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The Recognition of the Bifurcation Problem with Trivial Solutions

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Abstract: This paper studies the recognition criterion of the bifurcation problem with trivial solution. The t-equivalence is different from the strong equivalence studied by Golubitsky et al. The difference is that the second component of the differential homeomorphism is not identical. Consider the normal subgroup of t-equivalence group, we obtain the characterization of higher order terms $\mathcal{P}(h)$. In addition, we also explore the properties of intrinsic submodules and the finite determinacy of the bifurcation problem.

Keywords: bifurcation; equivalent group; recognition; intrinsic submodule

1. Introduction

Singularity theory offers an extremely useful approach to bifurcation problems. Many authors have studied the classifications of bifurcation problems up to some codimension in a given context by singularity theory. These classifications include the following three components:

(i) A list of normal forms, with some properties that all bifurcation problems up to the given codimension are equivalent to one of them.
(ii) Constructing and analyzing the universal unfolding of the normal forms.
(iii) The solutions to the recognition problem for the normal forms.

The recognition problem belongs to the third component and it is the one of the least explored aspects of bifurcation theory. We are interested in knowing precisely when a bifurcation problem is equivalent to a given normal form. This problem can often be reduced to the finite dimensions problem by the idea from singularity theory that is finite determinacy. Many smooth function germs are determined up to equivalence by finite coefficients in their Taylor expansion. The solutions to the recognition problem can be characterised as comprising those germs whose Taylor coefficients satisfy a finite number of polynomial constraints in the form of equalities and inequalities.

In recent years, bifurcation theory has been applied to many models of mathematical biology. In evolutionary theory, the environment changes are often reflected by the changing of the residents' ability to reproduce. In Reference [1], Smith and Price first studied the phenotypic traits in evolutionary game. Subsequently, the authors in References [2–5] explored the adaptive dynamics approach for studying evolution of phenotypic traits. In Reference [6], Vutha and Golubitsky applied singularity theory and adaptive dynamics theory to study evolutionarily stable strategies and convergence stable strategy of strategy functions, they gave the classification with a codimension up to 3 under the action of strategy equivalent group and the solutions to the recognition problems of these normal forms. Wang and Golubitsky studied the fitness functions in adaptive dynamics with dimorphism equivalence, they classified singularities up to topological codimension 2 and gave the solutions for recognition problems in Reference [7]. In addition, there are many applications such as [8–10]. These works imply the studying vitality by making connections with applications. From these results, the recognition problem about normal form is a very important facet.

The key step in recognition problems is to find precisely the higher order terms. Bruce, Du Plessis and Wall in Reference [11] studied the determination of map germs by means of unipotent equivalence and linear equivalence. In Reference [12], Gaffney applied the methods of Reference [11] to study the bifurcation problem with multiparameters. Melbourne in Reference [13] studied an equivariant bifurcation problem with one bifurcation parameter, and he proved that the equivalence group can be decomposed into a unipotent equivalence group and a linear equivalence group. Under the action of these two groups in turn, the recognition problem can be decomposed similarly.

Inspired by References [12,13], we study the recognition solutions to the bifurcation problem that have trivial solutions. The authors in Reference [14] have given the classification of the bifurcation problem with a trivial solution up to codimension 3. This type of bifurcation problem here is different from that which has been studied in detail in Reference [15]; since it has a trivial solution, the equivalence group should preserve the trivial solution. The equivalence is also different from it in Reference [15] because the second component of diffeomorphism is not identical any more. This difference makes it troubling to get the higher order terms of the bifurcation problem. Considering the normal subgroup of the equivalence group, we obtain the formula of the high order terms. The study of the bifurcation problem with trivial solutions has many applications. In fact, there are many models that have the bifurcation problem with trivial solutions—for instance, the nonlinear oscillations Model 2 in Reference [16] and the Model 7 in Reference [17], and so on.

The organisation of this paper is as follows. Section 2 gives the necessary preparations. Section 3 explores the invariable submodule and their properties under the action of equivalence group. In Section 4, we define the lower order terms and higher order terms, and study the properties of them. Considering the normal subgroup of equivalence group, we obtain the solutions to the recognition problem. Two examples are given to apply the methods above in the last section. For all undefined terms and symbols, the reader is referred to References [14,15]. Assume that the function germs in this paper are smooth.

2. Basic Concepts and Preliminaries

Let h be a smooth function germ defined near the origin that is $h : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. The set of all h is denoted as $\varepsilon_{x,\lambda}$ and we can verify that $\varepsilon_{x,\lambda}$ is a ring. In this ring, there are some germs that have trivial solutions. The bifurcation problem with trivial solutions has been defined in Reference [14]. Denote the set of all bifurcation problems that have trivial solutions as follows:

$$\varepsilon_{x,\lambda}\{x\} = \{h \in \varepsilon_{x,\lambda} \mid h(x,\lambda) = xf(x,\lambda),\ f \in \varepsilon_{x,\lambda}\}.$$

Then, $\varepsilon_{x,\lambda}\{x\}$ is a module over the ring $\varepsilon_{x,\lambda}$.

Let $h \in \varepsilon_{x,\lambda}\{x\}$; then, there exists $f \in \varepsilon_{x,\lambda}$ such that $h(x,\lambda) = f(x,\lambda)x$. Note that $h_x(0,0) = 0$, so $f(0,0) = 0$. Thus, the bifurcation problem with trivial solutions can also be represented as

$$\mathcal{M}_{x,\lambda}\{x\} = \{h \in \varepsilon_{x,\lambda}\{x\} \mid h(x,\lambda) = xf(x,\lambda),\ f \in \mathcal{M}_{x,\lambda}\},$$

where $\mathcal{M}_{x,\lambda}$ is the maximal ideal in the ring $\varepsilon_{x,\lambda}$ and briefly denoted as $\mathcal{M}$. Obviously, $\mathcal{M}\{x\}$ is a submodule of $\varepsilon_{x,\lambda}\{x\}$.

Lemma 1. *Let $J \subset \varepsilon_{x,\lambda}\{x\}$ is a submodule. Then, there exists an ideal $I \subset \varepsilon_{x,\lambda}$ such that $J = I\{x\}$. Conversely, this equality defines a submodule for every ideal $I \in \varepsilon_{x,\lambda}$.*

Proof. It can be easily proved by the definitions of ideal and submodule. □

If there is a k-dimensional subspace $V \subset \varepsilon_{x,\lambda}\{x\}$ such that $\varepsilon_{x,\lambda}\{x\} = J \oplus V$, we say that J has codimension k in $\varepsilon_{x,\lambda}\{x\}$. In Lemma 1, codim $J =$ codim I, codim I is computed in $\varepsilon_{x,\lambda}$.

Theorem 1. *A submodule $J \subset \varepsilon_{x,\lambda}\{x\}$ has a finite codimension if and only if $\mathcal{M}^k\{x\} \subset J$ for some positive integer k, where $\mathcal{M}$ is the maximal ideal in $\varepsilon_{x,\lambda}$.*

Proof. We can use Nakayama Lemma (see Reference [15], p. 71) to prove this theorem. □

Denote $\mathcal{G}$ as the set of all t-equivalences in Reference [14], that is,

$$\mathcal{G} = \{(S, X, \Lambda) \in \varepsilon_{x,\lambda} \times \varepsilon_{x,\lambda}\{x\} \times \mathcal{M}_\lambda \mid S(0,0) > 0,\ X_x(0,0) > 0,\ \Lambda_\lambda(0) > 0\},$$

where $X(0, \lambda) \equiv 0, \Lambda(0) = 0$ and $\mathcal{M}_\lambda$ is the maximal ideal in ε_λ. Here, $\Lambda = \Lambda(\lambda)$ is no longer the λ in Reference [15]. We can verify that $\mathcal{G}$ is a group. In addition, the action of $\mathcal{G}$ on $\mathcal{M}\{x\}$ induces the equivalence relation that is

$$\mathcal{G} \cdot h = \{g \in \mathcal{M}\{x\} \mid g \sim h\},$$

where the symbol $\sim$ is a t-equivalence.

Considering an arbitrary curve $\delta_t(h)(\delta_0 = 1)$ in $\mathcal{G} \cdot h$, let

$$p = \frac{d}{dt}\delta_t(h)|_{t=0}.$$

Then, the set of p is the orbit tangent space $T(h)$ of h that has been defined in Reference [14]:

$$T(h) =< h, xh_x >_{\varepsilon_{x,\lambda}} + \varepsilon_\lambda\{\lambda h_\lambda\}.$$

The orbit tangent space $T(h)$ is not a submodule of $\varepsilon_{x,\lambda}\{x\}$, so it brings difficulty in judging whether $T(h)$ has a finite codimension in the vector space $\varepsilon_{x,\lambda}\{x\}$. The codimension in this paper refers to the codimension as a vector subspace. In Reference [14], we have defined the codimension of a bifurcation problem h as the codimension of $T(h)$ in $\varepsilon_{x,\lambda}\{x\}$. The following theorem gives the judgement method of the finite codimension of a bifurcation problem.

Theorem 2. *Let $h \in \mathcal{M}\{x\}$. The submodule $< h, xh_x >$ has a finite codimension in $\varepsilon_{x,\lambda}\{x\}$ if and only if $T(h)$ has a finite codimension in $\varepsilon_{x,\lambda}\{x\}$.*

Proof. Since $< h, xh_x > \subset T(h)$, one direction of the implication is clear. The reverse implication will be proved by contradiction as follows.

Let $T(h)$ have a finite codimension. The first step proof in Reference [15] is to reduce h as a polynomial. Consider the equations

$$h = h_x = 0 \tag{1}$$

over the complex numbers.

Supposing $< h,\ xh_x >$ has infinite codimensions, then the solution set of Equation (1) contains a nonconstant smooth curve $(X(t), \Lambda(t))$ such that $X(0) = \Lambda(0) = 0$, where t is a real parameter. Thus,

$$h(X(t), \Lambda(t)) \equiv 0, \tag{2a}$$

$$h_x(X(t), \Lambda(t)) \equiv 0. \tag{2b}$$

Differentiating Equation (2a) with respect to t and apply Equation (2b), we have

$$h_\lambda(X(t), \Lambda(t)) \cdot \Lambda'(t) \equiv 0.$$

Therefore, either $h_\lambda(X(t),\Lambda(t)) \equiv 0$ or $\Lambda'(t) \equiv 0$. If $h_\lambda(X(t),\Lambda(t)) \equiv 0$, combined with Equations (2a) and (2b), the submodule $< h,\ h_x,\ h_\lambda >$ has infinite codimensions. Since this submodule contains $T(h)$, there is a contradiction. Thus, $\Lambda'(t) \equiv 0$. In addition, because $\Lambda(0) = 0$, then $\Lambda(t) \equiv 0$, thus

$$h(X(t),0) \equiv 0,\ h_x(X(t),0) \equiv 0. \tag{3}$$

Note that

$$J =< h(x,0),\ h_x(x,0),\ \lambda > . \tag{4}$$

$T(h) \subset J$, then J is the finite codimension. From Equation (4), J has a finite codimensions only when $< h(x,0),\ h_x(x,0) >$ has a finite codimension in ε_x. Thus, the only common zero of $h(x,0) = h_x(x,0) = 0$ is $x = 0$. It means that $X(t) \equiv 0$ in Equation (3), which contradicts the choice of $(X(t),\Lambda(t))$. Therefore, the submodule $< h,\ xh_x >$ must have finite codimensions. □

Let $T(h)$ have finite codimensions in $\varepsilon_{x,\lambda}\{x\}$; by Theorem 2, there exists a positive integer k such that $\mathcal{M}^k\{x\} \subset T(h)$. The finite codimension of $T(h)$ means h is finite determined.

Theorem 3. *Let $h \in \mathcal{M}\{x\}$ and $h = fx$ such that $\mathcal{M}^k\{x\} \subset T(h)$; then, h is t-equivalent to $(j^k f)x$, where $f \in \mathcal{M}$ and $j^k f$ is the k-jet of its Taylor expansion.*

Proof. Rewrite $h = (j^k f)x - r$, where $r \in \mathcal{M}^{k+1}\{x\}$. According to Theorem 3.3 in Reference [14], in order to prove h is t-equivalent to $(j^k f)x$, it suffices to show that $T(h) = T(h + tr)$, $0 \leq t \leq 1$. For any $f \in T(h + tr)$, there exist $a,\ b \in \varepsilon_{x,\lambda},\ c \in \varepsilon_\lambda$ such that

$$\begin{aligned} f &= a(h + tr) + b(xh_x + txr_x) + c(\lambda h_\lambda + t\lambda r_\lambda) \\ &= ah + bxh_x + c\lambda h_\lambda + tar + tbxr_x + tc\lambda r_\lambda. \end{aligned}$$

Since the last three terms on the right-hand side of the above equation, tar, $tbxr_x$, and $tc\lambda r_\lambda$ all belong to $\mathcal{M}^{k+1} \cdot \{x\} \subset T(h)$, then $f \in T(h)$. Thus, $T(h + tr) \subset T(h)$.

Conversely, the generators of $T(h)$ can be written as

$$\begin{aligned} h &= h + tr - tr \in T(h + tr) + \mathcal{M}^{k+1}\{x\} \subset T(h + tr) + \mathcal{M}T(h), \\ xh_x &= xh_x + txr_x - txr_x \in T(h + tr) + \mathcal{M}^{k+1}\{x\} \subset T(h + tr) + \mathcal{M}T(h), \\ \lambda h_\lambda &= \lambda h_\lambda + t\lambda r_\lambda - t\lambda r_\lambda \in T(h + tr) + \mathcal{M}^{k+1}\{x\} \subset T(h + tr) + \mathcal{M}T(h). \end{aligned}$$

Then, $T(h) \subset T(h + tr) + \mathcal{M}T(h)$. Thus, there exist $g_i \in T(h + tr)$, $\alpha_i^j \in \mathcal{M}$, where i, $j = 1, 2, 3$ such that

$$\begin{cases} h = g_1 + \alpha_1^1 h + \alpha_1^2 xh_x + \alpha_1^3 \lambda h_\lambda, \\ xh_x = g_2 + \alpha_2^1 h + \alpha_2^2 xh_x + \alpha_2^3 \lambda h_\lambda, \\ \lambda h_\lambda = g_3 + \alpha_3^1 h + \alpha_3^2 xh_x + \alpha_3^3 \lambda h_\lambda. \end{cases} \tag{5}$$

Rearranging the terms in the system Equation (5) to obtain the following matrix equation:

$$\begin{pmatrix} 1 - \alpha_1^1 & -\alpha_1^2 & -\alpha_1^3 \\ -\alpha_2^1 & 1 - \alpha_2^2 & -\alpha_2^3 \\ -\alpha_3^1 & -\alpha_3^2 & 1 - \alpha_3^3 \end{pmatrix} \begin{pmatrix} h \\ xh_x \\ \lambda h_\lambda \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}. \tag{6}$$

Denote the matrix in Equation (6) by

$$A = \begin{pmatrix} 1 - \alpha_1^1 & -\alpha_1^2 & -\alpha_1^3 \\ -\alpha_2^1 & 1 - \alpha_2^2 & -\alpha_2^3 \\ -\alpha_3^1 & -\alpha_3^2 & 1 - \alpha_3^3 \end{pmatrix}.$$

Since $\det A = 1 + \alpha, \alpha \in \mathcal{M}$, then $\det A(0,0) = 1$, A is invertible in $\varepsilon_{x,\lambda}$. Thus,

$$\begin{pmatrix} h \\ xh_x \\ \lambda h_\lambda \end{pmatrix} = A^{-1} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}. \tag{7}$$

By Equation (7), $T(h) \subset T(h+tr)$. From analysis of the above, the theorem is proved. □

3. Intrinsic Submodule

In this section, we define an intrinsic submodule and introduce some properties of it.

Definition 1. *Let $J \subset \varepsilon_{x,\lambda}\{x\}$ be a submodule. If $h \in J$* and *$g \sim h$, then $g \in J$, J is called an intrinsic submodule.*

From Definition 1, we can see that intrinsic submodule is invariant under the action of group $\mathcal{G}$. It can be easily verified that the sum of two intrinsic submodule is also intrinsic submodule, and so is the product.

Proposition 1. *Let $J \subset \varepsilon_{x,\lambda}\{x\}$ be an intrinsic submodule with a finite codimension and*

$$q(x,\lambda) = \sum_{\alpha} a_\alpha x^{\alpha_1} \lambda^{\alpha_2}.$$

Then, $q(x,\lambda)x \in J$ if and only if the monomial $x^{\alpha_1}\lambda^{\alpha_2}x \in J$ for every $a_\alpha \neq 0$.

Proof. If $x^{\alpha_1}\lambda^{\alpha_2}x \in J$ for all $a_\alpha \neq 0$, then $q(x,\lambda)x \in J$ naturally. The other implication is proved as follows. Letting $J = I\{x\}$, $I \in \varepsilon_{x,\lambda}$ is an ideal. We will show $x^l\lambda^m \in I$ for an arbitrary multi-index $\alpha = (l,m)$ satisfying $a_\alpha \neq 0$. Since J has a finite codimension, there exists $k \in \mathbb{Z}^+$ such that $\mathcal{M}^{k+1}\{x\} \subset J$. If $l + m > k$, then the desired conclusion holds trivially. Thus, we assume $l + m \leq k$; then, q can be reduced to a polynomial of degree k or less. Arranging the terms in q according to degree of x

$$q(x,\lambda) = q_0(\lambda) + q_1(\lambda)x + \cdots + q_k(\lambda)x^k. \tag{8}$$

We show that $q_j(\lambda)x^j \in I$ for $0 \leq j \leq k$. It is clear that

$$q(tx,\lambda) = q_0(\lambda) + tq_1(\lambda)x + \cdots + t^k q_k(\lambda)x^k$$

is in I for every $t > 0$. Differentiating k-times with respect to t, then $q_k(\lambda)x^k$ is in I. The claim is true by induction argument proceeding from the last term to the first.

Now consider $q_l(\lambda)$ the coefficient of x^l in Equation (8). Let

$$q_l(\lambda) = c_0 + c_1\lambda + \cdots + c_{k-l}\lambda^{k-l}. \tag{9}$$

Since $c_m = a_{lm} \neq 0$, the polynomial can not vanish identically. Let c_u be the first nonzero coefficient in Equation (9), then $u \leq m$. Hence,

$$q_l(\lambda) = \lambda^u p(\lambda), \tag{10}$$

where $p(0) \neq 0$, which means that $\frac{1}{p(\lambda)} \in \varepsilon_{x,\lambda}$, so that Equation (10) may be inverted. Thus,

$$x^l\lambda^m = \lambda^{m-u}(\lambda^u x^l) = \lambda^{m-u}\frac{1}{p(\lambda)}q_l(\lambda)x^l.$$

Since $q_l(\lambda)x^l \in I$ and I is an ideal, therefore $x^l\lambda^m \in I$ and $x^l\lambda^m x \in J$. □

Lemma 2. *Let $J \subset \varepsilon_{x,\lambda}\{x\}$ be an intrinsic submodule of finite codimension. If a germ $h \in J$, then $xh_x \in J$, and $\lambda h_\lambda \in J$.*

Proof. Let $h(x,\lambda) = xf(x,\lambda)$. Since J has a finite codimension, there exists $k \in \mathbb{Z}^+$ such that $\mathcal{M}^{k+1}\{x\} \subset J$—by Taylor Theorem

$$h(x,\lambda) = xj^k f(x,\lambda) + r(x,\lambda),$$

where $r \in \mathcal{M}^{k+1}\{x\}$. Obviously, xr_x and λr_λ belong to $\mathcal{M}^{k+1}\{x\} \subset J$, and we reduce h to the polynomial $(j^k f)x$. Thus, it is sufficient to prove the result for $h \in (\mathcal{M}^{k+1})^\perp\{x\} \cap J$.

Since J is intrinsic, then $h(tx,\lambda) \in (\mathcal{M}^{k+1})^\perp\{x\} \cap J$ for all $t > 0$. We obtain that

$$\rho(t) = \frac{h(tx,\lambda) - h(x,\lambda)}{t-1}$$

is in $(\mathcal{M}^{k+1})^\perp\{x\} \cap J$ for each t. However, $(\mathcal{M}^{k+1})^\perp\{x\} \cap J$ is a closed subspace of space $(\mathcal{M}^{k+1})^\perp\{x\}$. Thus, $\lim\limits_{t\to 1}\rho(t)$ is in $(\mathcal{M}^{k+1})^\perp\{x\} \cap J$; this limit is precisely $xh_x(x,\lambda)$.

Similarly, $h(x,t\lambda) \in J$ for all $t > 0$. Differentiating with respect to t and evaluating at $t = 1$ produces the germ $\lambda h_\lambda(x,\lambda)$, and $\lambda h_\lambda(x,\lambda)$ is in J. □

Proposition 2. *Let J be a submodule of $\varepsilon_{x,\lambda}\{x\}$ of finite codimension. Then, J is intrinsic if and only if it can be written as the form*

$$J = < x^{k_1}\lambda^{l_1},\ x^{k_2}\lambda^{l_2},\ \ldots,\ x^{k_s}\lambda^{l_s} > \{x\}. \tag{11}$$

Proof. Let $h(x,\lambda) = f(x,\lambda)x$ be in $\varepsilon_{x,\lambda}\{x\}$. Apply t-equivalence (S, X, Λ) to h.

Let $S(x,\lambda) = a(x,\lambda),\ X(x,\lambda) = b(x,\lambda)x$, and $\Lambda(\lambda) = c(\lambda)\lambda$, where

$$a(0,0) > 0,\ b(0,0) > 0, c(0) > 0.$$

Thus,

$$\begin{aligned} S(x,\lambda)h(X(x,\lambda),\Lambda(\lambda)) &= S(x,\lambda)f(X(x,\lambda),\Lambda(\lambda))X(x,\lambda) \\ &= a(x,\lambda)f(b(x,\lambda)x, c(\lambda)\lambda)b(x,\lambda)x. \end{aligned}$$

In particular, under equivalence, $x^k\lambda^l x$ is mapped into $a(x,\lambda)b^{k+1}(x,\lambda)c^l(\lambda)x^k\lambda^l x$, thus the submodule $< x^k\lambda^l > \{x\}$ is intrinsic. Since sums of intrinsic submodule are intrinsic, then Equation (11) defines an intrinsic submodule.

Conversely, since J has a finite codimension, there exists a positive integer k such that $\mathcal{M}^{k+1}\{x\} \subset J$. Substituting h by a polynomial, by Proposition 1, the result is obtained. □

Remark 1. *In Equation (11), we usually require that*

$$\begin{aligned} &(a)\ k_1 > k_2 > \cdots > k_s = 0, \\ &(b)\ 0 = l_1 < l_2 < \cdots < l_s. \end{aligned} \tag{12}$$

Definition 2. *If Equation (12) holds, monomials in Equation (11) are called the intrinsic generators of J.*

4. Statement of the Main Result

Letting $h \in \mathcal{M}\{x\}$, we define $\mathcal{S}(h)$ to be the smallest intrinsic submodule containing h.

Proposition 3. *Let $h \in \mathcal{M}\{x\}$ and $h(x,\lambda) = xf(x,\lambda)$ such that $T(h)$ has a finite codimension. Then,*
(a) *$\mathcal{S}(g) = \mathcal{S}(h)$, if g is t-equivalent to h.*
(b) *$\mathcal{S}(h)$ is an intrinsic submodule of finite codimension.*

$$\text{(c) } \mathcal{S}(h) = \sum_{\alpha=(\alpha_1,\alpha_2)} \{< x^{\alpha_1}\lambda^{\alpha_2} > \{x\} | D^{\alpha} f(0,0) \neq 0\}. \tag{13}$$

Proof. (a) By the definition of the smallest intrinsic submodule, we can get it.

(b) We need to show that $\mathcal{S}(h)$ has a finite codimension. Observing that $T(h)$ has a finite codimension, there exists some positive integer k such that $\mathcal{M}^k\{x\} \subset T(h)$. By the Theorem 3, h is t-equivalent to $(j^k f)x$. By (a), $\mathcal{S}(h) = \mathcal{S}((j^k f)x)$. Generally, we replace h by $(j^k f)x$.

To show that $\mathcal{S}(h)$ has a finite codimension, we will prove that $T(h) \subset \mathcal{S}(h)$. It is sufficient to prove $xh_x, \lambda h_\lambda \in ((\mathcal{M}^{k+1})^{\perp}\{x\}) \bigcap \mathcal{S}(h)$. By Lemma 2, this is true.

(c) h can be reduced to a polynomial as in (b). By Proposition 1 and $h \in S(h)$, $S(h)$ contains the monomial $x^{\alpha_1}\lambda^{\alpha_2}x$. Therefore, the right-hand side of Equation (13) is contained in $S(h)$. Conversely, assume that h belongs to the right-hand side of Equation (13), which is an intrinsic submodule. Since $S(h)$ is the smallest intrinsic submodule containing h, then $S(h)$ is contained on the right-hand side of Equation (13). □

Theorem 4. *Let $g(x,\lambda) = f(x,\lambda)x$ be equivalent to h, then*
(a) *$D^{\alpha} f(0,0) = 0$ for every monomial $x^{\alpha_1}\lambda^{\alpha_2}x \in \mathcal{S}(h)^{\perp}$.*
(b) *$D^{\alpha} f(0,0) \neq 0$ for each intrinsic generator $x^{\alpha_1}\lambda^{\alpha_2}x$ of $\mathcal{S}(h)$.*

Proof. (a) It is proved immediately by contradiction.

(b) By Proposition 3(c), the result is clear. □

Definition 3. *Let h be a finite codimension germ. Define the high order terms of h as follows:*

$$\mathcal{P}(h) = \{p \in \mathcal{M}\{x\} \mid g + p \in \mathcal{G} \cdot h, \ \forall\, g \in \mathcal{G} \cdot h\}.$$

Lemma 3. *$\mathcal{P}(h)$ is a submodule in $\varepsilon_{x,\lambda}\{x\}$.*

This lemma can be proved easily by the Definition of submodule.

Lemma 4. *The submodule $\mathcal{P}(h)$ is intrinsic.*

Proof. Let $p \in \mathcal{P}(h)$ and γ be a t-equivalence. We will show that $\gamma(p) \in \mathcal{P}(h)$. Suppose g is t-equivalent to h. By Lemma 12.2 (see Reference [15], p. 104), we have

$$\begin{aligned} T(g + t\gamma(p)) &= T(\gamma(\gamma^{-1}(g) + tp) \\ &= \gamma T(\gamma^{-1}(g) + tp). \end{aligned}$$

Since $\gamma^{-1}(g)$ is t-equivalent to g, then $\gamma^{-1}(g)$ is t-equivalent to h. In view of $p \in \mathcal{P}(h)$,

$$T(\gamma^{-1}(g) + tp) = T(\gamma^{-1}(g)).$$

Then, $T(g + t\gamma(p)) = \gamma T(\gamma^{-1}(g)) = T(g)$ and $\gamma(p) \in \mathcal{P}(h)$. □

Lemma 5. *Let J be an intrinsic submodule, then $J \subset \mathcal{P}(h)$ if $T(h+p) = T(h)$ for $p \in J$.*

Proof. By Lemma 4, it can be easily proved. □

Lemma 6. Itr$\{\mathcal{M}T(h)\} \subset \mathcal{P}(h)$ *if* codim$T(h) < \infty$.

Proof. Let $I = \mathrm{Itr}\{\mathcal{M}T(h)\}$. By Lemma 5, in order to prove $I \subset \mathcal{P}(h)$, it is sufficient to prove

$$T(h+p) = T(h),\ \forall\ p \in I.$$

Let $p \in I$. Since $T(h)$ has a finite codimension, by Theorem 2, $< h, xh_x >$ has a finite codimension, then there exists $k \in \mathbb{Z}^+$ such that $\mathcal{M}^k\{x\} \subset < h, xh_x > \subset T(h)$. Thus,

$$\mathcal{M}^{k+1}\{x\} \subset \mathrm{Itr}\{\mathcal{M}T(h)\}.$$

Hence, I has a finite codimension. Since $\mathcal{M}T(h) = \mathcal{M} < h, xh_x, \lambda h_\lambda >$, then $\mathcal{M}T(h)$ is a submodule. By Lemma 2, xp_x, $\lambda p_\lambda \in I$. Then, p, xp_x, and $\lambda p_\lambda \in \mathcal{M}T(h) = \mathcal{M} < h, xh_x, \lambda h_\lambda >$. By Nakayama's Lemma, we have

$$< h+p, x(h+p)_x, \lambda(h+p)_\lambda >_{\varepsilon_{x,\lambda}} = < h, xh_x, \lambda h_\lambda >_{\varepsilon_{x,\lambda}}.$$

The following proof of $T(h+p) = T(h)$ for $\forall\ p \in I$ is similar to Lemma 3, so it is omitted. □

Proposition 4. (a) *If $p \in \mathcal{P}(h)$ and g is equivalent to h, then $g + p$ is equivalent to g.*
(b) *If $T(h)$ has a finite codimension, then $\mathcal{P}(h)$ is an intrinsic submodule of $\varepsilon_{x,\lambda}\{x\}$ with a finite codimension.*

Proof. By Definition 3, (a) is obtained immediately.
(b) $T(h)$ has a finite codimension, then $\mathcal{M}^k\{x\} \subset T(h)$ for some $k \in \mathbb{Z}^+$. By Lemma 6,

$$\mathcal{M}^{k+1}\{x\} \subset \mathrm{Itr}\{\mathcal{M}T(h)\} \subset \mathcal{P}(h),$$

then $\mathcal{P}(h)$ has a finite codimension. Combining with Lemmas 3 and 4, $\mathcal{P}(h)$ is an intrinsic submodule with a finite codimension. □

Let

$$\widetilde{\mathcal{G}} = \{(S, X, \Lambda) \in \varepsilon_{x,\lambda} \times \varepsilon_{x,\lambda}\{x\} \times \mathcal{M}_\lambda \mid S(0,0) = 1,\ X_x(0,0) = 1,\ \Lambda_\lambda(0) = 1\}.$$

Then, $\widetilde{\mathcal{G}}$ is a normal subgroup of $\mathcal{G}$. Define the orbit tangent space $T(h, \widetilde{\mathcal{G}})$ with the action of $\widetilde{\mathcal{G}}$:

$$\begin{aligned} T(h, \widetilde{\mathcal{G}}) &= \{d(\delta_t h)/dt|_{t=0} \mid \delta_t \in \widetilde{\mathcal{G}}, \delta_0 = 1\} \\ &= < xh, \lambda h, x^2 h_x, x\lambda h_x > + \varepsilon_\lambda\{\lambda^2 h_\lambda\}, \end{aligned}$$

then $T(h, \widetilde{\mathcal{G}}) \subset \mathcal{M}T(h)$.

Definition 4. *For $h \in \mathcal{M}\{x\}$, denote sets as*

$$\mathcal{N}(h, \widetilde{\mathcal{G}}) = \{p \in \mathcal{M}\{x\} \mid h + p \in \widetilde{\mathcal{G}} \cdot h\}$$

and

$$\mathcal{P}(h, \widetilde{\mathcal{G}}) = \{p \in \mathcal{M}\{x\} \mid g + p \in \widetilde{\mathcal{G}} \cdot h \mid \forall\ g \in \widetilde{\mathcal{G}} \cdot h\}.$$

Proposition 5. $\mathcal{P}(h, \widetilde{\mathcal{G}}) = \mathrm{Itr}\mathcal{N}(h, \widetilde{\mathcal{G}})$ *if h has a finite codimension.*

Proof. It is sufficient to show that $\mathcal{P}(h, \widetilde{\mathcal{G}})$ is the unique maximal $\widetilde{\mathcal{G}}$-intrinsic subspace contained in $\mathcal{N}(h, \widetilde{\mathcal{G}})$. Closure under addition and scalar multiplication are similar to the proof of Proposition 3.8 in Reference [13].

Letting $p \in \mathcal{P}(h, \widetilde{\mathcal{G}}), \gamma \in \widetilde{\mathcal{G}}$, then

$$g + \gamma p = \gamma(\gamma^{-1} g + p) \in \widetilde{\mathcal{G}} \cdot h,$$

so $\gamma p \in \mathcal{P}(h, \widetilde{\mathcal{G}})$. Therefore, $\mathcal{P}(h, \widetilde{\mathcal{G}})$ is a $\widetilde{\mathcal{G}}$-intrinsic subspace. Clearly, $\mathcal{P}(h, \widetilde{\mathcal{G}}) \subset \mathcal{N}(h, \widetilde{\mathcal{G}})$. Suppose $Q \subset \mathcal{N}(h, \widetilde{\mathcal{G}})$, where Q is $\widetilde{\mathcal{G}}$-intrinsic. Let $p \in Q$ and $g = \gamma h, \gamma \in \widetilde{\mathcal{G}}$, then

$$g + p = \gamma h + p = \gamma(h + \gamma^{-1} p) \in \widetilde{\mathcal{G}} \cdot h.$$

Thus, $Q \subset \mathcal{P}(h, \widetilde{\mathcal{G}})$, and $\mathcal{P}(h, \widetilde{\mathcal{G}})$ is uniquely maximal in $\mathcal{N}(h, \widetilde{\mathcal{G}})$. □

Corollary 1. $\mathcal{P}(h, \widetilde{\mathcal{G}}) = \mathrm{Itr}(T(h, \widetilde{\mathcal{G}}))$ *if h has a finite codimension.*

Proof. By Corollary 3.6(b) in Reference [13] and Proposition 5, we have

$$M \subset \mathcal{P}(h, \widetilde{\mathcal{G}}) \ \ if \ \ and \ \ only \ \ if \ \ M \subset \mathrm{Itr}(T(h, \widetilde{\mathcal{G}}))$$

for any $\widetilde{\mathcal{G}}$-intrinsic subspace M. Setting $M = \mathcal{P}(h, \widetilde{\mathcal{G}})$ and $M = \mathrm{Itr}(T(h, \widetilde{\mathcal{G}}))$ in turn gives the result. □

We can also prove that the module $\mathcal{P}(h)$ is contained in the module $\mathcal{P}(h, \widetilde{\mathcal{G}})$ that is $\mathcal{P}(h) \subset \mathcal{P}(h, \widetilde{\mathcal{G}})$. By Corollary 1, $\mathcal{P}(h) \subset \mathrm{Itr} T(h, \widetilde{\mathcal{G}})$. Note that $\mathrm{Itr}(T(h, \widetilde{\mathcal{G}})) \subset \mathrm{Itr}(\mathcal{M}T(h))$, thus $\mathcal{P}(h) \subset \mathrm{Itr}(\mathcal{M}T(h))$. Combining with Lemma 6, we have proved the following theorem.

Theorem 5. *Let* $h \in \mathcal{M}\{x\}$ *be a germ such that* $T(h)$ *has a finite codimension, then* $\mathcal{P}(h) = \mathrm{Itr}\{\mathcal{M}T(h)\}$.

5. Examples

In this section, we apply the above results to solve the recognition problem for two classes of normal forms.

Example 1. *Let* $g(x, \lambda) = f(x, \lambda)x$ *be in* $\mathcal{M}\{x\}$*. Then, g is t-equivalent to* $h(x, \lambda) = (\varepsilon x^k + \delta\lambda)x$ *if and only if at* $x = \lambda = 0$

$$f = \frac{\partial f}{\partial x} = \cdots = \left(\frac{\partial}{\partial x}\right)^{k-1} f = 0,$$

and

$$\varepsilon = \mathrm{sgn}\left(\frac{\partial}{\partial x}\right)^k f \, , \, \delta = \mathrm{sgn}\frac{\partial f}{\partial \lambda}.$$

Proof. Firstly, by Proposition 3(c), $\mathcal{S}(h) =< x^k, \lambda > \{x\}$. By Theorem 4, if a germ g is t-equivalent to h, then

$$g(x, \lambda) = (ax^k + b\lambda)x + p(x, \lambda), \tag{14}$$

where $a \neq 0, b \neq 0$, and $p \in < x^{k+1}, x\lambda, \lambda^2 > \{x\}$.

Secondly, we have $T(h) =< x^k, \lambda > \{x\}$. By Theorem 5,

$$\mathcal{P}(h) = \mathrm{Itr}\{\mathcal{M}T(h)\} =< x^{k+1}, x\lambda, \lambda^2 > \{x\}.$$

Therefore, the term $p(x, \lambda)$ in Equation (14) has no influence on whether or not g is t-equivalent to h.

Then, g is t-equivalent to h if and only if $\tilde{g}(x, \lambda) = (ax^k + b\lambda)x$ is t-equivalent to h. Setting

$$S(x, \lambda) = \sqrt[k]{|a|} \, , \, X(x, \lambda) = \frac{1}{\sqrt[k]{|a|}} x \, , \, \Lambda(\lambda) = \frac{1}{|b|}\lambda \, ,$$

where $a \neq 0, b \neq 0$. Then,

$$S(x,\lambda)\tilde{g}(X(x,\lambda),\Lambda(\lambda)) = \left(\frac{a}{|a|}x^k + \frac{b}{|b|}\lambda\right)x = (\varepsilon x^k + \delta\lambda)x,$$

where $\varepsilon = \operatorname{sgn}\left(\frac{\partial}{\partial x}\right)^k f$, $\delta = \operatorname{sgn}\frac{\partial f}{\partial \lambda}$. $\square$

Example 2. *Let $g(x,\lambda) = f(x,\lambda)x$ be in $\mathcal{M}\{x\}$. Then, g is t-equivalent to $h(x,\lambda) = (\varepsilon x + \delta\lambda^k)x$ if and only if at $x = \lambda = 0$*

$$f = \frac{\partial f}{\partial \lambda} = \cdots = \left(\frac{\partial}{\partial \lambda}\right)^{k-1} f = 0$$

and

$$\varepsilon = \operatorname{sgn}\frac{\partial f}{\partial x}\ ,\ \delta = \operatorname{sgn}\left(\frac{\partial}{\partial \lambda}\right)^k f.$$

Proof. Firstly, $\mathcal{S}(h) =< x, \lambda^k > \{x\}$. Then,

$$g(x,\lambda) = (ax + b\lambda^k)x + p(x,\lambda),$$

where $a \neq 0, b \neq 0$, and $p \in< x^2, x\lambda, \lambda^{k+1} > \{x\}$.

Since $T(h) =< x, \lambda^k > \{x\}$, then $\mathcal{P}(h) = \operatorname{Itr}\{\mathcal{M}T(h)\} =< x^2, x\lambda, \lambda^{k+1} > \{x\}$. Finally, setting

$$S(x,\lambda) = |a|\ ,\ X(x,\lambda) = \frac{1}{|a|}x\ ,\ \Lambda(\lambda) = \frac{1}{\sqrt[k]{|b|}}\lambda\ ,$$

where $a \neq 0, b \neq 0$. Then,

$$S(x,\lambda)\tilde{g}(X(x,\lambda),\Lambda(\lambda)) = \left(\frac{a}{|a|}x + \frac{b}{|b|}\lambda^k\right)x = (\varepsilon x + \delta\lambda^k)x,$$

where $\varepsilon = \operatorname{sgn}\frac{\partial f}{\partial x}$, $\delta = \operatorname{sgn}\left(\frac{\partial}{\partial \lambda}\right)^k f$. $\square$

Subsequently, we will study the perturbations of the bifurcation problem under the action of the t-equivalence group and hope to find some models to apply the results.

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