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On Generalized (α, ψ, M_Ω)-Contractions with w-Distances and an Application to Nonlinear Fredholm Integral Equations

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Abstract: The study of asymmetric structures and their applications in mathematics is interesting. One of the types of asymmetric structures on a metric space has been initiated by Kada et al. (1996) and is known as a w-distance. That lack of symmetry attracts many researchers in fixed point theory. In this manuscript, we introduce a new type of contraction named generalized (α, ψ, M_Ω)-contractive mappings via w-distances, and then we prove some new related fixed point results, generalizing and improving the recent results of Lakzian et al. (2016) and others. At the end, we give some examples. To illustrate the usability of the new theory, we apply our obtained results to resolve a nonlinear Fredholm-integral-type equation.

Keywords: fixed point; w-distance; (α, ψ, M_Ω)-contractive mapping; fredholm integral equation

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1. Introduction

Recently, in [1], Kada et al. presented the definition of w-distances on metric spaces, generalizing many results in the literature such as the nonconvex minimization theorem of Takahashi [2], the Ekeland ϵ-variational principle, and the Caristi fixed point theorem; see also [3,4].

Definition 1 ([1])**.** *Let (K, l) be a metric space. A map $\Omega : K \times K \longrightarrow [0, \infty)$ is called a w-distance on K if the following assertions hold:*

(1) $\Omega(a, b) \le \Omega(a, c) + \Omega(c, b)$ *for all* $a, b, c \in K$*;*
(2) Ω *is lower semi-continuous in its second variable, i.e., if* $a \in K$ *and* $b_n \to b$ *in* K*, then* $\Omega(a, b) \le \liminf_{n\to\infty} \Omega(a, b_n)$*;*
(3) *For every* $\epsilon > 0$*, there is* $\gamma > 0$ *so that* $\Omega(c, a) \le \gamma$ *and* $\Omega(c, b) \le \gamma$ *imply* $l(a, b) \le \epsilon$*.*

Following Definition 1, a w-distance is asymmetric. The correlation of symmetry/asymmetry is inherent in the study of fixed point theory. Despite the lack of symmetry, the following lemma is useful in the sequel.

Lemma 1 ([1])**.** *Let Ω be a w-distance on a metric space (K, l) and $\{\varrho_n\}$ be a sequence in K.*

(i) *If* $\lim_{n\to\infty}\Omega(\varrho_n, a) = \lim_{n\to\infty}\Omega(\varrho_n, b) = 0$*, then* $a = b$*. In particular, if* $\Omega(c, a) = \Omega(c, b) = 0$*, then* $a = b$*.*

(ii) *If* $\Omega(\varrho_n, b_n) \leq \theta_n$ *and* $\Omega(\varrho_n, b) \leq \vartheta_n$*, where* $\{\theta_n\}$ *and* $\{\vartheta_n\}$ *are non-negative sequences tending both to* 0*, then* $\{b_n\}$ *is convergent to* b*.*

(iii) *If for any* $\varepsilon > 0$ *there is* N_ε *so that* $m > n > N_\varepsilon$ *implies* $\Omega(\varrho_n, \varrho_m) < \varepsilon$ *(or* $\lim_{n,m\to\infty}\Omega(\varrho_n, \varrho_m) = 0$*), then* $\{\varrho_n\}$ *is a Cauchy sequence.*

In the last years, there have been many results via w-distances (see [5–8]). Let Ψ be the family of non-negative functions ψ defined on $[0, \infty)$ such that

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{j=1}^{\infty}\psi^j(v) < \infty$ for all $v > 0$. Here, ψ^j is the j^{th} iterate of ψ.

Such a function ψ is known as a (c)-comparison function. In this case, $\psi(0) = 0$ and $\psi(v) < v$ for any $v > 0$. The notion of α-admissibility was first introduced in [9].

Definition 2 ([9])**.** *Let* $f : K \to K$ *be a self-mapping on a non-empty set* K *and* $\alpha : K \times K \to [0, \infty)$*. Such an* f *is called* α*-admissible if*

$$a, b \in K, \quad \alpha(a, b) \geq 1 \Longrightarrow \alpha(fa, fb) \geq 1. \tag{1}$$

In [9], the concept of (α, ψ)-contractions in the class of metric spaces was initiated. Variant (common) fixed point results dealing with this concept appeared (for example, see [10–20]). In the same direction, Lakzian et al. [21] initiated the concept of (α, ψ, Ω)-contractive mappings in metric spaces with w-distances.

Definition 3 ([21])**.** *Given* $T : K \to K$ *on a metric space* (K, l) *endowed with a* w*-distance* Ω*. Such a* T *is said to be an* (α, ψ, Ω)*-contraction if there are* $\alpha : K \times K \to [0, \infty)$ *and* $\psi \in \Psi$ *so that*

$$\alpha(a, b)\Omega(Ta, Tb) \leq \psi(\Omega(a, b)), \textit{ for all } a, b \in K. \tag{2}$$

Now, let (K, l) be a metric space with a w-distance Ω. Consider

$$M_\Omega(a, b) = \max\{\Omega(a, b), \Omega(a, fa), \Omega(b, fb), \frac{\Omega(a, fb) + \Omega(fa, b)}{2}\}. \tag{3}$$

If $\Omega = l$, set $M_\Omega = M$, where

$$M(a, b) = \max\{l(a, b), l(a, fa), l(b, fb), \frac{l(a, fb) + l(fa, b)}{2}\}. \tag{4}$$

We generalize Definition 3 as follows.

Definition 4. *Given* $f : K \to K$ *on a metric space* (K, l) *endowed with a* w*-distance* Ω*. Such an* f *is called a generalized* (α, ψ, M_Ω)*-contraction if there are* $\alpha : K \times K \to [0, \infty)$ *and* $\psi \in \Psi$ *so that*

$$\alpha(a, b)\Omega(fa, fb) \leq \psi(M_\Omega(a, b)), \textit{ for all } a, b \in X. \tag{5}$$

Such an f is called a generalized (ψ, M_Ω)-contraction if $\alpha \equiv 1$. If, in addition, $\Omega = l$, f is called a generalized (ψ, M)-contraction.

Using the concept of generalized (α, ψ, M_Ω)-contractions, we establish new fixed point theorems, generalizing some related ones, such as those of Samet et al. [9], Karapinar and Samet [22], Lakzian et al. [8,21], Banach [23], and many others in the literature. We also present some examples. At the end, by applying our obtained results, we ensure the existence of a solution of a nonlinear Fredholm integral equation.

2. Main Results

The first main result is stated as follows.

Theorem 1. *Let $f : X \to X$ be a generalized (α, ψ, M_Ω)-contraction on a complete metric space (X, d) endowed with a w-distance Ω. Assume that the following assertions hold:*

(i) *f is α-admissible;*

(ii) *there is $\nu_0 \in X$ so that $\alpha(\nu_0, f\nu_0) \geq 1$ and $\Omega(f^n(\nu_0), f^n(\nu_0)) = 0$, for each natural number n;*

(iii) *either f is continuous or $\inf\{\Omega(\mu, \zeta) + \Omega(\mu, f\mu) : \mu \in X\} > 0$ for every $\zeta \in X$ with $\zeta \neq f\zeta$.*

Then there is $u \in X$ such that $fu = u$.

Proof. By (ii), there is $\nu_0 \in X$ such that $\alpha(\nu_0, f\nu_0) \geq 1$. Define a sequence $\{\nu_n\}$ in X by $\nu_{n+1} = f\nu_n = f^{n+1}\nu_0$, for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $\nu_{n_0} = \nu_{n_0+1}$, then $u = \nu_{n_0}$ is a fixed point of f. The proof is completed. From now on, we assume that

$$\nu_n \neq \nu_{n+1} \text{ for all } n. \tag{6}$$

Since f is α-admissible, one writes

$$\alpha(\nu_0, \nu_1) = \alpha(\nu_0, f\nu_0) \geq 1 \Rightarrow \alpha(f\nu_0, f\nu_1) = \alpha(\nu_1, \nu_2) \geq 1.$$

By induction, we have

$$\alpha(\nu_n, \nu_{n+1}) \geq 1, \quad \text{for all } n \geq 0. \tag{7}$$

Step 1. We shall show that $\lim_{n\to\infty} \Omega(\nu_n, \nu_{n+1}) = 0$. Using (7) and Definition 4, we have

$$\begin{aligned} \Omega(\nu_n, \nu_{n+1}) &= \Omega(f\nu_{n-1}, f\nu_n) \\ &\leq \alpha(\nu_{n-1}, \nu_n)\Omega(f\nu_{n-1}, f\nu_n) \\ &\leq \psi(M_\Omega(\nu_{n-1}, \nu_n)), \end{aligned} \tag{8}$$

for all $n \geq 1$. By condition (ii), $\Omega(\nu_n, \nu_n) = 0$ for each natural number n. Thus,

$$\begin{aligned} M_\Omega(\nu_{n-1}, \nu_n) &= \max\{\Omega(\nu_{n-1}, \nu_n), \Omega(\nu_{n-1}, \nu_n), \Omega(\nu_n, \nu_{n+1}), \frac{\Omega(\nu_{n-1}, \nu_{n+1}) + \Omega(\nu_n, \nu_n)}{2}\} \\ &= \max\{\Omega(\nu_{n-1}, \nu_n), \Omega(\nu_n, \nu_{n+1}), \frac{\Omega(\nu_{n-1}, \nu_{n+1})}{2}\}. \end{aligned}$$

On the other hand, we have

$$\frac{\Omega(\nu_{n-1}, \nu_{n+1})}{2} \leq \frac{\Omega(\nu_{n-1}, \nu_n) + \Omega(\nu_n, \nu_{n+1})}{2} \leq \max\{\Omega(\nu_{n-1}, \nu_n), \Omega(\nu_n, \nu_{n+1})\}.$$

Therefore,

$$M_\Omega(\nu_{n-1}, \nu_n) = \max\{\Omega(\nu_{n-1}, \nu_n), \Omega(\nu_n, \nu_{n+1})\} \quad \text{for all } n \geq 1.$$

Suppose that $\Omega(\nu_{n-1}, \nu_n) < \Omega(\nu_n, \nu_{n+1})$ for some $n \geq 1$. Then (8) implies that

$$0 < \Omega(\nu_n, \nu_{n+1}) \leq \psi(M_\Omega(\nu_{n-1}, \nu_n)) = \psi(\Omega(\nu_n, \nu_{n+1})) < \Omega(\nu_n, \nu_{n+1}),$$

which is a contradiction. Thus, $M(\nu_{n-1}, \nu_n) = \Omega(\nu_{n-1}, \nu_n)$ for all $n \geq 1$. By induction, we obtain

$$\Omega(\nu_n, \nu_{n+1}) \leq \psi(\Omega(\nu_{n-1}, \nu_n)) \leq \psi^n(\Omega(\nu_0, \nu_1)) \quad \text{for all } n \geq 1. \tag{9}$$

From (Ψ_2), we get $\lim_{n\to\infty}\psi^n(\Omega(\nu_0,\nu_1))=0$, and so

$$\lim_{n\to\infty}\Omega(\nu_n,\nu_{n+1})=0. \tag{10}$$

Step 2. We claim that

$$\lim_{n,m\to\infty}\Omega(\nu_n,\nu_m)=0. \tag{11}$$

Applying (1) of Definition 1, (9), and (Ψ_2), we get for all $m,n\in\mathbb{N}$ with $m>n$,

$$\begin{aligned}\Omega(\nu_n,\nu_m) &\leq \Omega(\nu_n,\nu_{n+1})+\Omega(\nu_{n+1},\nu_{n+2})+\ldots+\Omega(\nu_{m-1},\nu_m)\\ &\leq \sum_{i=n}^{m-1}\psi^i(\Omega(\nu_0,\nu_1))\\ &\leq \sum_{i=n}^{\infty}\psi^i(\Omega(\nu_0,\nu_1))\to 0, \quad \text{when } n\to\infty.\end{aligned}$$

Therefore, by Lemma 1, $\{\nu_n\}$ is a Cauchy sequence in the complete metric space (X,d). Thus, there is $u\in X$ so that $\nu_n\to u$ as $n\to\infty$.

Step 3. Now we show that u is a fixed point of f.

Suppose that f is continuous. By Step 2, we have

$$u=\lim_{n\to\infty}\nu_{n+1}=\lim_{n\to\infty}f\nu_n=f(\lim_{n\to\infty}\nu_n)=fu.$$

Now, if $\inf\{\Omega(\mu,\zeta)+\Omega(\mu,f\mu):\mu\in X\}>0$ for each $\zeta\in X$ with $\zeta\neq f\zeta$. By (11), for any $\varepsilon>0$ there is $N_\varepsilon\in\mathbb{N}$ so that for $n>N_\varepsilon$, $\Omega(\nu_{N_\varepsilon},\nu_n)<\varepsilon$. But $\nu_n\to u$ and $\Omega(x,.)$ is lower semi-continuous, so using Definition 1, we have

$$\Omega(\nu_{N_\varepsilon},u)\leq \liminf_{n\to\infty}\Omega(\nu_{N_\varepsilon},\nu_n)\leq\varepsilon.$$

Putting $\varepsilon=1/k$ and $N_\varepsilon=n_k$, one writes

$$\lim_{k\to\infty}\Omega(\nu_{n_k},u)=0. \tag{12}$$

Assume that $u\neq fu$. Then

$$0<\inf\{\Omega(\mu,u)+\Omega(\mu,f\mu):\mu\in X\}\leq\Omega(\nu_n,u)+\Omega(\nu_n,\nu_{n+1}), \quad \text{for all } n\in\mathbb{N}.$$

Using (6) and (12), we get $\inf\{\Omega(\nu_n,u)+\Omega(\nu_n,\nu_{n+1}):n\in\mathbb{N}\}=0$. It is a contradiction with respect to the last inequality, i.e., $u=fu$. □

Example 1. *Let $X=[0,1]$ be endowed with the usual metric d. Define $\Omega(\mu,\tau)=4|\mu-\tau|$. Given $f:X\to X$ as*

$$f\mu=\begin{cases}0 & \text{if } \mu\in\{0,1\},\\ \frac{1}{2} & \text{if } \mu\in(0,1).\end{cases}$$

Also, define

$$\alpha(\mu,\tau)=\begin{cases}1 & \text{if } \mu,\tau\in(0,1),\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, f is an (α,ψ,M_Ω)-contractive mapping for any $\psi\in\Psi$. If $\mu,\tau\in(0,1)$, then $\alpha(\mu,\tau)=1$ and $f\mu=f\tau=\frac{1}{2}$, so $\Omega(f\mu,f\tau)=4|f\mu-f\tau|=0$. Thus,

$$\alpha(\mu,\tau)\Omega(f\mu,f\tau)\le\psi(M_\Omega(\mu,\tau)).$$

Otherwise, $\alpha(\mu,\tau)=0$, and so trivially,

$$\alpha(\mu,\tau)\Omega(f\mu,f\tau)\le\psi(M_\Omega(\mu,\tau))$$

holds. Now, let $\mu,\tau\in X$ be such that $\alpha(\mu,\tau)\ge1$. That is, $\mu,\tau\in(0,1)$, so $\alpha(f\mu,f\tau)=1$, i.e., f is α-admissible.

For $\nu_0=\frac{1}{2}\in X$, we have $\alpha(\nu_0,f\nu_0)=\alpha(\frac{1}{2},f\frac{1}{2})=\alpha(\frac{1}{2},\frac{1}{2})=1$ and $\Omega(f^n\nu_0,f^n\nu_0)=\Omega(\frac{1}{2},\frac{1}{2})=0$ for each $n\in\mathbb{N}$. Thus, all the hypotheses of Theorem 1 are satisfied. Here, 0 and $\frac{1}{2}$ are two fixed points of f. Note that $\alpha(\frac{1}{2},0)=0<1$.

However, f does not satisfy the contractive condition $\Omega(f\mu,f\tau)\le\psi(M_\Omega(\mu,\tau))$ for any $\psi\in\Psi$. Indeed, by taking $\mu=0$ and $\tau=\frac{1}{4}$, we have

$$\Omega(f\mu,f\tau)=2>\psi(\frac{3}{2})=\psi(M_\Omega(\mu,\tau)).$$

Example 2. *Let $X=\mathbb{R}$ be endowed with the usual metric d. Take*

$$\Omega(\mu,\tau)=|\int_\mu^\tau t^2dt|\quad(\mu,\tau\in\mathbb{R}).$$

Note that for $\mu\ne\tau$, we have $\Omega(\mu,\tau)>0$. Example 6 of [1] implies that Ω is a w-distance. Consider

$$\alpha(\mu,\tau)=\begin{cases}1 & \mu,\tau=0\\ \mu\tau & -1\le\mu,\tau<0\\ \frac{1}{2}(\mu^2+\mu\tau+\tau^2) & \mu,\tau\ge1\\ 0 & otherwise,\end{cases}$$

and

$$f\mu=\begin{cases}\sqrt[3]{\mu} & \mu\ge0\\ 0 & otherwise.\end{cases}$$

Let $\psi(t)=\frac{t}{2}$. Let $\mu,\tau\in X$ be such that $\alpha(\mu,\tau)\ge1$. Then $\mu,\tau=0$ or $\mu,\tau\ge1$.

In the case that $\mu,\tau=0$, we have $\alpha(f\mu,f\tau)=1$. While in the case $\mu,\tau\ge1$, we have $\alpha(\mu,\tau)=\frac{1}{2}(\mu^2+\mu\tau+\tau^2)\ge\frac{3}{2}$, which implies that $\sqrt[3]{\mu},\sqrt[3]{\tau}\ge1$. So

$$\alpha(f\mu,f\tau)=\alpha(\sqrt[3]{\mu},\sqrt[3]{\tau})=\frac{1}{2}(\sqrt[3]{\mu^2}+\sqrt[3]{\mu\tau}+\sqrt[3]{\tau^2})\ge\frac{3}{2}>1.$$

That is, f is α-admissible. On the other hand, if $-1\le\mu,\tau\le0$, we have $\Omega(f\mu,f\tau)=\Omega(0,0)=0$, and so

$$\alpha(\mu,\tau)\Omega(f\mu,f\tau)=0\le\psi(M_\Omega(\mu,\tau)).$$

If $\mu,\tau\ge1$, we have

$$\alpha(\mu,\tau)\Omega(f\mu,f\tau)=\frac{1}{2}(\mu^2+\mu\tau+\tau^2)|[\frac{t^3}{3}]_{\sqrt[3]{\mu}}^{\sqrt[3]{\nu}}|=\frac{1}{2}(\mu^2+\mu\tau+\tau^2)|\frac{\tau-\mu}{3}|=|\frac{\tau^3-\mu^3}{6}|\le\psi(M_\Omega(\mu,\tau)).$$

Otherwise, $\alpha(\mu,\tau)=0$ and so $\alpha(\mu,\tau)\Omega(f\mu,f\tau)=0\le\psi(M_\Omega(\mu,\tau))$. Therefore, f is an (α,ψ,M_Ω)-contraction. Moreover, f is continuous, $\alpha(0,f0)=1$ and $\Omega(f^n0,f^n0)=\Omega(0,0)=0$ for each $n\ge1$. Therefore, the conditions of Theorem 1 are true. Here, 0 and 1 are fixed points.

In Examples 1 and 2, the fixed point in Theorem 1 is not unique. To ensure its uniqueness, we need some additional properties. The following theorem describes this fact.

Theorem 2. *In addition to the hypotheses of Theorem 1, assume either*

(i) *for all fixed points u and v, we have $\alpha(u,v) \geq 1$ and $\Omega(u,u) = 0$; or,*

(ii) *ψ is continuous, and for all two fixed points u and v with $\Omega(u,u) = 0$, there is $\zeta \in X$ so that $\lim_{n\to\infty} \Omega(f^{n-1}\zeta, f^n\zeta) = 0$, $\alpha(\zeta,u) \geq 1$ and $\alpha(\zeta,v) \geq 1$.*

Then the fixed point of f is unique.

Proof. Let u be a fixed point of f (obtained by Theorem 1), and let v be such that $fv = v$. We shall show that $u = v$ either for case (i) or for case (ii).
Case (i): We have $\Omega(u,u) = 0$. Assume that $\Omega(u,v) > 0$. Then

$$M_\Omega(u,v) = \max\{\Omega(u,v), \Omega(u,fu), \Omega(v,fv), \frac{\Omega(u,fv) + \Omega(fu,v)}{2}\} = \Omega(u,v).$$

Then

$$\begin{aligned} 0 < \Omega(u,v) &\leq \alpha(u,v)\Omega(u,v) \\ &= \alpha(u,v)\Omega(fu,fv) \\ &\leq \psi(M_\Omega(u,v)) = \psi(\Omega(u,v)) < \Omega(u,v), \end{aligned}$$

which is a contradiction. Therefore, $\Omega(u,v) = 0 = \Omega(u,u) = 0$. By Lemma 1, $u = v$.
Case (ii): u and v are two fixed points of f with $\Omega(u,u) = 0$. Then there is ζ in X so that $\alpha(\zeta,u) \geq 1$ and $\alpha(\zeta,v) \geq 1$. The α-admissibility of f implies that

$$\alpha(f^n\zeta, u) \geq 1 \quad \textit{and} \quad \alpha(f^n\zeta, v) \geq 1, \quad n \geq 0.$$

Consider

$$\begin{aligned} \Omega(f^n\zeta, u) &= \Omega(f(f^{n-1}\zeta), fu) \\ &\leq \alpha(f^{n-1}\zeta, u)\Omega(f(f^{n-1}\zeta), fu) \\ &\leq \psi(M_\Omega(f^{n-1}\zeta, u)). \end{aligned}$$

Since $\Omega(u,u) = 0$ and $\lim_{n\to\infty} \Omega(f^{n-1}\zeta, f^n\zeta) = 0$, we conclude that

$$\lim_{n\to\infty} M_\Omega(f^{n-1}\zeta, u) = \lim_{n\to\infty} \Omega(f^{n-1}\zeta, u).$$

Using the continuity of ψ, $\lim_{n\to\infty} \Omega(f^n\zeta, u) \leq \psi(\lim_{n\to\infty} \Omega(f^{n-1}\zeta, u))$. Suppose that $\lim_{n\to\infty} \Omega(f^n\zeta, u) =: \alpha > 0$, then $0 < \alpha \leq \psi(\alpha) < \alpha$, which is a contradiction. Thus, $\lim_{n\to\infty} \Omega(f^n\zeta, u) = 0$. Similarly, $\lim_{n\to\infty} \Omega(f^n\zeta, v) = 0$, and by Lemma 1, we get $u = v$. That is, the uniqueness of the fixed point in each of the cases (i) and (ii) is ensured. □

Remark 1. *In Example 2, the elements 0 and 1 are fixed points of the considered mapping f. Note that $\alpha(1,1) = \frac{3}{2} > 1$. But, $\alpha(0,1) = \alpha(1,1) = 0 < 1$. In addition, there is no $\zeta \in X$ such that $\alpha(\zeta,0) \geq 1$. Thus, no condition in Theorem 2 holds. That's why we do not have a uniqueness of fixed point in Example 2.*

The following example shows that the presented results generalize and improve the previous ones of [21].

Example 3. *Let $(X,d) = (\mathbb{R}, |.|)$. Consider $\Omega(\mu,\zeta) = |\zeta|$ for all $\mu,\zeta \in \mathbb{R}$. Take*

$$\alpha(\mu,\zeta) = \begin{cases} 2 & \zeta \leq 0, \\ \frac{1}{2} & -\zeta \leq \mu \leq \zeta, \zeta > 0, \\ 1 & 0 < 2\zeta \leq \mu, \\ 0 & otherwise, \end{cases}$$

and

$$f\mu = \begin{cases} 5\mu & \mu \geq 0, \\ 0 & otherwise. \end{cases}$$

(i) Let $\alpha(\mu,\zeta) \geq 1$. Then $\alpha(\mu,\zeta) = 1$ or $\alpha(\mu,\zeta) = 2$. In the first case, we have $0 < 2\zeta \leq \mu$, and so $0 \leq 10\zeta \leq 5\mu$. Therefore, $\alpha(f\mu, f\zeta) = \alpha(5\mu, 5\zeta) = 1$. In the second case, we have $\zeta \leq 0$, and so $f\zeta = 0$. Now, since f is non-negative, we have $0 = 2f\zeta \leq f\mu$, and so $\alpha(f\mu, f\zeta) = 1$. We deduce that f is α-admissible.

(ii) For each $\mu \leq 0$, we have $\alpha(\mu, f\mu) = \alpha(\mu, 0) = 2 \geq 1$ and $\Omega(f^n\mu, f^n\mu) = \Omega(0,0) = 0$ for each $n \in \mathbb{N}$.

(iii) f is continuous. In addition, for each $w \in \mathbb{R}$ with $fw \neq w$, we have $w \neq 0$, and so $\inf\{\Omega(\mu,w) + \Omega(\mu, f\mu); \mu \in \mathbb{R}\} = \inf\{|w| + |f\mu|; \mu \in \mathbb{R}\} = |w| > 0$.

(iv) We claim that f is a generalized (α, ψ, M_Ω)-contraction (for $\psi(t) = \frac{t}{2}$).

If $\zeta \leq 0$, we have $f\zeta = 0$, and so $0 = \alpha(\mu,\zeta)\Omega(f\mu, f\zeta) \leq \psi(M_\Omega(\mu,\zeta))$.

If $\zeta > 0$ and $-\zeta \leq \mu \leq \zeta$, we have

$$\begin{aligned} \psi(M_\Omega(\mu,\zeta)) &= \frac{1}{2} Max\{|\zeta| + |5\mu| + |5\zeta| + |3\zeta|\} \\ &= \frac{1}{2} Max\{|5\mu| + |5\zeta|\} \\ &\geq \frac{1}{2}|5\zeta| \\ &= \alpha(\mu,\zeta)\Omega(f\mu, f\zeta). \end{aligned}$$

If $0 < 2\zeta \leq \mu$, we also have $\alpha(\mu,\zeta)\Omega(f\mu, f\zeta) = 5\zeta \leq \frac{5\mu}{2} \leq \psi(M_\Omega(\mu,\zeta))$.

Otherwise, we have $\alpha(\mu,\zeta) = 0$, and so the contraction (5) is valid.

All conditions of Theorem 2 hold, and $u = 0$ is the only fixed point of f. Note that the contraction of the reference [21] is not valid for this example. Indeed, for all ζ, μ with $0 < 2\zeta \leq \mu$, we have $\alpha(\mu,\zeta) = 1$, and so

$$\alpha(\mu,\zeta)\Omega(f\mu, f\zeta) = |5\zeta| > \psi(|\zeta|) = \psi(\Omega(\mu,\zeta)),$$

for each $\psi \in \Psi$.

The following examples illustrate Theorem 2.

Example 4. *Let $X = \{0\} \cup \{\frac{1}{2^n} : n \geq 1\}$ be endowed with the standard metric d. We define on X, $\Omega(\mu,\tau) = \tau$. Consider*

$$f\mu = \begin{cases} \frac{\mu}{2} & if\ \mu \in \{\frac{1}{2^n} : n \geq 1\}, \\ \\ 0 & if\ \mu = 0, \end{cases}$$

and

$$\alpha(\mu,\tau) = \begin{cases} 1 & if\ \mu = \tau = 0, \\ \\ \max\{\mu,\tau\} & otherwise. \end{cases}$$

Choose $\psi(t) = \frac{1}{2}t$. Note that f is an (α, ψ, M_Ω)-contractive mapping. Indeed,
Case 1: $\mu, \tau \in \{\frac{1}{2^n} : n \geq 1\}$. Here, $\alpha(\mu, \tau) = \max\{\mu, \tau\} < 1$. So

$$\alpha(\mu,\tau)\Omega(f\mu, f\tau) < \Omega(f\mu, f\tau) = \Omega(\frac{\mu}{2}, \frac{\tau}{2}) = \frac{\tau}{2} = \psi(\Omega(\mu,\tau)) \leq \psi(M_\Omega(\mu,\tau)).$$

Case 2: If $\mu = \tau = 0$, then $\alpha(\mu, \tau) = 1$ and $\Omega(f\mu, f\tau) = 0$. So

$$\alpha(\mu,\tau)\Omega(f\mu, f\tau) \leq \psi(M_\Omega(\mu,\tau)).$$

Case 3: $\mu = 0$ and $\tau \in \{\frac{1}{2^n} : n \geq 1\}$. Here, $\alpha(\mu, \tau) = \tau$.
Case 4: $\tau = 0$ and $\mu \in \{\frac{1}{2^n} : n \geq 1\}$. Then $\alpha(\mu, \tau) = \mu$.

Consequently, f is an (α, ψ, M_Ω)-contraction.

Now, let $\mu, \tau \in X$ be such that $\alpha(\mu, \tau) \geq 1$. Therefore, $\mu = \tau = 0$. So

$$\alpha(f\mu, f\tau) = \alpha(0,0) = 1.$$

That is, f is α-admissible.

Furthermore, taking $\nu_0 = 0$, we have $\alpha(\nu_0, f\nu_0) = \alpha(0,0) = 1$ and for any $n \geq 1$, $\Omega(f^n(\nu_0), f^n(\nu_0)) = \Omega(0,0) = 0$. For each $n \in \mathbb{N}$, recall that $\frac{1}{2^n} \neq f(\frac{1}{2^n})$, so

$$\inf\left\{\Omega\left(\frac{1}{2^m}, \frac{1}{2^n}\right) + \Omega\left(\frac{1}{2^m}, \frac{1}{2^{m+1}}\right) : m \in \mathbb{N}\right\} = \frac{1}{2^n} > 0.$$

Thus, we may apply Theorem 1. Here, $\mu = 0$ is the unique fixed point for f.

Example 5. *Let G be a locally compact group and $X = L^1(G)$. Consider*

$$\Omega(g,h) = \|g\|_1 + \|h\|_1, \quad (g,h) \in L^1(G)).$$

By Example 3 of [1], the function Ω is a w-distance. Denote $C_c(G)$ as the set of continuous functions on G. Define

$$\alpha(g,h) = \begin{cases} \frac{3}{2} & g,h \in C_c(G), \\ \frac{1}{2} & otherwise. \end{cases}$$

For an arbitrary $x \in G$, consider

$$\begin{array}{rcl} f_x : L^1(G) & \to & L^1(G) \\ g & \mapsto & \frac{1}{3}L_x g, \end{array}$$

where $L_x g(y) = g(x^{-1}y)$. Let $\psi(t) = \frac{t}{2}$. Now, for each $g \in C_c(G)$ and $x \in G$, we have $L_x g \in C_c(G)$. Hence, f_x is α-admissible. Moreover, since $\|L_x g\|_1 = \|g\|_1$, we have

$$\alpha(h,g)\Omega(\frac{1}{3}L_x h, \frac{1}{3}L_x g) \leq (\frac{3}{2})(\frac{1}{3})(\|h\|_1 + \|g\|_1) \leq \psi(M_\Omega(h,g)),$$

that is, f_x is an (α, ψ, M_Ω)-contraction mapping. Also, $\alpha(0, f_x 0) = \alpha(0,0) = \frac{3}{2} \geq 1$ and $\Omega(f_x^n 0, f_x^n 0) = \Omega(0,0) = \|0\|_1 + \|0\|_1 = 0$. Moreover, f_x is continuous. Therefore, all conditions of Theorems 1 and 2-(i) hold, and so $g = 0$ is the only fixed point.

Example 6. *Let $X = \mathbb{R}$ and d be the usual metric. Consider*

$$\Omega(\mu,\tau) = |\int_\mu^\tau t^2 dt| \quad (\mu, \tau \in \mathbb{R}),$$

and

$$\alpha(\mu,\tau)=\begin{cases}\mu\tau & -1\leq\mu,\tau<0,\\ \frac{1}{2}(\mu^2+\mu\tau+\tau^2) & \mu,\tau\geq 1,\\ 0 & otherwise.\end{cases}$$

Take

$$f\mu=\begin{cases}\sqrt[3]{\mu+1} & \mu\geq 0,\\ 0 & otherwise.\end{cases}$$

Let $\psi(t)=\frac{t}{2}$. Clearly, f is α-admissible.
Case 1: If $\mu,\tau\geq 1$, we have

$$\alpha(\mu,\tau)\Omega(f\mu,f\tau)=\frac{1}{2}(x^2+xy+y^2)|[\frac{t^3}{3}]^{\sqrt[3]{\tau+1}}_{\sqrt[3]{\mu+1}}|=\frac{1}{2}(\mu^2+\mu\tau+\tau^2)|\frac{\tau-\mu}{3}|=|\frac{\tau^3-\mu^3}{6}|\leq\psi(M_\Omega(\mu,\tau)).$$

Case 2: μ or τ is in $(-\infty,1)$. In this case, note that $\alpha(\mu,\tau)=0$ or $\Omega(f\mu,f\tau)=0$.

Consequently, f is an (α,ψ,M_Ω)-contraction. Note that $\alpha(1,f1)=\alpha(1,\sqrt[3]{2})\geq 1$ and $\Omega(f^n(1),f^n(1))=0$. Moreover, for every $\zeta\in X$ with $\zeta\neq f\zeta$, we have

$$\inf\{\Omega(\mu,\zeta)+\Omega(\mu,f\mu):\mu\in X\}=\inf\{|\int_\mu^\zeta t^2|+|\int_\mu^{f\mu}t^2|\}>0.$$

Note that there is a unique fixed point σ of f in $(1,2)$ (by taking $h(\mu)=\mu-\sqrt[3]{\mu+1}$, we have $h(2)>0$ and $h(1)<0$). Note that $\alpha(\sigma,\sigma)\geq 1$, and $\Omega(\sigma,\sigma)=0$.

Example 7. *Let $X=[0,1]$ be endowed by the usual metric d. Consider $\Omega=d$ and $f\mu=\frac{\mu+25}{50}$ for each $\mu\in X$. Define*

$$\alpha(\mu,\tau)=\begin{cases}24\mu\tau & if\ \mu,\tau\in[\frac{1}{4},1],\\ 1 & otherwise.\end{cases}$$

Clearly, $\alpha(\mu,\tau)\geq 1$ for all $\mu,\tau\in[0,1]$. Hence, f is α-admissible. In addition, if $\psi(t)=\frac{24t}{50}$, then for all $\mu,\tau\in X$,

$$\alpha(\mu,\tau)\Omega(f\mu,f\tau)\leq 24|\frac{x-y}{50}|\leq\psi(M_\Omega(\mu,\tau)).$$

So f is an (α,ψ,M_Ω)-contraction. Therefore, all conditions of Theorems 1 and 2-(i) are true. Here, $\mu=\frac{25}{49}$ is the unique fixed point of f.

Example 8. *Going back to Example 1, taking $f\mu=\frac{1}{2}$ for all $\mu\in[0,1]$, we have $\frac{1}{2}\in Fix(f)$. In addition, $\alpha(\frac{1}{2},\frac{1}{2})=1$, and f has a unique fixed point.*

Remark 2. *Theorem 2 remains true if we replace the fixed points u and v in conditions (i) and (ii), by all μ and τ in X.*

Putting $\alpha\equiv 1$ in Theorem 1, we state

Corollary 1. *Let Ω be a w-distance on a complete metric space (X,d) such that $\Omega(\mu,\mu)=0$, for all $\mu\in X$. Let $f:X\to X$ be a generalized (ψ,M_Ω)-contraction. Suppose either $\inf\{\Omega(\mu,\zeta)+\Omega(\mu,f\mu):\mu\in X\}>0$ for every $\zeta\in X$ with $\zeta\neq f\zeta$, or f is continuous. Then there is a unique $u\in X$ so that $fu=u$.*

By taking $\psi(t)=kt$, where $k\in[0,1)$ in Corollary 1, the following is a generalization of the Ćirić result via w-distances (see [24]).

Corollary 2. *Let Ω be a w-distance on a complete metric space (X, d) such that $\Omega(\varsigma, \varsigma) = 0$, for all $\varsigma \in X$. Let $f : X \to X$ be so that*

$$\Omega(f\mu, f\tau) \leq k\, M_\Omega(\mu, \tau)$$

for all $\mu, \tau \in X$, where $k \in [0, 1)$. Suppose either $\inf\{d(\mu, \zeta) + d(\mu, f\mu) : \mu \in X\} > 0$ for every $\zeta \in X$ with $\zeta \neq f\zeta$, or f is continuous. Then f has a unique fixed point.

We state the following technical lemma on metric spaces.

Lemma 2. *Let (X, d) be a complete metric space and $f : X \to X$ be a generalized (ψ, M)-contractive mapping. Suppose that ψ is continuous. Then $\inf\{d(\mu, \zeta) + d(\mu, f\mu) : \mu \in X\} > 0$ for every $\zeta \in X$ with $\zeta \neq f\zeta$.*

Proof. Suppose that there exists $\zeta \in X$ with $\zeta \neq f\zeta$ so that $\inf\{d(\mu, \zeta) + d(\mu, f\mu) : \mu \in X\} = 0$. Then there exists $\{\chi_n\}$ in X such that

$$\lim_{n\to\infty} \{d(\chi_n, \zeta) + d(\chi_n, f\chi_n)\} = 0.$$

Thus, $\lim_{n\to\infty} d(\chi_n, \zeta) = 0$ and $\lim_{n\to\infty} d(\chi_n, f\chi_n) = 0$. The triangular inequality implies that $\lim_{n\to\infty} d(f\chi_n, \zeta) = 0$, and so $f\chi_n \to \zeta$ as $n \to \infty$. Since f is a generalized (ψ, M)-contractive mapping (taking $\mu = \chi_n$ and $\tau = \zeta$), we have

$$d(f\chi_n, f\zeta) \leq \psi(M(\chi_n, \zeta)).$$

Letting $n \to \infty$ and using the continuity of ψ, we deduce that $0 < d(\zeta, f\zeta) \leq \psi(d(\zeta, f\zeta))$, which is a contradiction. Therefore, $f\zeta = \zeta$. $\square$

Taking $\Omega = d$ and $\alpha \equiv 1$ in Theorem 1, and from Lemma 2, we obtain the following.

Corollary 3. *Let (X, d) be a complete metric space and $f : X \to X$ be a generalized (ψ, M)-contractive mapping. Suppose that ψ is continuous. Then there is a unique $u \in X$ such that $fu = u$.*

Taking $\psi(t) = kt$, where $k \in [0, 1)$ in Corollary 3, we obtain the Ćirić result [24].

Corollary 4. *Let (X, d) be a complete metric space, and let $f : X \to X$ be so that*

$$d(f\mu, f\tau) \leq k\, M(\mu, \tau) \tag{13}$$

for all $\mu, \tau \in X$, where $k \in [0, 1)$. Then f has a unique fixed point.

2.1. Fixed Point in Partially Ordered Metric Spaces with w-Distances

Here, we will give some new fixed point results in ordered metric spaces equipped with w-distances. The triplet $(X, d, \preceq)$ is called an ordered metric space if

(i) d is a metric on X;
(ii) $\preceq$ is a partial order on X.

If $(X, \preceq)$ is a partially ordered set, then $\mu, \tau \in X$ are called comparable if $\mu \preceq \tau$ or $\tau \preceq \mu$. In addition, the mapping $T : X \to X$ is non-decreasing if for $\mu, \tau \in X$, $\mu \preceq \tau$ implies $T\mu \preceq T\tau$.

Corollary 5. *Let $(X, d, \preceq)$ be a partially ordered complete metric space equipped with a w-distance Ω. Let $f : X \to X$ be a nondecreasing continuous mapping so that*

(i) *for all $\mu, \tau \in X$ with $\mu \preceq \tau$,*

$$\Omega(f\mu, f\tau) \leq \psi(M_\Omega(\mu, \tau)), \tag{14}$$

where $\psi \in \Psi$;

(ii) there is $\nu_0 \in X$ so that $\nu_0 \preceq f\nu_0$ and $\Omega(f^n\nu_0, f^n\nu_0) = 0$ for each $n \geq 1$.

Then f has a fixed point.

Proof. Define

$$\alpha(\mu,\tau) = \begin{cases} 1, & \text{if } \mu \preceq \tau, \\ 0, & \text{otherwise.} \end{cases}$$

From (i), we have

$$\alpha(\mu,\tau)(\Omega(f\mu, f\tau)) \leq \psi(M_\Omega(\mu,\tau))$$

for all $\mu, \tau \in X$. Then f is a generalized (α, ψ, M_Ω)-contractive mapping. Now, let $\mu, \tau \in X$ be so that $\alpha(\mu,\tau) \geq 1$. This implies that $\mu \preceq \tau$. Since f is nondecreasing, $f\mu \preceq f\tau$, so $\alpha(f\mu, f\tau) = 1$, i.e., f is α-admissible. From (ii), there is $\nu_0 \in X$ so that $\nu_0 \preceq f\nu_0$ This implies that $\alpha(\nu_0, f\nu_0) = 1$. Therefore, all conditions of Theorem 1 hold, and so f has a fixed point. □

2.2. Cyclical Results

In this paragraph, we give some fixed point results via the cyclic concept. Our obtained results generalize the corresponding ones in [25,26].

Corollary 6. *Let $\{A_i\}_{i=1}^2$ be non-empty closed subsets of a complete metric space (X,d) endowed with a w-distance Ω. Given $f : Y = \cup_{i=1}^2 A_i \to Y$ so that*

(a) $f(A_1) \subseteq A_2$ and $f(A_2) \subseteq A_1$;

(b) for all $\mu, \tau \in Y$,

$$\Omega(f\mu, f\tau) \leq \psi(M_\Omega(\mu,\tau)),$$

where $\psi \in \Psi$. Suppose there is $\nu_0 \in Y$ so that $\Omega(f^n\nu_0, f^n\nu_0) = 0$ for each $n \geq 1$.

Then there is a fixed point $u \in \cap_{i=1}^2 A_i$ of f.

Proof. Define

$$\alpha(\mu,\tau) = \begin{cases} 1 & \text{if } (\mu,\tau) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ \\ 0 & \text{otherwise.} \end{cases}$$

From (b), one writes

$$\alpha(\mu,\tau)(\Omega(f\mu, f\tau)) \leq \psi(M_\Omega(\mu,\tau))$$

for all $\mu, \tau \in Y$. Thus, f is a generalized (α, ψ, M_Ω)-contraction.

Now, let $(\mu,\tau) \in Y \times Y$ be so that $\alpha(\mu,\tau) \geq 1$. If $(\mu,\tau) \in A_1 \times A_2$, then from (a), $(f\mu, f\tau) \in A_2 \times A_1$, which implies that $\alpha(f\mu, f\tau) \geq 1$. If $(\mu,\tau) \in A_2 \times A_1$, then from (a) we have $(f\mu, f\tau) \in A_1 \times A_2$, which implies that $\alpha(f\mu, f\tau) \geq 1$. In all cases, $\alpha(f\mu, f\tau) \geq 1$, that is, f is α-admissible. In addition, from (a), for any $\nu_0 \in A_1$, we have $(\nu_0, f\nu_0) \in A_1 \times A_2$, which implies $\alpha(\nu_0, f\nu_0) \geq 1$. Now, let $\{\nu_n\}$ be in Y so that $\alpha(\nu_n, \nu_{n+1}) \geq 1$ and $\nu_n \to \mu \in Y$ as $n \to \infty$. Thus,

$$(\nu_n, \nu_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \quad \forall n.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we get that

$$(\mu, \mu) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which implies that $\mu \in \cap_{i=1}^2 A_i$. We get immediately that $\alpha(\nu_n, \mu) \geq 1$ for all n.

Finally, let $\mu, \tau \in Y$ be two fixed points of f. From (a), we have $\mu, \tau \in \cap_{i=1}^{2} A_i$. So for any $z \in X$, we have $\alpha(\mu, z) \geq 1$ and $\alpha(\tau, z) \geq 1$, that is, condition (ii) in Theorem 2 is satisfied. All conditions of Theorem 1 hold, and then f has a fixed point in $\cap_{i=1}^{2} A_i$. $\square$

In the following, Corollary 6 is illustrated.

Example 9. *Let $X = \mathbb{R}$ be endowed with the usual metric. Define $\Omega(\mu, \tau) = |\tau|$. The subsets $A_1 = \{2k : k \in \mathbb{N}\} \cup \{0\}$ and $A_2 = \{2k-1 : k \in \mathbb{N}\} \cup \{0\}$ are non-empty closed subsets of (X, d). Given $Y = A_1 \cup A_2$ and $f : Y \to Y$ as*

$$f\mu = \begin{cases} 1 & \text{if } \mu \in A_1 - \{0\}, \\ 0 & \text{otherwise}, \end{cases}$$

then $f(A_1) \subseteq A_2$ and $f(A_2) \subseteq A_1$. Take $\psi(t) = \frac{t}{2}$. For all $\mu, \tau \in Y$, we have

$$\Omega(f\mu, f\tau) \leq \psi(M_\Omega(\mu, \tau)).$$

Thus, all the hypotheses of Corollary 6 are satisfied. Here, $0 \in \cap_{i=1}^{2} A_i$ is a fixed point of f.

3. Application

Here, we apply Theorem 1 to ensure the existence of a solution for the following nonlinear Fredholm integral equation (in short, NFIE):

$$\mu(t) = \int_a^b G(t,s) f(t, \mu(s)) ds, \tag{15}$$

where $G : [a, b] \times [a, b] \to [0, \infty)$ and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Consider $X = C[a, b]$ the set of all continuous functions from $[a, b]$ into $\mathbb{R}$. Define $T : X \to X$ as

$$(T\mu)(t) = \int_a^b G(t,s) f(t, \mu(s)) ds$$

for all $\mu \in X$ and $t \in [a, b]$. Take on X the complete metric:

$$d(\mu, \tau) = \sup_{\gamma \in [a,b]} |\mu(\gamma) - \tau(\gamma)|.$$

We endow on X the partial order:

$$\mu \preceq \tau \Longleftrightarrow \mu(\gamma) \leq \tau(\gamma), \quad \gamma \in [a, b].$$

Consider also on X the w-distance $\Omega : X \times X \to [0, \infty)$ given by

$$\Omega(\mu, \tau) = \sup_{t \in [a,b]} |\mu(t)| + \sup_{t \in [a,b]} |\tau(t)|$$

for all $\mu, \tau \in X$. Note that μ is a solution of the given Equation (15) if it is a fixed point of T. We shall prove that T has a fixed point under the following assumptions:

$(c1)$ There is $\psi \in \Psi$ so that for all $t \in [a, b]$ and $u \leq v \in \mathbb{R}$,

$$|f(t, u)| + |f(t, v)| \leq \psi(\max\{|u| + |v|, |Tu| + |u|, |Tv| + |v|, \frac{1}{2}(|Tv| + |u| + |Tu| + |v|)\});$$

$(c2)$ $\sup_{a\leq t\leq b}\int_a^b G(t,s)ds \leq 1;$

$(c3)$ for all $t \in [a,b]$, $f(t,.)$ is nondecreasing, that is, $u,v \in \mathbb{R}$, $u \leq v$ implies

$$f(t,u) \leq f(t,v);$$

$(c4)$ there is $\nu_0 \in X$ so that $\nu_0(t) \leq \int_a^b G(t,\nu_0(s))f(z,u(s))ds$ such that for all $n \geq 1$, $T^n\nu_0(t) = 0$ for each $t \in [a,b]$;

$(c5)$ $\inf\{\Omega(\mu,\zeta) + \Omega(\mu,T\mu) : \mu \in X\} > 0$ for every $\zeta \in X$ with $\zeta \neq T\zeta$.

Theorem 3. *Under the assumptions* $(c1)$–$(c5)$, *the NFIE (15) has a solution.*

It is foreseen to extend the above results by considering the frameworks of fractional calculus and Meir-Keeler contractions [27,28].

Proof. By condition $(c3)$, the mapping T is nondecreasing. By condition $(c4)$, there is $\nu_0 \in X$ so that $\nu_0 \preceq T\nu_0$ and $\Omega(T^n\nu_0, T^n\nu_0) = 0$ for each $n \geq 1$. Consider $\mu,\tau \in X$ and $t \in [a,b]$ such that $\mu \preceq \tau$. One writes

$$\begin{aligned}
&|(T\mu)(t)| + |(T\tau)(t)| \leq \int_a^b G(t,s))|f(t,\mu(s))| + |f(t,\tau(s))|ds \\
\leq\ & \int_a^b G(t,s)\psi(\max\{|\mu(s)| + |\tau(s)|, |T\mu(s)| + |\mu(s)|, |T\tau(s)| + |\tau(s)|, \frac{1}{2}(|T\tau(s)| + |\mu(s)| + |T\mu(s)| \\
&+|\tau(s)|)\})ds \\
\leq\ & \int_a^b G(t,s)\psi(\max\{\Omega(\mu,\tau), \Omega(T\mu,\mu), \Omega(T\tau,\tau), \frac{1}{2}(\Omega(T\tau,\mu) + \Omega(T\mu,\tau))\})ds \\
=\ & \psi(\max\{\Omega(\mu,\tau), \Omega(T\mu,\mu), \Omega(T\tau,\tau), \frac{1}{2}(\Omega(T\tau,\mu) + \Omega(T\mu,\tau))\})\int_a^b G(t,s)ds \\
\leq\ & \psi(\max\{\Omega(\mu,\tau), \Omega(T\mu,\mu), \Omega(T\tau,\tau), \frac{1}{2}(\Omega(T\tau,\mu) + \Omega(T\mu,\tau))\}).
\end{aligned}$$

This implies that (14) holds. Therefore, all conditions of Corollary 5 hold, and thus T has a fixed point. Hence, there is a solution of the NFIE (15). □

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