neutrinos M_R^M will be small with respect to M^D , or, more precisely, small with respect to the variation in the eigenvalues of M^D . Though we consider general M^D , we are inspired by the case

of galaxy cluster lensing where it has nearly equal eigenvalues with central value 1.5–1.9 eV [8–11]; in Section 3.1 we shall show that the entries of M_R^M then typically lie well below 1 meV.

2.4. Intermezzo: One Neutrino Family

In case of one family, the flavor vector is $\mathbf{N}_{fL} = (\nu_L, \nu_R^c)^T$. The entries of Equation (10) are scalars, so

$$M^{DM} = \begin{pmatrix} 0 & \tilde{m} \\ \tilde{m} & \bar{\mu} \end{pmatrix}. \quad (11)$$

Its eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\bar{\mu} \mp \sqrt{\tilde{m}^2 + \frac{1}{4}\bar{\mu}^2}. \quad (12)$$

The physical masses are their absolute values [12]. For $\bar{\mu}$ nonnegative, this leads to

$$m_1 = \sqrt{\tilde{m}^2 + \frac{1}{4}\bar{\mu}^2} - \frac{1}{2}\bar{\mu}, \quad m_2 = \sqrt{\tilde{m}^2 + \frac{1}{4}\bar{\mu}^2} + \frac{1}{2}\bar{\mu}. \quad (13)$$

The corresponding eigenvectors are

$$\mathbf{e}^{(1)} = \frac{1}{\sqrt{\tilde{m}^2 + m_1^2}} \begin{pmatrix} \tilde{m} \\ -m_1 \end{pmatrix} = \frac{1}{\sqrt{\tilde{m}^2 + m_2^2}} \begin{pmatrix} m_2 \\ -\tilde{m} \end{pmatrix}, \quad \mathbf{e}^{(2)} = \frac{1}{\sqrt{\tilde{m}^2 + m_1^2}} \begin{pmatrix} m_1 \\ \tilde{m} \end{pmatrix}. \quad (14)$$

For small $\bar{\mu}$ these are 45° rotations, i.e., maximal mixing of the active and sterile basis vectors. Formally we may undo the rotations over 45° , by considering

$$\mathbf{e}^{\tilde{a}} = \frac{\mathbf{e}^{(1)} + \mathbf{e}^{(2)}}{\sqrt{2}} \approx \begin{pmatrix} 1 \\ \bar{\mu}/4\tilde{m} \end{pmatrix}, \quad \mathbf{e}^{\tilde{s}} = \frac{\mathbf{e}^{(2)} - \mathbf{e}^{(1)}}{\sqrt{2}} \approx \begin{pmatrix} -\bar{\mu}/4\tilde{m} \\ 1 \end{pmatrix}, \quad (15)$$

where the approximations are to first order in $\bar{\mu}$, the pseudo Dirac regime. The first vector, $\mathbf{e}^{\tilde{a}}$, has its main weight on the first component, so it is mainly active, which we indicate by the tilde on a . The second one, $\mathbf{e}^{\tilde{s}}$, is mainly sterile. But unless $\bar{\mu} = 0$, the masses $m_{1,2}$ are different, so that $\mathbf{e}^{\tilde{a}}$ and $\mathbf{e}^{\tilde{s}}$ are not eigenvectors and have no physical meaning. In fact, the mass squares have the difference

$$\Delta m_{21}^2 = m_2^2 - m_1^2 = 2\bar{\mu}\sqrt{\tilde{m}^2 + \frac{1}{4}\bar{\mu}^2} \approx 2\bar{\mu}\tilde{m}. \quad (16)$$

An initially active state,

$$|\nu_a(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\tilde{m}\mathbf{e}^{(1)} + m_1\mathbf{e}^{(2)}}{\sqrt{\tilde{m}^2 + m_1^2}}, \quad (17)$$

with momentum p will at time t have oscillated into

$$|\nu_a(t)\rangle = \frac{\tilde{m}\mathbf{e}^{(1)}e^{-iE_1t} + m_1\mathbf{e}^{(2)}e^{-iE_2t}}{\sqrt{\tilde{m}^2 + m_1^2}}, \quad (18)$$

where $E_{1,2} = \sqrt{p^2 + m_{1,2}^2}$. The occurrence probability is

$$P_{aa}(t) = |\langle \nu_a(0) | \nu_a(t) \rangle|^2 = \frac{\bar{m}^4 + m_1^4 + 2\bar{m}^2 m_1^2 \cos \Delta E t}{(\bar{m}^2 + m_1^2)^2} \approx \frac{1 + \cos \Delta E t}{2}, \quad (19)$$

where for $p \gg \bar{m}$

$$\Delta E \equiv E_2 - E_1 \approx \frac{\Delta m_{21}^2}{2p} \approx \frac{\bar{\mu} \bar{m}}{p}. \quad (20)$$

In practice there will not be a pure initial state but some wave packet [12]. For $t \gg \hbar/\Delta E$ the cosine in Equation (19) will average out, so that the fraction of observable neutrinos is approximately $\frac{1}{2}$. In plain terms: for t large enough, half of the neutrinos are sterile and thus unobservable. For the solar neutrino problem the one-family approximation happens to work quite well [15] and the detection rates are well established. Hence for the pseudo Dirac model it would mean that twice as many neutrinos should be emitted as in the standard solar model. The corresponding doubling of heat generated by nuclear reactions is ruled out by the measurements of the solar luminosity, so the case is rarely discussed.

Only in the pure Dirac case, i.e., with Majorana mass $\bar{\mu} = 0$, the oscillations will not take place, since $m_{1,2} = \bar{m}$ and $\Delta E = 0$. When starting from an initial active state $\nu_a(0)$, it now equals $e^{\bar{a}}$, and this can be taken as eigenstate. The sterile state will merely be a spectator, “just sitting there and wasting its time”. This can be generalized to three families. If one would follow the Franciscan William of Ockham (Occam’s razor), it would be preferable for active neutrinos to be Majorana rather than Dirac with unobservable right handed partners.

The SvM differs from the neutrino sector in the SM by accounting for finite masses of its three Majorana neutrinos. Below we discuss a “Diracian” setup in which the sterile fields become physical, namely partly active, and the active fields partly sterile, even though the mass eigenstates have Dirac signature in vacuum.

2.5. Diagonalization of the Dirac Mass Matrix

We return to the three family case and its total mass matrix Equation (10) with $M_L^M = 0$. We notice that any 3×3 unitary matrix U can be decomposed as a product of five standard ones,

$$U = D' U^{DM}, \quad U^{DM} = U^D D^M, \quad U^D = U_1 U_2 U_3, \quad D^M = \text{diag}(e^{i\eta_1}, e^{i\eta_2}, e^{i\eta_3}). \quad (21)$$

The diagonal matrix D^M is called the Majorana phase matrix. Likewise we denote the diagonal phase matrix D' by $\text{diag}(e^{i\eta'_1}, e^{i\eta'_2}, e^{i\eta'_3})$ (for U in Equation (21) only five of the η_i and η'_i are needed; this can be seen by factoring out $e^{i\eta_1}$ from D^M and $e^{i\eta'_1}$ from D' and setting $\eta_1 \rightarrow \eta_1 - \eta'_1$. Both sides of Equation (21) thus involve nine free parameters). The matrix U^D is the product of

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} c_2 & 0 & s_2 e^{-i\delta} \\ 0 & 1 & 0 \\ -s_2 e^{i\delta} & 0 & c_2 \end{pmatrix}, \quad U_3 = \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (22)$$

where $c_i = \cos \theta_i$, $s_i = \sin \theta_i$ where the angles θ_i are termed in standard notation $\theta_1 = \theta_{23}$, $\theta_2 = \theta_{13}$ and $\theta_3 = \theta_{12}$. The Dirac phase δ is also called weak CP violation phase.

The complex valued Dirac mass matrix M^D can be diagonalized by two unitary matrices of the form (21), viz. $U_L = D'_L U_L^D D_L^M$ and $U_R = D'_R U_R^D D_R^M$. The result reads

$$M^D = U_R^{T\dagger} M^d U_L^\dagger, \quad M^d = \text{diag}(\bar{m}_1, \bar{m}_2, \bar{m}_3), \quad (23)$$

with the real positive \bar{m}_i (we denote Dirac mass eigenvalues by \bar{m}_i to distinguish them from the physical masses m_i , the eigenvalues in absolute value of the total mass matrix. Notice also that while the left hand side of Equation (23) has nine complex or 18 real parameters, the right hand side has $9 + 3 + 9$; but since M^d is diagonal, the diagonal matrices $D_L^{M^\dagger} = D_L^{M*}$ and D_R^{M*} only act as a product. Hence it is allowed to fix D_R^M before solving D_L^M , see below Equation (28). The number of parameters available for the diagonalization is then still 18). We identify U_L^D with the PMNS mixing matrix $U^D = U_1 U_2 U_3$ and D_L^M with the Majorana matrix $D^M = \text{diag}(e^{i\eta_1}, e^{i\eta_2}, e^{i\eta_3})$ employed in literature.

To connect the transformation (23) to M^{DM} , we introduce the 6×6 unitary matrix

$$U_{LR} = \begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix}, \quad (24)$$

and define, using that $M^{dT} = M^d$ since it is diagonal,

$$\mathcal{M} = U_{LR}^T M^{DM} U_{LR} = \begin{pmatrix} 0 & M^d \\ M^d & M^N \end{pmatrix}. \quad (25)$$

New active and sterile fields $\mathbf{n}_{aL} = U_L^\dagger \mathbf{v}_{fL}$, $\mathbf{n}_{sR} = U_R^T \mathbf{v}_{fR}$, merged as

$$\mathbf{n}_L = (n_{aL}^1, n_{aL}^2, n_{aL}^3, n_{sR}^{1c}, n_{sR}^{2c}, n_{sR}^{3c})^T, \quad (26)$$

express (8) as

$$\mathbf{N}_{fL} = (\mathbf{v}_{fL}^T, \mathbf{v}_{fR}^{cT})^T = U_{LR} \mathbf{n}_L, \quad \mathbf{n}_L = U_{LR}^\dagger \mathbf{N}_{fL}. \quad (27)$$

With these steps the right handed Majorana mass matrix transforms into

$$M^N = U_R^T M_R^M U_R. \quad (28)$$

Like M_R^M , it is complex symmetric, but since U_R was needed to diagonalize M^D , it will in general not result in a diagonal M^N . With the decomposition $U_R = D_R U_R^D D_R^M$ as in Equation (21), one can, however, use the phases in D_R^M to make the off-diagonal elements of M^N real and nonnegative (While the left hand side of Equation (23) has nine complex or 18 real parameters, the right hand side has $9 + 3 + 9$; but since M^d is diagonal, the diagonal matrices $D_L^{M^\dagger} = D_L^{M*}$ and D_R^{M*} only act as a product. Hence it is allowed to fix D_R^M before solving D_L^M , see below Equation (28). The number of parameters available for the diagonalization is then still 18. Moreover, for n lepton families there are $\frac{1}{2}n(n-1)$ independent complex valued off-diagonal elements and n Majorana phases, so making all off-diagonal elements real and nonnegative is possible for $n = 3$ or 2).

We denote the diagonal elements of the Majorana matrix M^N by $\bar{\mu}_i$, that may still be complex, and the real positive off-diagonal elements by μ_i . The right handed Majorana mass matrix M^N then takes the form

$$M^N = \begin{pmatrix} \bar{\mu}_1 & \mu_3 & \mu_2 \\ \mu_3 & \bar{\mu}_2 & \mu_1 \\ \mu_2 & \mu_1 & \bar{\mu}_3 \end{pmatrix}, \quad (29)$$

so that the total mass matrix \mathcal{M} reads

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & \bar{m}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{m}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{m}_3 \\ \bar{m}_1 & 0 & 0 & \bar{\mu}_1 & \mu_3 & \mu_2 \\ 0 & \bar{m}_2 & 0 & \mu_3 & \bar{\mu}_2 & \mu_1 \\ 0 & 0 & \bar{m}_3 & \mu_2 & \mu_1 & \bar{\mu}_3 \end{pmatrix}. \quad (30)$$

Except in the pure Dirac limit where $\mu_i = \bar{\mu}_i = 0$, the \mathbf{n}_L are not rotations of mass eigenstates.

2.6. Diracian Limit

For reasons explained above, we wish to achieve pairwise degeneracies in the masses. The standard Dirac limit, just taking μ_i and $\bar{\mu}_i \rightarrow 0$, is a trivial way to achieve this; we shall, however, need finite values for them and design the more subtle “Diracian” limit.

To start, we notice that the eigenvalues of the mass matrix (29) follow from $\det(\mathcal{M} - \lambda I) = 0$, where

$$\begin{aligned} \det(\mathcal{M} - \lambda I) = & (\lambda^2 - \bar{m}_1^2)(\lambda^2 - \bar{m}_2^2)(\lambda^2 - \bar{m}_3^2) - (\bar{\mu}_1 + \bar{\mu}_2 + \bar{\mu}_3)\lambda^5 - (\mu_1^2 + \mu_2^2 + \mu_3^2 - \bar{\mu}_1\bar{\mu}_2 - \bar{\mu}_2\bar{\mu}_3 - \bar{\mu}_3\bar{\mu}_1)\lambda^4 \\ & + [\bar{m}_1^2(\bar{\mu}_2 + \bar{\mu}_3) + \bar{m}_2^2(\bar{\mu}_3 + \bar{\mu}_1) + \bar{m}_3^2(\bar{\mu}_1 + \bar{\mu}_2) + \mu_1^2\bar{\mu}_1 + \mu_2^2\bar{\mu}_2 + \mu_3^2\bar{\mu}_3 - 2\mu_1\mu_2\mu_3 - \bar{\mu}_1\bar{\mu}_2\bar{\mu}_3]\lambda^3 \\ & + [\bar{m}_1^2(\mu_1^2 - \bar{\mu}_2\bar{\mu}_3) + \bar{m}_2^2(\mu_2^2 - \bar{\mu}_3\bar{\mu}_1) + \bar{m}_3^2(\mu_3^2 - \bar{\mu}_1\bar{\mu}_2)]\lambda^2 - (\bar{m}_1^2\bar{m}_2^2\bar{\mu}_3 + \bar{m}_2^2\bar{m}_3^2\bar{\mu}_1 + \bar{m}_3^2\bar{m}_1^2\bar{\mu}_2)\lambda. \end{aligned} \quad (31)$$

The criterion to get pairwise degeneracies in the eigenvalues (up to signs), is simply that the odd powers in λ vanish. Let us denote

$$\begin{aligned} \bar{\Delta}_1 &= \bar{m}_2^2 - \bar{m}_3^2, & \bar{\Delta}_2 &= \bar{m}_3^2 - \bar{m}_1^2, & \bar{\Delta}_3 &= \bar{m}_1^2 - \bar{m}_2^2, \\ \bar{M}_1 &= \frac{\bar{m}_2\bar{m}_3}{\bar{m}_1}, & \bar{M}_2 &= \frac{\bar{m}_3\bar{m}_1}{\bar{m}_2}, & \bar{M}_3 &= \frac{\bar{m}_1\bar{m}_2}{\bar{m}_3}, \end{aligned} \quad (32)$$

and express the $\bar{\mu}_i$ in a common dimensionless parameter \bar{u} through

$$\bar{\mu}_i = \frac{\bar{\Delta}_i}{\bar{M}_i} \bar{u}. \quad (33)$$

The relations $\sum_i \bar{m}_i^2 \bar{\Delta}_i = \sum_i \bar{\Delta}_i = 0$ make the coefficients of λ^5 and λ^1 of Equation (31) vanish, respectively. To condense further notation, we express the μ_i into dimensionless non-negative parameters u_i ,

$$\mu_i = \sqrt{|\bar{\Delta}_1 \bar{\Delta}_2 \bar{\Delta}_3|} \frac{u_i}{\bar{m}_i \sqrt{|\bar{\Delta}_i|}}. \quad (34)$$

For normal ordering of the \bar{m}_i (notice that these are Dirac masses, not the physical masses), $\bar{m}_1 < \bar{m}_2 < \bar{m}_3$ implies $\bar{\Delta}_1 < 0$, $\bar{\Delta}_2 > 0$, $\bar{\Delta}_3 < 0$, hence $\bar{\Delta}_1 \bar{\Delta}_2 \bar{\Delta}_3 > 0$; this is also the case for the inverted ordering $\bar{m}_3 < \bar{m}_1 < \bar{m}_2$ whence $\bar{\Delta}_1 > 0$, $\bar{\Delta}_2 < 0$, $\bar{\Delta}_3 < 0$. It thus holds that

$$\mu_1 \mu_2 \mu_3 = \frac{\bar{\Delta}_1 \bar{\Delta}_2 \bar{\Delta}_3}{\bar{m}_1 \bar{m}_2 \bar{m}_3} u_1 u_2 u_3. \quad (35)$$

Equating the λ^3 coefficient of Equation (31) to zero requires

$$\bar{u}^3 - (1 + u^2)\bar{u} + 2u_1 u_2 u_3 = 0, \quad u^2 \equiv \frac{\bar{\Delta}_1}{|\bar{\Delta}_1|} u_1^2 + \frac{\bar{\Delta}_2}{|\bar{\Delta}_2|} u_2^2 + \frac{\bar{\Delta}_3}{|\bar{\Delta}_3|} u_3^2 = \frac{\bar{\Delta}_1}{|\bar{\Delta}_1|} (u_1^2 - u_2^2) - u_3^2. \quad (36)$$

This cubic equation has the solutions for $n = -1, 0, 1$, and positive or negative u^2 ,

$$\bar{u} = \frac{-i}{\sqrt{3}} \left[e^{2\pi i n/3} u_+^{1/3} - e^{-2\pi i n/3} (1+u^2) u_+^{-1/3} \right], \quad (37)$$

$$u_+ = \sqrt{D} + i\sqrt{27}u_1u_2u_3, \quad D = (1+u^2)^3 - 27u_1^2u_2^2u_3^2.$$

We restrict ourselves to real solutions; there is always one. Then the matrix \mathcal{M} is real-valued. All solutions are real when $D > 0$, which occurs in particular when the u_i are small, i.e., in the pseudo-Dirac case. Then there exist the large solutions $n = \pm 1$ with $\bar{u} \approx \pm 1$, which in both cases leads to the eigenvalues $\lambda_i^\pm \approx \pm M_i$ for $i = 1, 2, 3$. For small u_i the $n = 0$ solution has a small \bar{u} and $\bar{\mu}_i$, viz.

$$\bar{u} \approx 2u_1u_2u_3 = 2 \frac{\mu_1\mu_2\mu_3}{\bar{\Delta}_1\bar{\Delta}_2\bar{\Delta}_3} \bar{m}_1\bar{m}_2\bar{m}_3, \quad \bar{\mu}_i \approx 2 \frac{\mu_1\mu_2\mu_3}{\bar{\Delta}_1\bar{\Delta}_2\bar{\Delta}_3} \bar{m}_i^2 \bar{\Delta}_i = 2u_1u_2u_3 \frac{\bar{\Delta}_i}{M_i}. \quad (38)$$

The Diracian limit, defined by Equations (33), (34) and (37), reduces Equation (31) to a cubic polynomial in λ^2 ,

$$(\lambda^2 - \bar{m}_1^2)(\lambda^2 - \bar{m}_2^2)(\lambda^2 - \bar{m}_3^2) + \lambda^2 \bar{\Delta}_1 \bar{\Delta}_2 \bar{\Delta}_3 \left[\frac{(\lambda^2 - \bar{m}_1^2)(\bar{u}^2 - u_1^2)}{\bar{m}_1^2 \bar{\Delta}_1} + \frac{(\lambda^2 - \bar{m}_2^2)(\bar{u}^2 - u_2^2)}{\bar{m}_2^2 \bar{\Delta}_2} + \frac{(\lambda^2 - \bar{m}_3^2)(\bar{u}^2 - u_3^2)}{\bar{m}_3^2 \bar{\Delta}_3} \right] = 0. \quad (39)$$

Its analytical roots are intricate, but they are easily calculated numerically. Denoting them as m_i^2 , the squares of the physical masses, the eigenvalues of \mathcal{M} are $\lambda_{2i-1} = -m_i$ and $\lambda_{2i} = +m_i > 0$ for $i = 1, 2, 3$. From $\det \mathcal{M} = -\bar{m}_1^2 \bar{m}_2^2 \bar{m}_3^2$ it holds that $m_1 m_2 m_3 = \bar{m}_1 \bar{m}_2 \bar{m}_3$. The eigenvectors are set by

$$\sum_{k=1}^6 \mathcal{M}_{jk} e_k^{(i)} = \lambda_i e_j^{(i)}, \quad (i = 1, \dots, 6). \quad (40)$$

and they are real and orthonormal. They can be expressed as

$$\mathbf{e}^{(2i-1)} = \frac{\mathbf{e}^{(i\tilde{a})} - \mathbf{e}^{(i\tilde{s})}}{\sqrt{2}}, \quad \mathbf{e}^{(2i)} = \frac{\mathbf{e}^{(i\tilde{a})} + \mathbf{e}^{(i\tilde{s})}}{\sqrt{2}}, \quad (41)$$

with orthonormal $\mathbf{e}^{(i\tilde{a})}$ and $\mathbf{e}^{(i\tilde{s})}$ for $i = 1, 2, 3$. For small μ_i and $\bar{\mu}_i$ the $\mathbf{e}^{(i\tilde{a})}$ and $\mathbf{e}^{(i\tilde{s})}$ read to first order

$$\begin{aligned} \mathbf{e}^{(1\tilde{a})} &= \left(1, 0, 0, \frac{\bar{\mu}_1}{4\bar{m}_1}, \bar{m}_1 \frac{\mu_3}{\bar{\Delta}_3}, -\bar{m}_1 \frac{\mu_2}{\bar{\Delta}_2} \right)^T, & \mathbf{e}^{(1\tilde{s})} &= \left(-\frac{\bar{\mu}_1}{4\bar{m}_1}, \bar{m}_2 \frac{\mu_3}{\bar{\Delta}_3}, -\bar{m}_3 \frac{\mu_2}{\bar{\Delta}_2}, 1, 0, 0 \right)^T, \\ \mathbf{e}^{(2\tilde{a})} &= \left(0, 1, 0, -\bar{m}_2 \frac{\mu_3}{\bar{\Delta}_3}, \frac{\bar{\mu}_2}{4\bar{m}_2}, \bar{m}_2 \frac{\mu_1}{\bar{\Delta}_1} \right)^T, & \mathbf{e}^{(2\tilde{s})} &= \left(-\bar{m}_1 \frac{\mu_3}{\bar{\Delta}_3}, -\frac{\bar{\mu}_2}{4\bar{m}_2}, \bar{m}_3 \frac{\mu_1}{\bar{\Delta}_1}, 0, 1, 0 \right)^T, \\ \mathbf{e}^{(3\tilde{a})} &= \left(0, 0, 1, \bar{m}_3 \frac{\mu_2}{\bar{\Delta}_2}, -\bar{m}_3 \frac{\mu_1}{\bar{\Delta}_1}, \frac{\bar{\mu}_3}{4\bar{m}_3} \right)^T, & \mathbf{e}^{(3\tilde{s})} &= \left(\bar{m}_1 \frac{\mu_2}{\bar{\Delta}_2}, -\bar{m}_2 \frac{\mu_1}{\bar{\Delta}_1}, -\frac{\bar{\mu}_3}{4\bar{m}_3}, 0, 0, 1 \right)^T. \end{aligned} \quad (42)$$

Notice that the i and $i + 3$ components of $\mathbf{e}^{(i\tilde{a})}$ and $\mathbf{e}^{(i\tilde{s})}$ stem with the one-family case Equation (15).

With the first three components of these vectors relating to active neutrinos and the last three to sterile ones, it is seen that for small μ_i and $\bar{\mu}_i$ the $\mathbf{e}^{(i\tilde{a})}$ are mainly active and the $\mathbf{e}^{(i\tilde{s})}$ mainly sterile, which we indicate by the tildes. The $\mathbf{e}^{(2i-1)}$ and $\mathbf{e}^{(2i)}$ are 45° rotations of the $\mathbf{e}^{(i\tilde{a})}$ and $\mathbf{e}^{(i\tilde{s})}$, which is maximal mixing. In the standard Dirac limit, it is customary to work with Dirac states and not with Majorana states. Likewise, in our Diracian limit the mass degeneracies allow the rotations to be circumvented by working with the $\mathbf{e}^{(i\tilde{a})}$ and $\mathbf{e}^{(i\tilde{s})}$ themselves. Indeed, there holds the exact decomposition

$$\mathcal{M} = \sum_{i=1}^6 \lambda_i \mathbf{e}^{(i)} \mathbf{e}^{(i)T} = \sum_{i=1}^3 m_i \left[\mathbf{e}^{(i\bar{a})} \mathbf{e}^{(i\bar{s})T} + \mathbf{e}^{(i\bar{s})} \mathbf{e}^{(i\bar{a})T} \right]. \quad (43)$$

These steps allow us to retrieve the standard Dirac expressions in the limit where the Majorana masses $\mu_i, \bar{\mu}_i$ vanish, whence the $\mathbf{e}^{(i\bar{a})}$ and $\mathbf{e}^{(i\bar{s})}$ become purely active and purely sterile states, respectively.

For neutrino oscillation probabilities in vacuum (see Sections 2.4 and 3.2) one needs the eigenvalues m_i^2 and eigenvectors $\mathbf{e}^{(i\bar{a})}$ and $\mathbf{e}^{(i\bar{s})}$ of \mathcal{M}^2 ,

$$\mathcal{M}^2 = \sum_{i=1}^3 m_i^2 \left[\mathbf{e}^{(i\bar{a})} \mathbf{e}^{(i\bar{a})T} + \mathbf{e}^{(i\bar{s})} \mathbf{e}^{(i\bar{s})T} \right]. \quad (44)$$

In terms of \mathbf{n}_L defined above Equation (27) and related there to the flavor states (8), the fields for the mass eigenstates are

$$v_{\bar{a}L}^i = \sum_{j=1}^6 e_j^{(i\bar{a})} n_{jL}, \quad v_{\bar{s}R}^i = \sum_{j=1}^6 e_j^{(i\bar{s})} n_{jL}^c, \quad (i = 1, 2, 3). \quad (45)$$

Here $j = 1, 2, 3$ label active fields and $j = 4, 5, 6$ sterile ones. Hence the fields $v_{\bar{a}L}^i$ annihilate chiral left handed, *mainly* active neutrinos and create similar right handed antineutrinos, while the $v_{\bar{s}R}^i$ annihilate chiral right handed, *mainly* sterile neutrinos and create similar left handed antineutrinos.

The mass term of the 6 Majorana fields now takes the form of 3 Dirac terms,

$$\begin{aligned} \mathcal{L}_m &= \frac{1}{2} \sum_{i=1}^3 m_i \left[v_{\bar{a}L}^{iT} C^\dagger v_{\bar{s}R}^{ic} + (v_{\bar{s}R}^{ic})^T C^\dagger v_{\bar{a}L}^i \right] + h.c. = \frac{1}{2} \sum_{i=1}^3 m_i \left(v_{\bar{a}L}^i \overline{v_{\bar{s}R}^i}^T - \overline{v_{\bar{s}R}^i} v_{\bar{a}L}^i \right) + h.c. \\ &= - \sum_{i=1}^3 m_i \left(\overline{v_{\bar{s}R}^i} v_{\bar{a}L}^i + \overline{v_{\bar{a}L}^i} v_{\bar{s}R}^i \right) = - \sum_{i=1}^3 m_i \overline{v^i} v^i, \end{aligned} \quad (46)$$

because fermion fields anticommute and left and right handed fields are orthogonal. The here introduced Dirac fields,

$$v^i = v_{\bar{a}L}^i + v_{\bar{s}R}^i, \quad (i = 1, 2, 3), \quad (47)$$

combine left and right handed chiral fields, as usual. They are the mass eigenstates. In this basis the Dirac–Majorana neutrino Lagrangian is a sum of Dirac terms,

$$\mathcal{L} = \sum_{i=1}^3 \left(\overline{v^i} i \overleftrightarrow{\not{\partial}} v^i - m_i \overline{v^i} v^i \right). \quad (48)$$

2.7. Charged and Neutral Current

Neutrinos also enter the currents coupled to the W and Z gauge bosons, which are part of the covariant derivatives in the Lagrangian, see Equation (A11) below. The W boson couples to the charged weak current. On the flavor basis it reads

$$\mathcal{L}_{CC} = -\frac{g}{2\sqrt{2}} \left[W_\mu J_{CC}^{\mu\dagger} + W_\mu^\dagger J_{CC}^\mu \right], \quad J_{CC}^\mu = 2 \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} \gamma^\mu \ell_{\alpha L}, \quad J_{CC}^{\mu\dagger} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\ell_{\alpha L}} \gamma^\mu \nu_{\alpha L}, \quad (49)$$

with g the weak coupling constant. The neutral weak current reads on the flavor basis

$$\mathcal{L}_{NC} = -\frac{g}{2\cos\theta_w} Z_\mu J_{NC}^\mu, \quad J_{NC}^\mu = \sum_{\alpha=e,\mu,\tau} \bar{\nu}_{\alpha L} \gamma^\mu \nu_{\alpha L}, \quad (50)$$

with θ_w the weak or Weinberg angle.

To express these in the mass eigenstates, we define \mathcal{A} and \mathcal{S} as matrices consisting of the active components of the 6-component eigenvectors $\mathbf{e}^{(i\bar{a})}$ and $\mathbf{e}^{(i\bar{s})}$, respectively,

$$\mathcal{A}_{ji} = e_j^{(i\bar{a})}, \quad \mathcal{S}_{ji} = e_j^{(i\bar{s})}, \quad (j, i = 1, 2, 3), \quad (51)$$

and, likewise, \mathcal{A}^s and \mathcal{S}^s for the sterile components

$$\mathcal{A}_{ji}^s = e_{j+3}^{(i\bar{a})}, \quad \mathcal{S}_{ji}^s = e_{j+3}^{(i\bar{s})}, \quad (j, i = 1, 2, 3). \quad (52)$$

From (42) we read off that for small μ_i and $\bar{\mu}_i$

$$\mathcal{A}_{ji} \approx \delta_{ij}, \quad \mathcal{S}_{ji} \approx -\delta_{ij} \frac{\bar{\mu}_i}{4\bar{m}_i} + \sum_{k=1}^3 \varepsilon_{ijk} \bar{m}_j \frac{\mu_k}{\Delta_k}, \quad (53)$$

$$\mathcal{S}_{ji}^s \approx \delta_{ij}, \quad \mathcal{A}_{ji}^s \approx \delta_{ij} \frac{\bar{\mu}_i}{4\bar{m}_i} + \sum_{k=1}^3 \varepsilon_{ijk} \bar{m}_j \frac{\mu_k}{\Delta_k}. \quad (54)$$

From the orthonormality of the eigenvectors it follows that the real valued 3×3 matrices \mathcal{A} and \mathcal{S} satisfy the unitarity relation

$$\mathcal{A}\mathcal{A}^T + \mathcal{S}\mathcal{S}^T = 1_{3 \times 3}, \quad (55)$$

while $\mathcal{A}^T \mathcal{A} + \mathcal{S}^T \mathcal{S} \neq 1_{3 \times 3}$.

From Equations (7), (8), (21), (22), (24) and (27), and denoting $D^M \equiv D_L^M$, we have $\nu_{fL} = D_L U^D D^M \mathbf{n}_{aL}$. As shown below Equation (50), the diagonal phase matrix D_L' can be absorbed in the fields. Inverting Equation (45) leaves Equation (48) invariant and expresses the flavor eigenstates as superpositions of mass eigenstates $\nu_{mL} = (\nu_{aL}, \nu_{sR}^c)$. In vector notation, and using $\bar{\nu}_{sR}^c = -\nu_{sR}^T C^\dagger$, one has

$$\begin{aligned} \nu_{fL} &= U^{DM} (\mathcal{A} \nu_{aL} + \mathcal{S} \nu_{sR}^c) = A \nu_{aL} + S \nu_{sR}^c, \\ \bar{\nu}_{fL} &= (\bar{\nu}_{aL} \mathcal{A}^T + \bar{\nu}_{sR}^c \mathcal{S}^T) U^{DM\dagger} = \bar{\nu}_{aL} A^\dagger + \bar{\nu}_{sR}^c S^\dagger, \\ &= (\bar{\nu}_{aL} \mathcal{A}^T - \nu_{sR}^T C^\dagger \mathcal{S}^T) U^{DM\dagger} = \bar{\nu}_{aL} A^\dagger - \nu_{sR}^T C^\dagger S^\dagger. \end{aligned} \quad (56)$$

Here $U^{DM} = U^D D^M$, with U^D is the standard PMNS matrix, see (21), while $D^M = \text{diag}(e^{i\eta_1}, e^{i\eta_2}, e^{i\eta_3})$ is the Majorana matrix of the three-neutrino problem; its η_i are Dirac phases now (a word on nomenclature: the Majorana phases in the matrix D^M stem from the $3 + 0$ SvM , without sterile neutrinos. While they become physical Dirac phases in the $3 + 3$ Dirac–Majorana neutrino standard model (DM ν SM), there appear no true $3 + 3$ Majorana phases, so we propose to keep this name for them. Hence the DM ν SM has three physical phases: one Dirac and two “Majorana” phases. They all appear in the CP-invariance breaking part of the neutrino oscillation probabilities, see Equation (81). We also introduced

$$A = U^{DM} \mathcal{A}, \quad S = U^{DM} \mathcal{S}, \quad AA^\dagger + SS^\dagger = 1_{3 \times 3}. \quad (57)$$

The sterile field $\nu_{sR}^c = (\nu_{sR}^{1c}, \nu_{sR}^{2c}, \nu_{sR}^{3c})$ similar to (56) reads

$$\nu_{sR}^c = D_R' U_R^D D_R^M (\mathcal{A}^s \nu_{\bar{a}} + \mathcal{S}^s \nu_{\bar{s}}^c). \quad (58)$$

The only current knowledge of the involved matrix elements lies in (54).

The flavor eigenstates can also be written as single sums over mass eigenstates,

$$\nu_{\alpha L} = \sum_{i=1}^6 U_{\alpha i} \nu_{mL}^i, \quad \nu_{fL} = U \nu_{mL}, \quad \nu_{mL} = (\nu_{\bar{a}L}^1, \nu_{\bar{a}L}^2, \nu_{\bar{a}L}^3, \nu_{\bar{s}R}^{1c}, \nu_{\bar{s}R}^{2c}, \nu_{\bar{s}R}^{3c})^T, \quad (59)$$

with the 3×6 PMNS matrix U having elements

$$U_{\alpha,i} = A_{\alpha i} = (U^{DM} \mathcal{A})_{\alpha i}, \quad U_{\alpha,i+3} = S_{\alpha i} = (U^{DM} \mathcal{S})_{\alpha i}, \quad (UU^\dagger)_{\alpha\beta} = \delta_{\alpha\beta}, \quad (\alpha, \beta = 1, 2, 3). \quad (60)$$

the latter deriving from (55), while $U^\dagger U \neq 1_{6 \times 6}$, because U represents the three active rows of a unitary 6×6 matrix which also involves \mathcal{A}^s and \mathcal{S}^s . Hence the GIM theorem that J_{NC}^μ has the same form on flavor and mass basis, does not hold [12].

Inserted in the currents the relations (56) yield

$$J_{CC}^\mu = 2\bar{\nu}_{mL} U^\dagger \gamma^\mu \ell_L = 2(\bar{\nu}_{\bar{a}L} A^\dagger + \bar{\nu}_{\bar{s}R}^c S^\dagger) \gamma^\mu \ell_L = 2(\bar{\nu}_{\bar{a}L} \mathcal{A}^T - \nu_{\bar{s}R}^T \mathcal{C}^\dagger \mathcal{S}^T) U^{DM\dagger} \gamma^\mu \ell_L \quad (61a)$$

$$J_{CC}^{\mu\dagger} = 2\bar{\ell}_L \gamma^\mu U \nu_{mL} = 2\bar{\ell}_L \gamma^\mu (A \nu_{\bar{a}L} + S \nu_{\bar{s}R}^c) = 2\bar{\ell}_L \gamma^\mu U^{DM} (\mathcal{A} \nu_{\bar{a}L} + \mathcal{S} \nu_{\bar{s}R}^c), \quad (61b)$$

$$J_{NC}^\mu = \bar{\nu}_{\bar{a}L} \gamma^\mu \mathcal{A}^T \mathcal{A} \nu_{\bar{a}L} - \bar{\nu}_{\bar{s}R} \gamma^\mu \mathcal{S}^T \mathcal{S} \nu_{\bar{s}R} - \nu_{\bar{s}R}^T \mathcal{C}^\dagger \gamma^\mu \mathcal{S}^T \mathcal{A} \nu_{\bar{a}L} - \nu_{\bar{a}L}^\dagger \gamma^\mu \mathcal{C} \mathcal{A}^T \mathcal{S} \nu_{\bar{s}R}^{\dagger T}. \quad (61c)$$

2.8. Lepton Number for Sterile Neutrinos

There is an ambiguity in defining the lepton number of the sterile neutrinos. The lepton number of neutrinos is investigated by making the transformation

$$\nu_{aL} \rightarrow e^{iL_a \phi} \nu_{aL}, \quad \nu_{sR} \rightarrow e^{iL_s \phi} \nu_{sR}, \quad (62)$$

This leaves the kinetic terms invariant and for the standard choice $L_s = L_a = 1$ also the Dirac mass terms (2). Only the Majorana mass terms (1) and (3) will vary by factors $e^{\pm 2i\phi}$: they violate lepton number conservation by $\Delta L = \pm 2$. This approach connects the lepton number $L_a = 1$ of active neutrinos also to sterile neutrinos, hence $L_{\bar{\nu}_s} = -1$ for sterile antineutrinos (charge conjugated sterile ones). This assigns lepton number $+1$ to the components $j = 1, 2, 3$ of \mathbf{N}_{fL} of Equation (8) and \mathbf{n}_L of Equation (26), but -1 to the components $j = 4, 5, 6$. Then the mixing (45), or its reverse (56), (58), enforced by the nonvanishing right handed Majorana mass matrix, makes it impossible to consistently connect a lepton number to the particles connected to the mass eigenstates $\nu_{\bar{a}}^i$ and $\nu_{\bar{s}}^i$.

The opposite choice $L_a = 1, L_s = -1$ circumvents this problem for general models with active and sterile neutrinos. According to (8), (45) and (56) the lepton number $L_a = 1$ of ν_a particles is consistent with $L_{\bar{a}} = 1$ of a $\nu_{\bar{a}}$ particle and $L_{\bar{s}} = -1$ of $\nu_{\bar{s}}$ particles. This choice is henceforward consistent with (58). The benefit of this convention is that in pion and neutron decay both channels $\pi^- \rightarrow \mu + \bar{\nu}_{\bar{a}L}$, $\pi^- \rightarrow \mu + \nu_{\bar{s}R}$, and $n \rightarrow p + e + \bar{\nu}_{\bar{e}}, n \rightarrow p + e + \nu_{\bar{s}}$, respectively, satisfy lepton number conservation.

The minor price to pay is that now both the Dirac mass term and the Majorana terms violate lepton number conservation by two units, so that the unsolvable problem of lepton number violation remains unsolved. Indeed, as we shall discuss below, the Majorana mass terms still allow for neutrinoless double β decay, where a nucleus decays by emitting two electrons (or two positrons) but no (anti)neutrinos.

3. Applications

3.1. Estimates for the Dirac and Majorana Masses

For later use we present the eigenvalues up to third order in $\mu_{1,2,3}$ and $\bar{\mu}_{1,2,3}$, viz.

$$\begin{aligned}\lambda_1^\pm &= \pm \bar{m}_1 \left(1 - \frac{\mu_2^2}{2\bar{\Delta}_2} + \frac{\mu_3^2}{2\bar{\Delta}_3} \right) + \frac{\bar{\mu}_1}{2} - \frac{\bar{m}_1^2 \mu_1 \mu_2 \mu_3}{\bar{\Delta}_2 \bar{\Delta}_3}, \\ \lambda_2^\pm &= \pm \bar{m}_2 \left(1 - \frac{\mu_3^2}{2\bar{\Delta}_3} + \frac{\mu_1^2}{2\bar{\Delta}_1} \right) + \frac{\bar{\mu}_2}{2} - \frac{\bar{m}_2^2 \mu_1 \mu_2 \mu_3}{\bar{\Delta}_3 \bar{\Delta}_1}, \\ \lambda_3^\pm &= \pm \bar{m}_3 \left(1 - \frac{\mu_1^2}{2\bar{\Delta}_1} + \frac{\mu_2^2}{2\bar{\Delta}_2} \right) + \frac{\bar{\mu}_3}{2} - \frac{\bar{m}_3^2 \mu_1 \mu_2 \mu_3}{\bar{\Delta}_1 \bar{\Delta}_2}.\end{aligned}\quad (63)$$

Due to Equation (38) the last terms cancel, to make the $m_i = |\lambda_i^\pm|$ pairwise degenerate. Employing the averages $\bar{m}^2 = \frac{1}{3}(\bar{m}_1^2 + \bar{m}_2^2 + \bar{m}_3^2)$ and $m^2 = \frac{1}{3}(m_1^2 + m_2^2 + m_3^2)$, the mass-squared differences $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ become approximately,

$$\Delta_1 \equiv \Delta m_{23}^2 = \bar{\Delta}_1 + \bar{m}^2 \left(\frac{2\mu_1^2}{\bar{\Delta}_1} - \frac{\mu_2^2}{\bar{\Delta}_2} - \frac{\mu_3^2}{\bar{\Delta}_3} \right), \quad \Delta_2 \equiv \Delta m_{31}^2 = \bar{\Delta}_2 + \bar{m}^2 \left(\frac{2\mu_2^2}{\bar{\Delta}_2} - \frac{\mu_3^2}{\bar{\Delta}_3} - \frac{\mu_1^2}{\bar{\Delta}_1} \right), \quad (64a,b)$$

$$\Delta_3 \equiv \Delta m_{12}^2 = \bar{\Delta}_3 + \bar{m}^2 \left(\frac{2\mu_3^2}{\bar{\Delta}_3} - \frac{\mu_1^2}{\bar{\Delta}_1} - \frac{\mu_2^2}{\bar{\Delta}_2} \right), \quad (64c)$$

provided that $\mu_i^2 \ll |\bar{\Delta}_i|$. It holds that $-\Delta_3 = \Delta m_{\text{sol}}^2 = (7.53 \pm 0.18) 10^{-5} \text{eV}^2$ [16]. Normal ordering $m_1 < m_2 < m_3$ is connected to $-\Delta_1 = \Delta m_{\text{atm}}^2 = (2.44 \pm 0.06) 10^{-3} \text{eV}^2$ [16] and $\Delta_2 = -\Delta_1 - \Delta_3 > 0$, while inverse ordering $m_3 < m_1 < m_2$ leads to $\Delta_1 = \Delta m_{\text{atm}}^2$ and $\Delta_2 < 0$. Cluster lensing puts forward a value $\bar{m} \sim 1.5\text{--}1.9 \text{ eV}$ for the absolute scale of the neutrino masses [8–11].

With ε_{ijk} the Levi-Civita symbol, there hold the exact relations

$$\bar{m}_i^2 = \bar{m}^2 + \frac{1}{3} \sum_{j,k=1}^3 \varepsilon_{ijk} \bar{\Delta}_k, \quad m_i^2 = m^2 + \frac{1}{3} \sum_{j,k=1}^3 \varepsilon_{ijk} \Delta_k. \quad (65)$$

Let us investigate Equation (64) for normal ordering. The effects of the Majorana masses are anticipated to occur at the level of Δm_{sol}^2 . We fix \bar{m}_2 and express $\bar{m}_{1,3}$ in $d_{1,3}$ as

$$\bar{m}_1^2 = \bar{m}_2^2 - d_3 \Delta m_{\text{sol}}^2, \quad \bar{m}_3^2 = \bar{m}_2^2 + \Delta m_{\text{atm}}^2 - d_1 \Delta m_{\text{sol}}^2, \quad (66)$$

so that

$$-\bar{\Delta}_1 = \Delta m_{\text{atm}}^2 - d_1 \Delta m_{\text{sol}}^2, \quad -\bar{\Delta}_3 = d_3 \Delta m_{\text{sol}}^2. \quad (67)$$

With $\bar{m} \approx \bar{m}_2$ we set also

$$\Theta_3 = \frac{\mu_3 \bar{m}}{d_3 \Delta m_{\text{sol}}^2} = \frac{\Delta m_{\text{atm}}^2}{\Delta m_{\text{sol}}^2} \frac{u_3}{d_3}, \quad \mu_3 = d_3 \Theta_3 \frac{\Delta m_{\text{sol}}^2}{\bar{m}}, \quad u_3 = d_3 \Theta_3 \frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2}. \quad (68)$$

With $d_{1,3}$ and Θ_3 fixed we can determine $\mu_{1,2}$ or, equivalently, $u_{1,2}$. Imposing $\Delta m_{\text{sol}}^2 \ll \Delta m_{\text{atm}}^2 \ll \bar{m}^2$, we deduce from Equation (64c) and from the difference of (64a) and (64b) that

$$\begin{aligned} u_1^2 &= \frac{2d_1 - 5(1 - d_3)}{12d_3} + \Theta_3^2, & \mu_1 &= \sqrt{|\bar{\Delta}_2 \bar{\Delta}_3|} \frac{u_1}{\bar{m}_1}, \\ u_2^2 &= \frac{2d_1 + 7(1 - d_3)}{12d_3} - \Theta_3^2, & \mu_2 &= \sqrt{|\bar{\Delta}_3 \bar{\Delta}_1|} \frac{u_2}{\bar{m}_2}, \\ u^2 &= u_2^2 - u_1^2 - u_3^2 = \frac{1 - d_3}{d_3} - 2\Theta_3^2 - d_3^2 \Theta_3^2 \left(\frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2} \right)^2. \end{aligned} \quad (69)$$

These are expressions of order unity and exact to leading order in $d_1, d_3 - 1$ and Θ_3 . Hence the typical scale is $\mu_{1,2} \sim \sqrt{\Delta m_{\text{atm}}^2 \Delta m_{\text{sol}}^2} / \bar{m} = 4.3 \cdot 10^{-4} \text{eV}^2 / \bar{m}$ and $\mu_3 \sim 8 \cdot 10^{-5} \text{eV}^2 / \bar{m}$.

In the eigenvectors (42) the coefficients

$$\frac{\bar{\mu}_1}{4\bar{m}} \approx -\frac{\bar{\mu}_2}{4\bar{m}} \approx -u_1 u_2 u_3 \frac{\Delta m_{\text{atm}}^2}{2\bar{m}^2}, \quad \frac{\bar{\mu}_3}{4\bar{m}} \approx u_1 u_2 u_3 \frac{\Delta m_{\text{sol}}^2}{2\bar{m}^2}, \quad (70)$$

are typically rather small. Hence the mixing matrix S will essentially involve elements $\pm \Theta_{1,2,3}$

$$\Theta_{1,2} = \frac{\bar{m} \mu_{1,2}}{|\bar{\Delta}_{1,2}|} \approx u_{1,2} \sqrt{\frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2}} = 0.18 u_{1,2}, \quad \Theta_3 \approx 32 \frac{u_3}{d_3}. \quad (71)$$

with Θ_3 introduced in (68). If one of the Θ_i dominates but is still small, it can be seen as a mixing angle. In particular, $\Theta_3 \sim 0.1$ is possible, which is relevant for the ANITA events to be discussed below.

3.2. Neutrino Oscillations in Vacuum

We consider neutrino oscillations in the plane wave approximation. See Ref. [17] for an excellent discussion of its merits. In the notation (59), (60) an initially pure active state vector reads

$$|\nu_{\alpha L}(0)\rangle = \sum_{i=1}^6 U_{\alpha i}^* |\nu_{mL}^i\rangle. \quad (72)$$

It evolves after time t into

$$|\nu_{\alpha L}(t)\rangle = \sum_{i=1}^6 U_{\alpha i}^* e^{-i\phi_i} |\nu_{mL}^i\rangle, \quad (73)$$

with the Lorentz invariant phase at $|\mathbf{r}| \approx ct$ given by

$$\phi_i = \frac{E_i t - \mathbf{p} \cdot \mathbf{r}}{\hbar} \approx \left(\sqrt{1 + \frac{m_i^2 c^2}{p^2}} - 1 \right) \frac{pct}{\hbar} \approx \frac{m_i^2 c^3 t}{2\hbar p}. \quad (74)$$

This result can be motivated for a wave packet [17]. The amplitude for transition into active state β is

$$\langle \nu_{\beta L} | \nu_{\alpha L}(t) \rangle = \sum_{i=1}^6 U_{\beta i} U_{\alpha i}^* e^{-i\phi_i} = \sum_{i,k,l=1}^3 U_{\beta k}^D U_{\alpha l}^{D*} (\mathcal{A}_{ki} \mathcal{A}_{li} + \mathcal{S}_{ki} \mathcal{S}_{li}) e^{i(\eta_k - \eta_l) - i\phi_i}. \quad (75)$$

where we used that $\phi_{2i-1} = \phi_{2i}$. The transition probability after time t may be expressed as two terms,

$$P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{DM} = |\langle \nu_{\beta L} | \nu_{\alpha L}(t) \rangle|^2 = P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^D + P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^M, \quad (76)$$

where the first one

$$P_{\nu_\alpha \rightarrow \nu_\beta}^D = \delta_{\alpha\beta} - \sum_{i,j=1}^3 U_{\beta i}^D U_{\alpha i}^{D*} U_{\beta j}^{D*} U_{\alpha j}^D (1 - e^{i\phi_j - i\phi_i}), \quad (77)$$

represents the standard “Dirac” result and where the Majorana masses add the “Majorana” expression

$$P_{\nu_\alpha \rightarrow \nu_\beta}^M = \sum_{i,j,k,l,m,n=1}^3 U_{\beta k}^D U_{\alpha l}^{D*} U_{\beta m}^{D*} U_{\alpha n}^D e^{i(\eta_k - \eta_l - \eta_m + \eta_n)} (e^{i\phi_j - i\phi_i} - 1) \times [(\mathcal{A}_{ki}\mathcal{A}_{li} + \mathcal{S}_{ki}\mathcal{S}_{li})(\mathcal{A}_{mj}\mathcal{A}_{nj} + \mathcal{S}_{mj}\mathcal{S}_{nj}) - \delta_{ki}\delta_{li}\delta_{mj}\delta_{nj}]. \quad (78)$$

The fact that $\sum_{\beta=1}^3 P_{\nu_\alpha \rightarrow \nu_\beta}^M \leq 0$ reflects oscillation into sterile states. While the Majorana phases η_i cancel in $P_{\nu_\alpha \rightarrow \nu_\beta}^D$ as usual, they remain present in $P_{\nu_\alpha \rightarrow \nu_\beta}^M$. This occurs in the DMνSM because they are upgraded to physical Dirac phases (a word on nomenclature: the Majorana phases in the matrix D^M stem from the 3 + 0 SνM, without sterile neutrinos. While they become physical Dirac phases in the 3 + 3 DMνSM, there appear no true 3 + 3 Majorana phases, so we propose to keep this name for them. Hence the DMνSM has three physical phases: one Dirac and two “Majorana” phases. They all appear in the CP-invariance breaking part of the neutrino oscillation probabilities, see Equation (81)). $P_{\nu_\alpha \rightarrow \nu_\beta}^{DM}$ has the schematic δ -dependence $1 + \cos \delta + \sin \delta + \cos 2\delta + \sin 2\delta$, but the $\cos 2\delta$, $\sin 2\delta$ terms are turned into $\cos \delta$, $\sin \delta$ terms by taking $\eta_3 \rightarrow \eta'_3 = \delta + \eta_3$. This is equivalent to replace U^D of (21) by

$$\tilde{U}^D = U_1 U_2 U_3 \times \text{diag}(1, 1, e^{i\delta}) = \begin{pmatrix} c_2 c_3 & c_2 s_3 & s_2 \\ -c_1 s_3 - s_1 s_2 c_3 e^{i\delta} & c_1 c_3 - s_1 s_2 s_3 e^{i\delta} & s_1 c_2 e^{i\delta} \\ s_1 s_3 - c_1 s_2 c_3 e^{i\delta} & -s_1 c_3 - c_1 s_2 s_3 e^{i\delta} & c_1 c_2 e^{i\delta} \end{pmatrix}, \quad (79)$$

wherein δ enters only in the schematic form $1 + e^{i\delta}$. Absolute Majorana phases have no physical meaning; indeed, the expression (78) involves them only through their differences. The CP violation effect,

$$\Delta P_{\nu_\alpha \leftrightarrow \nu_\beta}^{CP} = P_{\nu_\alpha \rightarrow \nu_\beta}^{DM} - P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta}^{DM} = P_{\nu_\alpha \rightarrow \nu_\beta}^{DM} - P_{\nu_\beta \rightarrow \nu_\alpha}^{DM}, \quad (80)$$

can be read off from the above by switching $\alpha \leftrightarrow \beta$. The terms with $\sin(2\eta_i - \eta_j - \eta_k)$, $\cos(2\eta_i - \eta_j - \eta_k)$ with $j = k$ and $j \neq k$ from (78) cancel, leaving the dependence on δ , the η_i and t of the form

$$\Delta P_{\nu_\alpha \leftrightarrow \nu_\beta}^{CP}(t) = \sum_{k=1}^3 \left[d_k \sin \delta + \sum_{i \neq j=1}^3 d_{ijk} \sin(\delta + \eta_i - \eta_j) + \sum_{i > j=1}^3 c_{ijk} \sin(\eta_i - \eta_j) \right] \sin \frac{\Delta_k t}{2q}. \quad (81)$$

Choosing $\eta_1 = 0$ this vanishes only for the trivial values δ , η_2 , η_3 equal to 0 or π . It confirms that in the DMνSM two of the Majorana phases η_i of the SνM are physical Dirac phases.

3.3. Neutrino Oscillations in Matter

Relativistic neutrinos have energy $E_i = (q^2 + m_i^2)^{1/2} \approx q + m_i^2/2q$. Neutrino oscillations in vacuum are ruled by the Hamiltonian $H_0 = E$, which reads on the flavor basis

$$(H_0)_{\alpha\beta} \approx \sum_{i=1}^6 U_{\alpha i} \left(q + \frac{m_i^2}{2q} \right) |v_{mL}^i\rangle \langle v_{mL}^i| U_{i\beta}^\dagger. \quad (82)$$

The q term leads to $q\delta_{\alpha\beta}$ and can be omitted, as it plays no role for the eigenfunctions. For propagation in matter one adds the matter potential. The charged and neutral currents induce the scalar potentials

$$V_{CC} = \sqrt{2}G_F n_e, \quad V_{NC} = -\frac{1}{2}\sqrt{2}G_F n_n, \quad (83)$$

involving the electron number density $n_e = n_{e^-} - n_{e^+}$ and the neutron number density n_n , and yielding

$$V_e = V_{CC} + V_{NC}, \quad V_\mu = V_\tau = V_{NC}, \quad (84)$$

The potential of the active neutrinos is diagonal on the flavor basis, while the sterile ones do not sense any. This results in the total matter potential on the flavor basis

$$V_m = \text{diag}(V_e, V_\mu, V_\tau, 0, 0, 0). \quad (85)$$

From Equation (25) and its real, symmetric nature it follows that

$$M^{DM} = U_{LR}^* \mathcal{M} U_{LR}^\dagger, \quad M^{DM\dagger} = U_{LR} \mathcal{M} U_{LR}^T. \quad (86)$$

Hence the m_i^2 , the eigenvalues of \mathcal{M}^2 , arise from

$$M^{DM\dagger} M^{DM} = U_{LR} \mathcal{M}^2 U_{LR}^\dagger. \quad (87)$$

The total matter Hamiltonian therefore reads on the flavor basis

$$H_m = \frac{1}{2q} M^{DM\dagger} M^{DM} + V_m. \quad (88)$$

Let us set $\mathcal{V}_m = 2q U_{LR}^\dagger V_m U_{LR} = (\mathcal{V}_m^a, \mathcal{V}_m^s)$ with $\mathcal{V}_m^s = \text{diag}(0, 0, 0)$ and

$$\mathcal{V}_m^a = 2q U_L^\dagger V_m^a U_L, \quad V_m^a = \text{diag}(V_e, V_\mu, V_\tau). \quad (89)$$

The factor D_L' in the decomposition (21) for U_L , also drops out from \mathcal{V}_m^a since V_m^a is diagonal, hence it can be totally omitted. Equation (88) can be expressed as

$$H_m = U_{LR} \mathcal{H}_m U_{LR}^\dagger, \quad \mathcal{H}_m = \frac{\mathcal{M}^2 + \mathcal{V}_m}{2q} = \frac{1}{2q} \begin{pmatrix} M^{d2} + \mathcal{V}_m^a & M^d M^N \\ M^N M^d & M^{d2} + M^{N2} \end{pmatrix}. \quad (90)$$

First solving the eigenmodes of \mathcal{H}_m and then going to the flavor basis allows us to evaluate the effects of oscillations on the active neutrinos without having knowledge of the undetermined matrix U_R . The eigenfunctions do not alter upon subtracting $(\bar{m}^2/2q) 1_{6 \times 6}$ from \mathcal{H}_m , after which all elements are small.

Inside matter the Diracian properties are lost, there are just six Majorana states with different masses. While the neutral current potential V_{NC} can be omitted in the limit $M^N \rightarrow 0$, this is not allowed in general. In matter one has real potentials V_α , $\alpha = e, \mu, \tau$. Due to V_{CC} the matrix \mathcal{V}_m^a is complex hermitian. The hermitian \mathcal{H}_m has six different positive eigenvalues but complex valued eigenmodes.

Inside the Sun, neutrino transport is dominated by a mostly electron-neutrino mode [15]; in the 3 + 3 model this is represented by two nearly degenerate, nearly maximally mixed Majorana modes. The resonance condition in the standard solar model now splits up as a condition for each of them.

The so-called solar abundance problem stems from the inconsistency between the standard solar model parameterized by the best description of the photosphere and the one parameterized to optimize agreement with helioseismic data sensitive to interior composition [18]. The biggest deviations in the

solar composition are of relative order 1% and occur at $\sim 0.7R_{\odot}$. Our modified resonance conditions offer hope for an improved description of the data.

3.4. Pion Decay

One of the simplest elementary particle reactions is

$$\pi^- \rightarrow W^- \rightarrow \mu + \bar{\nu}_{\mu R}. \quad (91)$$

It describes a negatively charged π^- particle, $\pi^- = (d\bar{u})$, consisting of a down quark (charge $-e/3$) and an anti-up quark (charge $-2e/3$), decaying into a W^- boson (charge $-e$), which in its turn decays into a muon (charge $-e$) and an muon-antineutrino (charge 0). Related reactions are $\pi^- \rightarrow e + \bar{\nu}_e$, $\pi^+ \rightarrow \mu^+ + \nu_{\mu}$ and $\pi^+ \rightarrow e^+ + \nu_e$. In the DM ν SM, the current $J_{CC}^{\mu\dagger} = 2\bar{\ell}_L\gamma^{\mu}U\nu_{mL} = 2\bar{\ell}_L\gamma^{\mu}(A\nu_{\bar{a}L} + S\nu_{\bar{s}R}^c)$ replaces Equation (91) by decays with any of the 6 mass eigenstates $(\bar{\nu}_m^i)_R$ emitted. They can be grouped as

$$\pi^- \rightarrow W^- \rightarrow \mu + \bar{\nu}_{\bar{a}R}^i, \quad (i = 1, 2, 3), \quad \pi^- \rightarrow W^- \rightarrow \mu + \bar{\nu}_{\bar{s}R}^{i-3} \quad (i = 4, 5, 6). \quad (92)$$

The $\nu_{\bar{a}L}^i$ and the charge conjugated $\nu_{\bar{s}R}^{i-3}$ fields have identical chiral structure (up to a phase factor, see Ref. [12], Equations (2.139) vs. (2.356)), differing only by their creation and annihilation operators. Hence all decay channels involve the standard chiral factors, and a new factor, the sum over final neutrino states, $\sum_{i=1}^6 |U_{\mu i}|^2 = \sum_{i=1}^3 |(U^D D^M \mathcal{A})_{\mu i}|^2 + \sum_{i=1}^3 |(U^D D^M \mathcal{S})_{\mu i}|^2$. It equals

$$(UU^{\dagger})_{\alpha\beta} = (U^D D^M (\mathcal{A}\mathcal{A}^T + \mathcal{S}\mathcal{S}^T) D^{M\dagger} U^{D\dagger})_{\alpha\beta} = \delta_{\alpha\beta}, \quad (93)$$

for $\alpha = \beta = \mu$. (In this equality we employed Equation (55)). So charged pions decay in the DM ν SM at the same rate as in the SM. Neutral pion decay does not involve neutrinos, so it is also not modified.

3.5. Neutron Decay

A neutron $n = (ddu)$ consists of two down quarks and one up quark, and a proton $p = (duu)$ of one down and two up quarks. Neutron decay $n \rightarrow p + e + \bar{\nu}_e$ involves a transition from a down quark to an up quark producing a virtual W^- boson, which decays into an electron and an electron antineutrino. As coded in the charged current $J_{CC}^{\mu\dagger}$ of (49), it occurs in the DM ν SM in two channels, $n \rightarrow p + e + \bar{\nu}_{\bar{a}}$ and $n \rightarrow p + e + \nu_{\bar{s}}$. Both decay channels involve the standard chiral factors, and a new factor, the sum over final neutrino states $\sum_i |(U^D D^M \mathcal{A})_{ei}|^2 + \sum_i |(U^D D^M \mathcal{S})_{ei}|^2$. This is the $\alpha = \beta = e$ element of Equation (93), so it is equal to unity. Hence the neutron lifetime in the DM ν SM stems with the one in the S ν M.

The main decay channel is $n \rightarrow p + e + \bar{\nu}_{\bar{a}}$. With our convention $L_{\nu_{\bar{s}}} = -1$, also the channel $n \rightarrow p + e + \nu_{\bar{s}}$ conserves the lepton number. The latter occurs at a slower rate due to the small term $\mathcal{S}\mathcal{S}^T$ in (93). We could not convince ourselves that it would be ineffective in beam experiments and hence be capable to explain the neutron decay anomaly between beam and bottle measurements [19].

3.6. Muon Decay

With the neutron decay going into two channels, muon decay $\mu \rightarrow e + \bar{\nu}_{eR} + \nu_{\mu L}$ goes into four,

$$\mu \rightarrow e + \bar{\nu}_{\bar{a}R} + \nu_{\bar{a}L}, \quad \mu \rightarrow e + \bar{\nu}_{\bar{a}R} + \bar{\nu}_{\bar{s}L}, \quad \mu \rightarrow e + \nu_{\bar{s}R} + \nu_{\bar{a}L}, \quad \mu \rightarrow e + \nu_{\bar{s}R} + \bar{\nu}_{\bar{s}L}, \quad (94)$$

with rates of leading schematic order 1, $\mathcal{S}\mathcal{S}^T$, $\mathcal{S}\mathcal{S}^T$ and $(\mathcal{S}\mathcal{S}^T)^2$, respectively, adding up to the SM result.

3.7. Neutrinoless Double β -Decay

In a simultaneous double neutron decay (double β -decay) the emission of two electrons involves the schematic neutrino terms $\nu_{\bar{a}L}^{i2} + \nu_{\bar{s}R}^{ic2} + \nu_{\bar{a}L}^i \nu_{\bar{s}R}^{ic}$, corresponding to the emission of two mostly active

antineutrinos, two mostly sterile neutrinos, or one of each. With $L_{\nu_{\bar{a}}} = 1$ and $L_{\nu_{\bar{s}}} = -1$, all three channels conserve the lepton number.

In the standard neutrino model also neutrinoless double β -decay is possible. Then only the $\nu_{\bar{a}L}^{i2}$ term occurs, subject to the Majorana condition $\nu_{\bar{a}L}^{i,c} = \nu_{\bar{a}L}^i$; with $\mathcal{A}_{ij} \rightarrow \delta_{ij}$ and $S_{ij} \rightarrow 0$, it yields an amplitude proportional to $m_{ee} = \sum_{i=1}^3 U_{ei}^D 2e^{2i\eta_i} m_i$ for small m_i . The GERDA search puts a bound $|m_{ee}| \leq 0.15\text{--}0.33$ eV [20]. Does this rule out the DM ν SM for $m \sim 2$ eV? Not, as we show now.

In our situation with Diracian neutrinos neither $\nu_{\bar{a}L}^{i2}$ nor $\nu_{\bar{s}R}^{i,c2}$ contributes, but neutrinoless double- β decay does arise from the $\nu_{\bar{a}L}\nu_{\bar{s}R}^c$ terms. All spinor terms are again as in the SvM. The only change occurs in the effective mass, which now reads

$$m_{ee} = [A\underline{m}S^T + S\underline{m}A^T]_{ee} = [U^D D^M (\underline{A}S^T + S\underline{m}A^T) D^M U^{D\dagger}]_{ee}, \quad \underline{m} = \text{diag}(m_1, m_2, m_3). \quad (95)$$

It involves cancellations, since S is nearly asymmetric while A and \underline{m}/m are close to the identity matrix. But the cancellations are maximal. From the definitions (51) we can go back to the six eigenvectors $\mathbf{e}^{(i)}$ of Equations (40) and (41). Recalling that $\lambda_{2i} = m_i$ and $\lambda_{2i-1} = -m_i$, ($i = 1, 2, 3$), it follows that

$$(\underline{A}S^T + S\underline{m}A^T)_{jk} = \sum_{i=1}^6 \lambda_i e_j^i e_k^i = \mathcal{M}_{jk}. \quad (96)$$

From (30) it is seen that $\mathcal{M}_{jk} = 0$ for $j, k = 1, 2, 3$, because we neglected the left handed Majorana mass matrix M_L^M . In general $m_{ee} = M_{ee}^{DM} = (M_L^M)_{ee}$ [13]. Hence $m_{ee} = 0$ for this leading order diagram.

Nevertheless, neutrinoless double β -decay, involving lepton number violation $\Delta L = 2$, is not forbidden in the DM ν SM. It occurs in the m_i^3/q^2 correction to the m_i in (95) stemming from the internal propagator $m_i/(q^2 - m_i^2) = m_i/q^2 + m_i^3/q^4 + \dots$. But the suppression factor m_i^2/q^2 makes its measurement impractical for realistic $q \sim \text{MeV} - \text{GeV}$. Loop effects may fare better, but are also tiny. If a finite m_{ee} is established, it points at new high-energy physics.

In conclusion, the non-detection of neutrinoless double β -decay is compatible with the DM ν SM.

3.8. Small Twin-Oscillation

If the degeneracy of the solar twin modes is slightly lifted by non-cancellation of the last two terms in each line of Equation (63), one gets

$$\Delta\bar{m}_1^2 \equiv (\lambda_1^+)^2 - (\lambda_1^-)^2 \approx \bar{m}\bar{\mu}_1 - \frac{2\bar{m}^3\mu_1\mu_2\mu_3}{\bar{\Delta}_2\bar{\Delta}_3}. \quad (97)$$

From the standard solar model we know that oscillations should not occur underway to Earth [2], so that $|\Delta\bar{m}_1^2| \lesssim 10^{-12} \text{ eV}^2$. For the supernova SN1987A at distance of 51.4 kpc the absence of twin-oscillations even implies that $|\Delta\bar{m}_1^2| \lesssim 10^{-22} \text{ eV}^2$; the alternative is that twin-oscillation did take place, and only half of the emitted neutrinos arriving here on Earth were active and could be detected. The implied doubling of power emitted in neutrinos then requires an adjustment of the SN1987A explosion model.

The similarly defined $\Delta\bar{m}_2^2$ and $\Delta\bar{m}_3^2$ may be larger. Either of them may describe the MiniBooNE anomaly [5] (disputed by MINOS [6] and still debated [7]) with $\Delta m^2 \approx 0.04 \text{ eV}^2$. But a more elegant approach hereto is to keep the 3 Diracian neutrinos and add a fourth sterile one.

3.9. Sterile Neutrino Creation in the Early Universe

The creation of sterile neutrinos in cosmology is an important process based on loss of coherence in oscillation processes. It is well studied, see e.g., [14], and is important when sterile neutrinos are to make up half of the cluster dark matter [8–11].

The charged currents in Equations (61a,b) allow the creation of sterile neutrinos out of active ones via $e^+ + \nu_e \rightarrow W^+ \rightarrow e^+ + \bar{\nu}_s$, and creation of sterile antineutrinos out of active ones in the process $e + \bar{\nu}_e \rightarrow W^- \rightarrow e + \nu_s$, that is, by four-Fermi processes with virtual W^\pm exchange. The first term in Equation (61c) describes interaction of active neutrinos with Z, the second of sterile ones, and the last two the exchange of active versus sterile neutrinos, and vice versa. In particular the creation of sterile neutrinos out of active ones is possible in two channels via the four-Fermi process $\nu_\alpha + \bar{\nu}_\alpha \rightarrow \nu_s + \bar{\nu}_s$ with the exchange of a virtual Z boson. All these processes conserve lepton number. As is seen from the sterile component in the flavor eigenstate (56) or from the charged and neutral currents (61), to achieve the sterile neutrino creation a finite matrix \mathcal{S} is needed. Hence it does not occur in the standard Dirac limit where both Majorana mass matrices M_L^M and M_R^M vanish.

3.10. Muon $g - 2$ Anomaly

The gyromagnetic factor of the muon is $g_\mu = 2(1 + a_\mu)$. Dirac theory yields $g_\mu = 2$ and a_μ is the anomaly due to quantum effects. The leading term is Schwinger's famous result,

$$a_\mu = \frac{\alpha}{2\pi} + \dots = 0.00116 + \dots, \quad (98)$$

where a_μ is known up to its 9th digit, but there is a $\sim 3.5\sigma$ discrepancy between measurement and prediction, $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 288(63)(43) \cdot 10^{-11}$ [16], where the first error is statistical and the second systematic.

Our interest lies in the contribution of neutrinos, which occurs in a simple triangle diagram with virtual W bosons. Ref. [21] presents the result for an arbitrary number of sterile neutrinos. For neutrino masses well below M_W it reads

$$a_\mu^\nu = (a_\mu^\nu)^{\text{SM}} \sum_{i=1}^{3+N_s} U_{\mu i} U_{\mu i}^* = (a_\mu^\nu)^{\text{SM}} (UU^\dagger)_{\mu\mu} = (a_\mu^\nu)^{\text{SM}} = \frac{G_F}{\sqrt{2}} \frac{5m_\mu^2}{12\pi^2} = 389 \cdot 10^{-11}. \quad (99)$$

The unitarity relation (60), viz. $(UU^\dagger)_{\alpha\beta} = \delta_{\alpha\beta}$, is valid even beyond our $N_s = 3$ case [12]. So the DMνSM reproduces the one-loop outcome of the SM, as well as the dominant two-loop electroweak contributions of Ref. [22].

3.11. ANITA Detection of UHE Cosmic Neutrino Events

Scattering of ultra high energy (UHE) cosmic rays on cosmic microwave background photons puts the GZK limit on their maximal energy [23,24] and acts as a source for EeV (10^{18} eV) (anti)neutrinos via the creation and decay of charged pions [25], as considered in Section 3.4.

The Antarctic Impulsive Transient Antenna (ANITA) is a balloon experiment at the South Pole that detects the radio pulses emitted when UHE cosmic neutrinos interact with the Antarctic ice sheet. In a set of $\gtrsim 30$ cosmic ray events, ANITA has discovered an upward going event with energy $E \sim 0.6$ EeV [26] and one with ~ 0.56 EeV [27]. Both are consistent with the cascade caused by a τ lepton created beneath the ice surface (see [28] for a possible explanation due to sub-surface reflection in the ice for a downward event). But the SM connects a relatively large neutrino-nucleon cross section $\sigma \sim G_F^2 m_N E$ to an UHE neutrino [29], so that the probability for it to traverse a large path L through the Earth to reach Antarctica is $P_T \sim \exp(-n\sigma L)$ is small, where n is the effective density of nuclei. The numbers are $P_T \sim 4 \cdot 10^{-6}$ and $2 \cdot 10^{-8}$, respectively [26,27]. In Ref. [30] it is pointed out that a sterile neutrino with a smaller cross section, viz. $\sigma \rightarrow \sigma \sin^2 \Theta$, with Θ the mixing angle with respect to active neutrinos, may be involved. To explain the events and relate them to the detections at IceCube, AUGER and Super-Kamiokande, these authors fix Θ at 0.1 [30].

For typical models a sterile neutrino with such a large mixing angle should have been discovered already. In the DMνSM the situation is different, however. It contains the reaction $\nu_s \rightarrow \tau + W^+$, where the W^+ is quickly lost locally but the τ escapes and decays while creating a shower. In the

approximation where one mixing angle dominates, an emitted electron neutrino will have components of strength $\sin^2 \Theta$ on mostly sterile states. Inside the Earth they are scattered less, and, given that they enter the other side of the Earth, can be measured in the τ -flavor mode with modified probability $\tilde{P}_T \sim \sin^2 \Theta \exp(-n \sigma L \sin^2 \Theta)$ and modified flux $\Phi_{\text{sterile}}/\Phi_{\text{active}} \sim \sin^4 \Theta \exp(-n \sigma L \sin^2 \Theta)$. The estimates of Section 3.1 show that the value $\Theta = 0.1$ is reasonable for the component Θ_3 of the mixing matrix \mathcal{S} , see in particular the expression for Θ_3 in Equation (71). Hence the DM ν SM supports the sterile-neutrino interpretation of ANITA events.

For determining the DM ν SM parameters, it seems worth to include UHE data.

4. Summary and Outlook

Since the maximal mixing of pseudo Dirac neutrinos runs into observational problems, neutrino mass is often supposed to stem from a high-energy sector beyond the standard model (BSM), for instance by the seesaw mechanism [12,14]. We show that the mixing effects can be suppressed in the DM ν SM, the minimal extension of the SM with three sterile neutrinos (3 + 3 model) with both a Dirac and a right handed Majorana mass matrix. Indeed, to have the six physical masses condense in three degenerate pairs poses three conditions, which leaves three Dirac and three Majorana masses free. In this Diracian limit the neutrino mass eigenstates act as Dirac particles like the other fermions in the SM. There is no change in the pion, neutron and muon decay, nor on the muon $g - 2$ problem. Compared to the general case, less mixing occurs since members of the same Dirac pair undergo no mutual oscillation. For small Majorana masses the left handed mass eigenstate is still mostly active, and the right handed one still mostly sterile. A flavor eigenstate has a component on mass eigenstates with a mostly sterile character. With mixing angles up to 0.2–0.3, this allows to explain the ANITA ultra high energy events. Hence for determining the DM ν SM parameters, it is natural to include UHE data.

In the Diracian limit the model keeps some of its Majorana properties. Neutrino oscillations in matter involve the usual six nondegenerate Majorana states. Lepton number is not conserved. Neutrinoless double- β decay remains possible, be it at an impractically small rate. Sterile neutrino generation in the early cosmos is possible at temperatures in the few MeV range.

It is interesting to investigate whether processes involving the neutral current can further test the model. They are relevant e.g., in nonresonant sterile neutrino production in the early universe.

By connecting lepton number $L = 1$ to (mostly) active neutrinos but $L = -1$ to (mostly) sterile neutrinos, neutron decay and double β -decay conserve the lepton number, while lepton number violation is restricted to feeble neutrinoless double- β decay. Should that be observed, it would invalidate our assumption of negligible left handed Majorana mass matrix, and prove the presence of BSM physics in the high energy sector.

The SM has 19 parameters while six neutrino parameters are established and two anticipated (the SM in Equation (A11) has three gauge coupling constants; two Higgs self couplings; six quark masses; three charged lepton masses; three strong mixing angles and a strong Dirac phase. Parameter 19 is the strong CP angle. The established neutrino parameters are two mass-squared differences, three weak mixing angles and, to some extent, the weak Dirac phase [16]. The two weak Majorana phases can in the S ν M only be measured via neutrinoless double β decay, but in the DM ν SM in many ways). The DM ν SM adds three further Majorana masses. In the limit where they vanish, the sterile partners decouple and the standard neutrino model emerges. The extra Majorana masses and Dirac role of the “Majorana” phases may alleviate some of the tensions in solar, reactor and other neutrino problems.

From a philosophical point of view, we do not consider the values of the Dirac and Majorana masses and phases as problematic properties in urgent need of an explanation, but rather as further mysteries of the standard model.

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Appendix A. Gamma and Charge Conjugation Matrices

The four 4×4 anticommuting γ -matrices were introduced by Dirac. They play a role in the description of, e.g., the chiral left handed and right handed electron and positron.

In the convention of Giunti and Kim [12] the Lorentz indices $\mu = 0, 1, 2, 3$ label the coordinates $x^\mu = (ct, x, y, z)$. The anticommutation relations read for any representation of the γ matrices,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) = \eta_{\mu\nu}. \quad (\text{A1})$$

The γ^5 matrix has the properties

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \{\gamma^\mu, \gamma^5\} = 0, \quad (\gamma^5)^2 = 1. \quad (\text{A2})$$

Left and right handed chiral projectors are, respectively,

$$P_L = \frac{1}{2}(1 - \gamma^5), \quad P_R = \frac{1}{2}(1 + \gamma^5). \quad (\text{A3})$$

Chiral left handed fields are $\nu_L = P_L \nu$ and chiral right handed ones $\nu_R = P_R \nu$. The projections are orthogonal, viz. $P_R P_L = P_L P_R = 0$, while $P_L^2 = P_L$ and $P_R^2 = P_R$.

Hermitian conjugation brings $\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$, summarized as

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 = \gamma_\mu = \eta_{\mu\nu} \gamma^\nu, \quad \gamma^{5\dagger} = \gamma^5. \quad (\text{A4})$$

The charge conjugation matrix \mathcal{C} has the properties

$$\mathcal{C}^\dagger = \mathcal{C}^{-1}, \quad \mathcal{C}^T = -\mathcal{C}. \quad (\text{A5})$$

It is defined up to an overall phase factor, which plays no physical role, and connected to transpositions,

$$\gamma^{\mu T} = -\mathcal{C}^\dagger \gamma^\mu \mathcal{C}, \quad \gamma^{5T} = \mathcal{C}^\dagger \gamma^5 \mathcal{C}. \quad (\text{A6})$$

The Pauli matrices are

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A7})$$

The charge conjugate of any spinor ν has the properties

$$\begin{aligned} \nu^c &= \mathcal{C} \bar{\nu}^T = -\gamma^0 \mathcal{C} \nu^{\dagger T}, & \nu^{c\dagger} &= -\nu^T \mathcal{C}^\dagger \gamma^0, & \nu &= -\gamma^0 \mathcal{C} \nu^{c\dagger T}, \\ \nu^T &= -\nu^{c\dagger} \gamma^0 \mathcal{C}, & \nu^\dagger &= -\nu^T \mathcal{C}^\dagger \gamma^0, & \nu^{T\dagger} &= -\mathcal{C}^\dagger \gamma^0 \nu^c. \end{aligned} \quad (\text{A8})$$

For four-component spinors ν_i and ν_j (with $i = j$ allowed) the contraction $\nu_i^T \mathcal{C}^\dagger \nu_j = \sum_{k,l=1}^4 \nu_{ik} \mathcal{C}_{kl}^\dagger \nu_{jl}$ is a nonvanishing scalar, since the fermion fields $\nu_{i,j}$ anticommute and \mathcal{C} is antisymmetric. The relation

$$(\nu_i^T \mathcal{C}^\dagger \nu_j)^\dagger = \nu_j^\dagger \mathcal{C} \nu_i^T = \nu_j^{c\dagger} \mathcal{C}^\dagger \nu_i^c, \quad (\text{A9})$$

assures that the Majorana mass Lagrangian (3) is hermitean.

In the chiral representation the γ and \mathcal{C} matrices have the 2×2 blocks

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 0 & -\sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma_i & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \\ \gamma^5 &= \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad P_L = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \quad P_R = \begin{pmatrix} \sigma^0 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}\quad (\text{A10})$$

Appendix B. The Standard Model with Sterile Neutrinos

For completeness we present the Lagrangian of the standard model with Dirac–Majorana neutrinos in a compact form (leaving out the strong CP violating term),

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}\mathbf{A}^{\mu\nu}\mathbf{A}_{\mu\nu} - \frac{1}{4}\mathbf{G}^{\mu\nu}\mathbf{G}_{\mu\nu} + D_\mu\Phi^\dagger D^\mu\Phi - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 \\ &+ \overline{\mathbf{Q}}_L i\not{D}\mathbf{Q}_L + \overline{\mathbf{q}}_R^U i\not{D}\mathbf{q}_R^U + \overline{\mathbf{q}}_R^D i\not{D}\mathbf{q}_R^D + \overline{\mathbf{L}}_L i\not{D}\mathbf{L}_L + \overline{\ell}_R i\not{D}\ell_R \\ &- \overline{\mathbf{Q}}_L \Phi Y^D \mathbf{q}_R^D - \overline{\mathbf{q}}_R^D Y^{D\dagger} \Phi^\dagger \mathbf{Q}_L - \overline{\mathbf{Q}}_L \tilde{\Phi} Y^U \mathbf{q}_R^U - \overline{\mathbf{q}}_R^U Y^{U\dagger} \tilde{\Phi}^\dagger \mathbf{Q}_L - \overline{\mathbf{L}}_L \Phi Y^\ell \ell_R - \overline{\ell}_R Y^{\ell\dagger} \Phi^\dagger \mathbf{L}_L\end{aligned}\quad (\text{A11})$$

$$+ \overline{\nu}_R i\not{D}\nu_R - \overline{\mathbf{L}}_L \tilde{\Phi} Y^\nu \nu_R - \overline{\nu}_R Y^{\nu\dagger} \tilde{\Phi}^\dagger \mathbf{L}_L + \frac{1}{2}\nu_R^T \mathcal{C}^\dagger M_R^{M\dagger} \nu_R + \frac{1}{2}\nu_R^c{}^T \mathcal{C}^\dagger M_R^M \nu_R^c.\quad (\text{A12})$$

Up to (A11) included, this represents the SM itself: the first line contains the $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ gauge fields, respectively, and the Higgs kinetic and potential energy. The second line contains the kinetic terms for three families of quarks, charged leptons and active neutrinos; the third line lists the quark couplings to the Higgs field with 3×3 Yukawa matrices $Y^{U,D}$ and the charged lepton couplings to the Higgs field with Yukawa matrix Y^ℓ . Equation (A12) exhibits the kinetic term for 3 sterile neutrinos and the Yukawa couplings between the active leptons, the Higgs field and the sterile neutrinos with a 3×3 Yukawa matrix Y^ν . Equation (A12) also contains the right handed Majorana mass terms of Equation (3).

The Diracian limit of the main text refers a special form for M_R^M .

The quark doublets in Equation (A11) contain the left handed up, down, charm, strange, top and bottom quarks,

$$\mathbf{Q}_L = \begin{pmatrix} Q_{1L} \\ Q_{2L} \\ Q_{3L} \end{pmatrix}, \quad Q_{1L} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad Q_{2L} = \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \quad Q_{3L} = \begin{pmatrix} t_L \\ b_L \end{pmatrix}.\quad (\text{A13})$$

The lepton doublets contain the left handed electron, muon and tau (tau lepton, tauon), and their active neutrinos: the left handed electron, mu and tau neutrino,

$$\mathbf{L}_L = \begin{pmatrix} L_{1L} \\ L_{2L} \\ L_{3L} \end{pmatrix}, \quad L_{1L} = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad L_{2L} = \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \quad L_{3L} = \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}.\quad (\text{A14})$$

The right handed quark and lepton singlets are grouped as

$$\mathbf{q}_R^U = \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}, \quad \mathbf{q}_R^D = \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix}, \quad \ell_R = \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}, \quad \nu_R = \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}.\quad (\text{A15})$$

The covariant derivatives D_μ and $\not{D} = \gamma^\mu D_\mu$ in Equation (A11) contain currents from gauge fields, except for $\not{D}\nu_R = \not{\partial}\nu_R$, since ν_R is gauge invariant. Indeed, the weak currents J_{CC} from Equation (49) and J_{NC} from Equation (50) arise as parts of $\overline{\mathbf{L}}_L i\not{D}\mathbf{L}_L$. The strong currents from $\overline{\mathbf{Q}}_L i\not{D}\mathbf{Q}_L$ do not involve the neutrino sector. In the unitary gauge the normal and conjugated Higgs doublets read, respectively,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix}, \quad \tilde{\Phi} = i\sigma_2 \Phi^{*T} = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H \\ 0 \end{pmatrix}, \quad (\text{A16})$$

where $v = \sqrt{-\mu^2/\lambda}$ is the vacuum expectation value and H the dynamical Higgs field.

After spontaneous symmetry breaking the 3×3 Dirac mass matrices for the *Down* = $\{d, s, b\}$ and *Up* = $\{u, c, t\}$ quarks are $M_q^D = vY^D/\sqrt{2}$ and $M_q^U = vY^U/\sqrt{2}$; for the charged leptons $M^\ell = vY^\ell/\sqrt{2}$ and the Dirac mass matrix for the active neutrinos is $M^\nu = vY^\nu/\sqrt{2}$ (it is denoted as M^D in the main text). From unitary transformations of the fields it follows that the matrices M_q^D and M^ℓ can be taken diagonal with the respective particle masses as entries. Next, the diagonalization of M_q^U is performed with the CKM mixing matrix and of $M^D = M^\nu$ with the PMNS mixing matrix of Equation (21).

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