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The Unbounded Fuzzy Order Convergence in Fuzzy Riesz Spaces

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Abstract: The fuzzy order convergence in fuzzy Riesz spaces is defined only for fuzzy order bounded nets. The aim of this paper is to define and study unbounded fuzzy order convergence and some of its applications. Furthermore, some theoretical concepts like the fuzzy weak order unit and fuzzy ideals are studied in relation to unbounded fuzzy order convergence.

Keywords: fuzzy Riesz space; fuzzy order convergence; unbounded fuzzy order convergence; fuzzy weak order unit

1. Introduction

Zadeh [1] proposed the notion of fuzzy relations by generalizing the concepts of reflexivity, antisymmetry, and transitivity. Later, Venugopalan [2] developed an efficient structure of fuzzy partial ordered sets. Since then, many authors have studied fuzzy ordering and relations by adopting different approaches [3–8].

Vector space is widely used in modeling different kinds of real-life scenarios. Beg and Islam [9] gave the concept of fuzzy ordered linear spaces and studied their general properties and in [10,11] introduced the notion of fuzzy Riesz spaces and Archimedean fuzzy Riesz spaces. Beg [12] defined and characterized the fuzzy positive operator and in [13] gave further details on fuzzy order relations. Hong [14] defined and studied many concepts like fuzzy Riesz subspaces, fuzzy ideals, fuzzy bands, and fuzzy band projections. Recently, Park et al. [15] introduced the notion of Riesz fuzzy normed spaces.

Classically, vector spaces are used with other tools like topology, norm, and metric. Different types of convergence have many direct applications like in solving nonlinear equations [16,17]. Whereas, in a series of papers [18–22], the unbounded order convergence is defined for the nets that are not necessarily to be order bounded in Banach lattices, this is a different approach than convergence in norm or topology. In function spaces, unbounded order convergence is the same as pointwise convergence. Indeed, It is easily seen that in c_0 or $l_p(1 \leq p < \infty)$, unbounded order convergence of nets is an analogy to coordinate-wise convergence, and in measure spaces, unbounded order convergence of sequences is the same as convergence almost everywhere.

In order to handle imprecise and vague scenarios more effectively, a novel concept of unbounded fuzzy order convergence is proposed to deal with unbounded fuzzy ordered nets. In this regard, first the fuzzy order convergence [9,14] is redefined to overcome certain flaws, and based on this, unbounded fuzzy order convergence is defined. Then, an in depth theoretical investigation is done to study its various properties. However, it is much harder to determine the unbounded fuzzy order convergence of a given net. To resolve this issue, the notion of the fuzzy weak order unit is proposed to reduce the labor of checking unbounded fuzzy order convergence. Thus, unbounded fuzzy order convergence is nicely characterized in fuzzy Dedekind complete Riesz spaces.

To further develop the theory for practical use, the completeness of a fuzzy Riesz space is also explored with respect to unbounded fuzzy order convergence. For this purpose, the fuzzy ideals and fuzzy bands are studied in connection with fuzzy order convergence, and some results are given in the end as applications. The results and theory developed in this paper not only helped us to achieve the main goal of defining and characterizing unbounded fuzzy order convergence in fuzzy Riesz spaces but can also be used to develop and study more theoretical concepts like locally convex-solid fuzzy Riesz spaces, fuzzy Banach lattices, and unbounded fuzzy norm convergence in fuzzy Banach lattices, for better practical applications.

The paper is organized as follows. Section 2 provides the preliminary concepts necessary to understand the proposed work. Section 3 is for unbounded fuzzy convergence and its basic properties. Section 4 is devoted to defining and studying the fuzzy weak order unit in Dedekind complete fuzzy Riesz space. Finally, some concluding remarks for possible future work are given in Section 5.

2. Preliminaries

We recall some basic concepts from [9,10,14].

Definition 1. *A fuzzy order μ on a set K is a fuzzy set on $K \times K$ with the understanding that k precedes g if and only if $\mu(k,g) > 1/2$ for $k, g \in K$ and the following conditions are also satisfied:*

(i) $\forall\, k \in K$ $\mu(k,k) = 1$ *(reflexivity);*
(ii) *for* $k, g \in K$ $\mu(k,g) + \mu(g,k) > 1$ *implies* $k = g$ *(antisymmetric);*
(iii) *for* $k, h \in K$ $\mu(k,h) \geq \bigvee_{g \in K}[\mu(k,g) \wedge \mu(g,h)]$ *(transitivity).*

The space (K, μ) is called fuzzy ordered set (FOS).

Definition 2. *Let (K, μ) be an FOS, for $k \in K$, two related fuzzy sets $\uparrow k$ and $\downarrow k$ are known as $(\uparrow k)(g) = \mu(k,g)$ and $(\downarrow k)(g) = \mu(g,k)$ for $g \in K$, respectively. For $C \subseteq K$, then two fuzzy sets $U(C)$ and $L(C)$ are defined as follows.*

$$U(C)(g) = \begin{cases} 0, & \text{if } (\uparrow k)(g) \leq 1/2 \text{ for some } k \in C; \\ (\cap_{k \in C} \uparrow k)(g), & \text{otherwise.} \end{cases}$$

$$L(C)(g) = \begin{cases} 0, & \text{if } (\downarrow k)(g) \leq 1/2 \text{ for some } k \in C; \\ (\cap_{k \in C} \downarrow k)(g), & \text{otherwise.} \end{cases}$$

Let $(C)^u$ denote the set of all upper bounds of C, and $k \in (C)^u$ if $U(C)(k) > 0$. Analogously, $(C)^l$ denotes the set of all lower bounds, and $k \in (C)^l$ if $L(C)(k) > 0$. In addition, $d \in K$ is known as the supremum of C in K if (i) $d \in (C)^u$ and (ii) $g \in (C)^u$ implies $g \in (d)^u$. The infimum is defined analogously. A subset C is said to be fuzzy order bounded if $(C)^u$ and $(C)^l$ are non-empty.

Definition 3. *A real vector space K with fuzzy order μ is known as a fuzzy ordered vector space (FOVS) if μ satisfies:*

(i) *for* $k, g \in K$ *if* $\mu(k,g) > 1/2$ *then* $\mu(k,g) \leq \mu(k+h, g+h)$ *for all* $h \in K$*;*
(ii) *for* $k, g \in K$ *if* $\mu(k,g) > 1/2$ *then* $\mu(k,g) \leq \mu(\lambda k, \lambda g)$ *for all* $0 \leq \lambda \in \mathbb{R}$*.*

Let (K, μ) be an FOVS. Then $k \in K$ is known as positive if $\mu(0,k) > 1/2$ and negative if $\mu(k,0) > 1/2$. Moreover, K^+ will be referred to as the set of positive elements in K. $C \subseteq K$ is called a directed upwards set if for each finite subset D of A we have $C \cap (D)^u \neq \emptyset$. The directed downwards set is defined analogously. Furthermore, a net $k_\lambda \uparrow k$ for each $\lambda \in \Lambda$ reads as the net (k_λ) being directed upwards to k, i.e., for $\lambda_0 \leq \lambda$, we have $\mu(k_{\lambda_0}, k_\lambda) > 1/2$, and $\sup\{k_\lambda\} = k$. $k_\lambda \downarrow k$ is defined analogously.

An FOVS (K, μ) is said to be Archimedean if $\mu(nk, g) > 1/2$ for all $n \in \mathbb{N}$ implies that $\mu(k, 0) > 1/2$ for all $k, g \in K$. Therefore, $\{\frac{k}{n}\} \downarrow 0$ and $\{nk\}$ is unbounded from above for all $0 \neq k \in K^+$.

Definition 4. *An FOVS (K, μ) is said to be a fuzzy Riesz space (FRS) if $k \vee g = \sup\{k, g\}$ and $k \wedge g = \inf\{k, g\}$ exist in K for all $k, g \in K$.*

For $k \in K$, $k^+ = k \vee 0$ and $k^- = (-k) \vee 0$ are defined to be positive and negative parts of k, respectively, whereas the absolute value of k is defined as $|k| = (-k) \vee k$. Furthermore, let $k_1, k_2 \in K$ be called orthogonal or disjoint if $k_1 \wedge k_2 = 0$, and written as $k_1 \perp k_2$. In addition, for $C_1, C_2 \subset K$ are called disjoint and denoted by $C_1 \perp C_2$ if $k_1 \perp k_2 = 0$ for each $k_1 \in C_1$ and $k_2 \in C_2$. Moreover, if $\emptyset \neq C \subseteq K$, then its disjoint complement is defined as $C^d = \{k \in K : k \perp g \ for\ each\ g \in C\}$.

Proposition 1. *If k and g are elements of an FRS (K, μ), then*

(i) $k = k^+ - k^-$;
(ii) $k^+ \wedge k^- = 0$;
(iii) $|k| = k^+ + k^-$;
(iv) $|k| = 0 \Leftrightarrow k = 0$;
(v) $\mu(||k| - |g||, |k - g|) > 1/2$.

Definition 5. *A vector subspace L of an FRS (K, μ) is known as a fuzzy Riesz subspace if L is closed under fuzzy Riesz operations $\vee$ and $\wedge$.*

Definition 6. *An FRS (K, μ) is called:*

(i) *fuzzy order complete if each non-empty subset of K has a supremum and infimum in K;*
(ii) *fuzzy $\sigma-$ order complete if each nonempty countable subset of K has a supremum and infimum in K;*
(iii) *fuzzy Dedekind complete if each non-empty subset of K which is bounded from above has a supremum in K;*
(iv) *fuzzy $\sigma-$ Dedekind complete if each nonempty countable subset of K that is bounded from above has a supremum in K.*

3. Unbounded Fuzzy Order Convergence

Although the notion of fuzzy order convergence is a central tool in studying the fuzzy Riesz spaces, the definition given in [9,14] has some limitations that we now try to overcome. Later, it will be generalized as unbounded fuzzy order convergence to open new horizons, and its various properties are studied.

Definition 7. *A net $(k_\lambda)_{\lambda \in \Lambda}$ in an FRS (K, μ) is said to be fuzzy order convergent (fo-convergent for short) to $k \in K$ denoted $k_\lambda \xrightarrow{fo} k$ if there exists another net $(g_\lambda)_{\lambda \in \Lambda}$ in K^+ directed downwards to zero such that $\mu(|k_\lambda - k|, g_\lambda) > 1/2$ for each $\lambda \in \Lambda$.*

But the above definition cannot truly fulfill the concepts of convergence, as intuitively, if we add some terms at the start of net, then convergence should not be changed. The following example illustrates our point.

Example 1. *Let (K, μ) be an Archimedean FRS. Then for $k \in K^+$, the net $\{\frac{k}{n}\} \downarrow 0$. Therefore, $\frac{k}{n} \xrightarrow{fo} 0$ according to Definition 7. On the other hand, negative integers are added and placed between 1 and 2 in the index set. Thus, the new index set is denoted as*

$$\Lambda = \{1, -1, -2, -3, ..., 2, 3, 4, ...\},$$

and the extended net (g_n) is defined as

$$g_n = \begin{cases} k, & \text{if } n = 1 \\ |n|k, & \text{if } n \in -\mathbb{N}; \\ \frac{k}{n}, & \text{otherwise.} \end{cases}$$

Clearly, (g_n) is not fuzzy order convergent to zero according to Definition 7.

This deficiency is pointed out in [18] for the classical order convergence. Therefore, a new definition is proposed to overcome this issue.

Definition 8. *A net $(k_\lambda)_{\lambda\in\Lambda}$ in an FRS (K,μ) is said to be fuzzy order convergent (fo-convergent for short) to $k \in K$ denoted $k_\lambda \xrightarrow{fo} k$ if there exists another net $(g_\gamma)_{\gamma\in\Gamma}$ in K^+ directed downwards to zero, and for each $\gamma \in \Gamma$ there exist $\lambda_0 \in \Lambda$ such that $\mu(|k_\lambda - k|, g_\gamma) > 1/2$ whenever $\lambda \geq \lambda_0$.*

One can check—the extended net in Example 1 is fuzzy order convergent to zero according to Definition 8. Clearly, Definition 7 implies Definition 8. From here on, fuzzy order convergence will be considered according to Definition 8 without reference. The generalization of fo-convergence is given below.

Definition 9. *A net $(k_\lambda)_{\lambda\in\Lambda}$ in an FRS (K,μ) is said to be unbounded fuzzy order convergent (ufo-convergent for short) to $k \in K$ denoted $k_\lambda \xrightarrow{ufo} k$ if $|k_\lambda - k| \wedge g \xrightarrow{fo} 0$ for each $g \in K^+$.*

Note that fo-convergence implies ufo-convergence. The ufo-convergence has a number of nice characterizing conditions.

Proposition 2. *If $(k_\lambda)_{\lambda\in\Lambda}$ and $(g_\gamma)_{\gamma\in\Gamma}$ are nets in an FRS (K,μ), then the following statements are true:*

(i) $k_\lambda \xrightarrow{ufo} k$ *iff* $(k_\lambda - k) \xrightarrow{ufo} 0$;
(ii) *if* $k_\lambda \xrightarrow{ufo} k$ *and* $g_\gamma \xrightarrow{ufo} g$, *then* $ak_\lambda + bg_\gamma \xrightarrow{ufo} ak + bg$ *for each* $a, b \in \mathbb{R}$;
(iii) *if* $k_\lambda \xrightarrow{ufo} k$ *and* $k_\lambda \xrightarrow{ufo} g$, *then* $k = g$;
(iv) *if* $k_\lambda \xrightarrow{ufo} k$, *then*

(a) $(k_\lambda)^+ \xrightarrow{ufo} k^+$;
(b) $(k_\lambda)^- \xrightarrow{ufo} k^-$.

Furthermore, (a) and (b) imply that

$$|k_\lambda| \xrightarrow{ufo} |k|.$$

(v) *If a positive net* $k_\lambda \xrightarrow{ufo} k$ *and* $\mu(k_\lambda, g_\gamma) > 1/2$, $g_\gamma \xrightarrow{ufo} g$, *then* $\mu(k,g) > 1/2$.

Proof. (i) Suppose $k_\lambda \xrightarrow{ufo} k$. Then $|(k_\lambda - k) - 0| \wedge g = |k_\lambda - k| \wedge g \xrightarrow{fo} 0$ for each $g \in K^+$, hence $(k_\lambda - k) \xrightarrow{ufo} 0$. The converse can be proved analogously.
(ii) Suppose $k_\lambda \xrightarrow{ufo} k$ and $g_\gamma \xrightarrow{ufo} g$. Then we have

$$\mu(|(k_\lambda + g_\gamma) - (k+g)| \wedge h, (|k_\lambda - k| + |g_\gamma - g|) \wedge h) > 1/2$$

and

$$\mu((|k_\lambda - k| + |g_\gamma - g|) \wedge h, |k_\lambda - k| \wedge h + |g_\gamma - g| \wedge h) > 1/2$$

for each λ, γ, and $h \in K^+$. It follows that $k_\lambda + g_\gamma \xrightarrow{fo} k + g$. Fix $a \in \mathbb{R}$, and let $h \in K^+$. Check that $|ak_\lambda - ak| \wedge h = |a||k_\lambda - k| \wedge h$. If $|a| \leq 1$, then $\mu(|a||k_\lambda - k| \wedge h, |k_\lambda - k| \wedge h) > 1/2$ and $|k_\lambda - k| \wedge h \xrightarrow{fo} 0$. If $|a| > 1$ then $|h| \leq |a|h$ and $\mu(|a||k_\lambda - k| \wedge h, |a||k_\lambda - k| \wedge |a|h) > 1/2$ and $|a|(|k_\lambda - k| \wedge h) \xrightarrow{fo} 0$. In each case $ak_\lambda \xrightarrow{ufo} ak$.

1. Let $\mu(|k-g|, |k-k_\lambda| + |g-k_\lambda|) > 1/2$ for each λ. Let $h = |k-g|$. Observe that $|k-g| = |k-g| \wedge h$. Also
$$\mu(|k-g| \wedge h, |k-k_\lambda| \wedge h + |g-k_\lambda| \wedge h) > 1/2.$$
Hence, $k = g$.
2. Suppose $|k_\lambda - k| \xrightarrow{ufo} 0$. As $\mu(|(k_\lambda)^+ - k^+|, |k_\lambda - k|) > 1/2$ for each λ. So $|(k_\lambda)^+ - k^+| \xrightarrow{ufo} 0$. Hence, $(k_\lambda)^+ \xrightarrow{ufo} k^+$. Thus $-k_\lambda \xrightarrow{ufo} -k$ this gives that $(k_\lambda)^- \xrightarrow{ufo} k^-$. The final statement follows from $\mu(||k_\lambda| - |k||, |k_\lambda - k|) > 1/2$.
3. By Proposition 2, $k_\lambda = |k_\lambda| \xrightarrow{ufo} |k|$. $k = |k|$ by uniqueness of fuzzy order limit. As $\mu(0, g_\gamma - k_\lambda) > 1/2$, then $g_\gamma - k_\lambda \xrightarrow{ufo} g - k$, and we have $\mu(k, g) > 1/2$.

□

Remark 1. *Let $(k_\lambda)_{\lambda \in \Lambda}$ be a fuzzy order bounded net in a Dedekind complete FRS (K, μ). Then $k_\lambda \xrightarrow{fo} k$ iff $k = \limsup_\lambda (k_\lambda) = \liminf_\lambda (k_\lambda)$. Moreover, two sequences (k_n) and (k_m) are called* disjoint *if $|k_n| \wedge |k_m| = 0$ or ($k_n \perp k_m$) holds for $m \neq n$. The ufo-convergence for disjoint sequences in σ-Dedekind complete FRS is discussed in the following proposition.*

Proposition 3.

- *(i) Let $(k_n)_{n \in N}$ be a disjoint sequence in a σ-Dedekind complete FRS (K, μ). Then $k_n \xrightarrow{ufo} 0$ in K.*
- *(ii) Let $(k_n)_{n \in N}$ be a sequence in an FRS (K, μ). If $k_n \xrightarrow{ufo} 0$, then $\inf_m (k_{n_m}) = 0$ for each increasing sequence (n_m) of natural numbers.*

Proof. (i) Fix $k \in K^+$. We will show that $\limsup_n (|k_n| \wedge k) = 0$. Indeed, let $g \in K^+$ such that $\mu(g, \sup_n (|k_n| \wedge k)) > 1/2$. Therefore,
$$\mu(g \wedge |k_n|, (\sup_{n+1}(|k_{n+1}| \wedge k) \wedge |k_n|) > 1/2 \textit{ and } \sup_{n+1}(|k_{n+1}| \wedge |k_n| \wedge k) = 0.$$
Thus, $g \wedge |k_n| = 0$ for all $n \in \mathbb{N}$. It follows that
$$g = g \wedge \sup_{n \geq 1}(|k_n| \wedge k) = \sup_{n \geq 1}(g \wedge |k_n| \wedge k) = 0.$$
Hence, $|k_n| \wedge k \xrightarrow{fo} 0$.

(ii) Suppose $k_n \xrightarrow{ufo} 0$. Take (n_m) as an increasing sequence of natural numbers. Clearly, $k_{n_m} \xrightarrow{ufo} 0$. Let $\mu(k, k_{n_m}) > 1/2$ for each $m \in \mathbb{N}$, and $k \in K^+$. Therefore, $k = k_{n_m} \wedge k \xrightarrow{fo} 0$ implies that $k = 0$. Hence, $\inf_m (k_{n_m}) = 0$.

□

4. Fuzzy Weak Order Unit

Our next goal is to reduce the task of checking ufo-convergence at every positive vector to a single special vector, which will be defined as fuzzy weak order unit that allows us to nicely characterize ufo-convergence. Before that, we present some important theoretical concepts as follows.

Definition 10. *Let (K, μ) be an FRS.*

(i) *A subset C of K is said to be fuzzy order closed (fo-closed for short) if for any net $(k_\lambda) \subset C$ and $k \in K$ with $k_\lambda \xrightarrow{fo} k$ in K implies $k \in C$.*
(ii) *A subset S of K is called fuzzy solid if $\mu(|k|, |g|) > 1/2$ and $g \in S$ implies $k \in S$.*
(iii) *A fuzzy solid vector subspace I of K is called a fuzzy ideal of K.*
(iv) *A fuzzy order closed ideal in K is said to be a fuzzy band.*

For $k \in K$, the fuzzy band generated by k is known as the principal fuzzy band and defined as $B_k = \{g \in K : |g| \wedge n|k| \uparrow |g|\}$ by Corollary 5.4 in [14]. The fuzzy band generated by a non-zero positive element is discussed as follows.

Definition 11. *Let (K, μ) be an FRS, and $0 \neq w \in K^+$ is said to be a fuzzy weak order unit if the fuzzy band generated by w satisfies $B_w = K$, or equivalently, $k \wedge nw \uparrow k : n \in \mathbb{N}$ for each $k \in K^+$.*

Proposition 4. *Let (K, μ) be an Archimedean FRS. Then $0 \neq w \in K^+$ is a fuzzy weak order unit if and only if $k \perp w$ implies $k = 0$ for each $k \in K^+$.*

Proof. It follows from the definition of fuzzy weak order unit, Theorems 4.7 and 5.8 in [14]. □

Proposition 4 leads to the following result.

Proposition 5. *Let (K, μ) be a Dedekind complete FRS with a fuzzy weak order unit w. Then $k_\lambda \xrightarrow{ufo} 0$ iff $|k_\lambda| \wedge w \xrightarrow{fo} 0$.*

Proof. Suppose $k_\lambda \xrightarrow{ufo} 0$. Take any $g \in K^+$. As K is fuzzy Dedekind complete, then

$$(\limsup_{\lambda}(|k_\lambda| \wedge g)) \wedge w = (\limsup_{\lambda}(|k_\lambda| \wedge w)) \wedge g = 0 \wedge g = 0.$$

Thus, w being a fuzzy weak order unit implies that $\limsup_\lambda(|k_\lambda| \wedge g) = 0$. Hence, $|k_\lambda| \wedge g \xrightarrow{fo} 0$. The converse follows from Definition 9. □

We defined and studied the properties of the fuzzy component in Dedekind complete FRS in which ufo-convergence is nicely characterized.

Definition 12. *Let (K, μ) be an FRS. A vector $k \in K^+$ is said to be the fuzzy component of w whenever $k \wedge (w - k) = 0$ for $w \in K^+$.*

Remark 2. *For $k \in K$, w_k denotes the fuzzy component of w in the fuzzy band generated by k. So for each $\alpha \in \mathbb{R}$, $w_{(k-\alpha w)^+}$ is the fuzzy component of w in the fuzzy band generated by $(k - \alpha w)^+$, and we set $e(\alpha) = w_{(k-\alpha w)^+}$. In a Dedekind complete FRS (K, μ) with $k \in K^+$ and letting $e = w_{k^+}$, then $k_e = k^+$.*

Now, many lemmas are proved in order to characterize the ufo-convergence with the help of fuzzy components.

Lemma 1. *If (K, μ) is a Dedekind complete FRS and $k \in K^+$, then*

$$\mu(e(\alpha), \frac{1}{\alpha}k) > 1/2$$

for $\alpha > 0$.

Proof. Remark 2 yields that $(k - \alpha w)_{e(\alpha)} = (k - \alpha w)^+$ and $\mu(0, (k - \alpha w)^+) > 1/2$. Therefore, $(k - \alpha w)_{e(\alpha)} = k_{e(\alpha)} - \alpha w_{e(\alpha)}$ and $\mu(0, k - \alpha e(\alpha)) > 1/2$ implies $\mu(e(\alpha), \frac{1}{\alpha}k) > 1/2$. □

Lemma 2. *If (K,μ) is a Dedekind complete FRS and (k_λ) a net in K^+, then $\wedge_\lambda w_{k_\lambda} = 0$ implies $\wedge_\lambda k_\lambda = 0$. But the converse is not true.*

Proof. For each λ, $\mu(k_\lambda \wedge w, w_{k_\lambda}) > 1/2$, so $(\wedge_\lambda k_\lambda) \wedge w = \wedge_\lambda (k_\lambda \wedge w)$ and $\mu(\wedge_\lambda (k_\lambda \wedge w), \wedge_\lambda w_{k_\lambda}) > 1/2$. Thus, $(\wedge_\lambda k_\lambda) \wedge w = 0$. Hence, $\wedge_\lambda k_\lambda = 0$. To see that the converse is false, take a set $k_n = \frac{1}{n} w$ for all $n \in \mathbb{N}$. □

Lemma 3. *Let (K,μ) be a Dedekind complete FRS and (k_λ) a net in K^+. Then $\wedge_\lambda w_{(k_\lambda - \alpha w)^+} = 0$ for each $\alpha > 0$ iff $\wedge_\lambda k_\lambda = 0$.*

Proof. For the forward implication, we show that $\mu(\wedge_\lambda k_\lambda, \alpha w) > 1/2$ for each $\alpha > 0$. Fix α. By Lemma 2, $\wedge_\lambda (k_\lambda - \alpha w)^+ = 0$. So $\mu(\wedge_\lambda (k_\lambda - \alpha w), 0) > 1/2$, which implies $\mu(\wedge_\lambda k_\lambda, \alpha w) > 1/2$.

The converse follows from Lemma 1. □

Lemma 4. *If (K,μ) is a Dedekind complete FRS and (k_λ) a net in K^+, then $w_{k_\lambda} \xrightarrow{fo} 0$ implies $k_\lambda \xrightarrow{ufo} 0$. The converse is not true.*

Proof. The proof is essentially the same as for Lemma 2 with the use of Lemma 5. □

Now, the characterization of ufo-convergence is established in the following theorem.

Theorem 1. *Let (K,μ) be a Dedekind complete FRS and (k_λ) a net in K^+. Then $w_{(k_\lambda - \alpha w)} \xrightarrow{fo} 0$ for each $\alpha > 0$ iff $k_\lambda \xrightarrow{ufo} 0$.*

Proof. For the forward implication, suppose the net (k_λ) is fuzzy order bounded. We show that

$$\mu(\limsup_\lambda k_\lambda, \alpha w) > 1/2 \; \forall \; \alpha > 0.$$

Fix α. By Lemma 4, $(k_\lambda - \alpha w)^+ \xrightarrow{fo} 0$. In particular, $\limsup_\lambda (k_\lambda - \alpha w)^+ = 0$, thus $\mu(\limsup_\lambda (k_\lambda - \alpha w), 0) > 1/2$ such that $\mu(\limsup_\lambda k_\lambda, \alpha w) > 1/2$.

Now drop the supposition that (k_λ) is fuzzy order bounded. For every $\alpha > 0$,

$$\mu(w_{(k_\lambda \wedge w - \alpha w)^+}, w_{(k_\lambda - \alpha w)^+}) > 1/2 \text{ and } w_{(k_\lambda - \alpha w)^+} \xrightarrow{fo} 0.$$

Since $k_\lambda \wedge w$ is fuzzy order bounded, then $k_\lambda \wedge w \xrightarrow{fo} 0$.

The backward implication is followed from Lemma 1. □

Fuzzy Ideals and Completeness with Respect to Ufo-Convergence

The fuzzy ideal is a useful structure with important properties that can help to study ufo-convergence. To work in the fuzzy ideal is much easier than to work in the whole space. Indeed, it is shown that ufo-convergence in the fuzzy ideal is equivalent to ufo-convergence in the complete space.

Remark 3. *Let (K,μ) be an FRS, I be a fuzzy ideal of K, and $(k_\lambda) \subset I$. If $k_\lambda \xrightarrow{fo} 0$ in I, then $k_\lambda \xrightarrow{fo} 0$ in K. Conversely, If (k_λ) is fuzzy order bounded in I and $k_\lambda \xrightarrow{fo} 0$ in K, then $k_\lambda \xrightarrow{fo} 0$ in I.*

Proposition 6. *Let (K,μ) be a Dedekind complete FRS and (k_λ) a net in fuzzy ideal I of K. Then $k_\lambda \xrightarrow{ufo} 0$ in I iff $k_\lambda \xrightarrow{ufo} 0$ in K.*

Proof. Suppose $k_\lambda \xrightarrow{ufo} 0$ in K. Then for any $g \in I^+$ such that $|k_\lambda| \wedge g \xrightarrow{fo} 0$ in K, Remark 3 yields $|k_\lambda| \wedge g \xrightarrow{fo} 0$ in I. Hence, $k_\lambda \xrightarrow{ufo} 0$ in I. Conversely, take any $g \in I^+$, then $|k_\lambda| \wedge g \xrightarrow{fo} 0$ in I, and again by Remark 3, $|k_\lambda| \wedge g \xrightarrow{fo} 0$ in K. It follows that for any $g \in I^+$ and positive $h \in I^d$ such that $|k_\lambda| \wedge (g+h) \xrightarrow{fo} 0 = |k_\lambda| \wedge g \xrightarrow{fo} 0$ in K.

For any $u \in K^+$ and $z \in (I \oplus I^d)^+$, we have $u \wedge z \in (I \oplus I^d)^+$. Therefore, by Remark 3, $|k_\lambda| \wedge (u+z) \xrightarrow{fo} 0$ in K, or equivalently,

$$\limsup_\lambda (|k_\lambda| \wedge u) \wedge z = \limsup_\lambda (|k_\lambda| \wedge (u \wedge z)) = 0.$$

Theorem 4.7 (i) in [14] yields that $(I \oplus I^d)^d = \{0\}$. Thus,

$$\limsup_\lambda (|k_\lambda| \wedge u) = 0.$$

Hence, $|k_\lambda| \wedge u \xrightarrow{fo} 0$ in K. □

The closeness of ufo-convergence is defined and discussed as follows.

Definition 13. *Let (K, μ) be an FRS. For $C \subset K$ is said to be unbounded fuzzy order closed (ufo-closed for short) if for any net $(k_\lambda) \subset C$ and $k \in K$ with $k_\lambda \xrightarrow{ufo} k$ in K implies $k \in C$.*

Proposition 7. *Let L be a fuzzy Riesz subspace of an FRS (K, μ). Then L is ufo-closed in K iff L is fo-closed in K.*

Proof. The forward implication is obvious.

Conversely, suppose L is fo-closed in K. Let $(g_\lambda) \subseteq L$ and $k \in K$ such that $g_\lambda \xrightarrow{ufo} k$ in K. By Lemma 2(iv), $|g_\lambda| \xrightarrow{ufo} |k|$ in K. Therefore, without loss of generality, consider $(g_\lambda) \subseteq L^+$ and $k \in K^+$. Observe that for each $u \in K^+$, then

$$\mu(|g_\lambda \wedge u - k \wedge u|, |g_\lambda - k| \wedge u) > 1/2 \textit{ and } |g_\lambda - k| \wedge u \xrightarrow{fo} 0 \text{ in } K. \tag{1}$$

Consequently, for any $g \in L^+$, $g_\lambda \wedge g \xrightarrow{fo} k \wedge g$ in K. As L is fo-closed, then $k \wedge g \in L$. On the other hand, for any $u \in (L^d)^+$, then $g_\lambda \wedge u = 0$ for each λ, so that by (1), $k \wedge u = 0$. Thus, $k \in L^{dd}$, which is fuzzy band generated by L in K.

It follows that there is a net (u_γ) in the fuzzy ideal generated by L^+ such that $u_\gamma \uparrow k$ in K. Moreover, for each γ there exists $z_\gamma \in L$ such that $\mu(u_\gamma, z_\gamma) > 1/2$. So

$$\mu(u_\gamma \wedge k, z_\gamma \wedge k) > 1/2 \textit{ and } \mu(z_\gamma \wedge k, k) > 1/2$$

implies that $u_\gamma \uparrow k$ in K. Therefore, $z_\gamma \wedge k \xrightarrow{fo} k$ in K. Hence, $z_\gamma \wedge k \in L$ and L is fo-closed, then $k \in L$. □

5. Conclusions

In the present paper, fuzzy order convergence is generalized as unbounded fuzzy order convergence. Many other concepts like fuzzy weak order units and fuzzy components are studied, and many related results are proved. In addition, some applications of unbounded fuzzy order convergence are provided. For future research, one can define and explore the notions of fuzzy locally convex solid Riesz spaces and study different fuzzy norms with respect to fuzzy ordering to develop fuzzy norm lattices, which will lead to the fuzzy Banach lattices.

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