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A Study of Boundedness in Fuzzy Normed Linear Spaces

Tudor Bînzar ¹, Flavius Pater ¹ and Sorin Nădăban ^{2,*}

¹ Department of Mathematics, Politehnica University of Timișoara, Regina Maria 1, RO-300004 Timișoara, Romania

² Department of Mathematics and Computer Science, Aurel Vlaicu University of Arad, Elena Drăgoi 2, RO-310330 Arad, Romania

* Correspondence: snadaban@gmail.com; Tel.: +40-745-818-447

Received: 18 June 2019; Accepted: 8 July 2019; Published: 15 July 2019



Abstract: In the present paper some different types of boundedness in fuzzy normed linear spaces of type $(X, N, *)$, where $*$ is an arbitrary t-norm, are considered. These boundedness concepts are very general and some of them have no correspondent in the classical topological metrizable linear spaces. Properties of such bounded sets are given and we make a comparative study among these types of boundedness. Among them there are various concepts concerning symmetrical properties of the studied objects arisen from the classical setting appropriate for this journal topics. We establish the implications between them and illustrate by examples that these concepts are not similar.

Keywords: fuzzy norm; fuzzy normed linear space; fuzzy metric space; fuzzy bounded set

MSC: 46S40

1. Introduction

Fuzzy normed linear spaces, briefly FNL spaces, were first introduced by Katsaras, who introduced some general types of fuzzy topological linear spaces [1,2]. In fact, a fuzzy norm of Katsaras's type is associated to each absolutely convex and absorbing fuzzy set. In 1992, Felbin [3] introduced another concept of fuzzy norm defined on a vector space by putting in correspondence to each element of the linear space, a fuzzy real number. Inspired by Cheng and Mordeson [4], in 2003, Bag and Samanta [5] defined a more suitable notion of fuzzy norm, even if it could be more refined, made simpler or even made more general (see [6–10]).

In this context, there are two concepts of boundedness, one of them introduced by Bag and Samanta [5] and the other one introduced by Sadeqi and Kia [11] in 2009. On the other hand, as any fuzzy norm induces naturally a fuzzy metric, for studying boundedness we can also use the notion of F-bounded introduced by George and Veeramani for fuzzy metric spaces (see [12]). The notion of fuzzy totally bounded set was first dealt with by Sadeqi and Kia [11]. Numerous applications have emerged from fuzzy sets theory. To name a few recent ones, we would refer to where the fuzzy set theory merged with chaos theory [13]. This approach may potentially improve some recent results in chaos theory application, e.g., designing chaotic sensors, see [14]. Also, applications of fuzzy set theory may be considered within the actual scope of neuroscience like in [15].

In this paper we emphasize different properties of such bounded sets. Moreover, we will make a comparative study among these concepts of boundedness. We establish the implications between them and we illustrate by examples that these concepts are not similar. Our context is very general because we work with fuzzy normed linear space of type $(X, N, *)$, where $*$ is an arbitrary t-norm, as they were considered by Nădăban and Dzitac in paper [9].

Structurally, the paper comprises the following: we begin with the preliminary section, then, in Section 2, we study fuzzy bounded sets. This concept of boundedness corresponds to the classical boundedness, as it is shown in Theorem 4. In Section 3, we present bounded sets. We prove that the union and the sum of two bounded sets are also bounded and so is the closure of a bounded set. We characterize the boundedness of a set of Cartesian product of FNL spaces. F-bounded sets are considered in Section 4 and in the next section we present different properties of fuzzy totally bounded sets. We highlight that in Theorem 6 it is proved that any compact set is fuzzy totally bounded. The last section is very important. In Theorem 8 we obtain the implications between these types of boundedness. In Theorem 10 is presented an example of a F-bounded set which is not fuzzy bounded. Finally, in Proposition 12 is given an example of a fuzzy bounded set that is not fuzzy totally bounded.

2. Preliminaries

Definition 1 ([16]). A binary operation

$$* : [0, 1] \times [0, 1] \rightarrow [0, 1]$$

is called triangular norm (*t*-norm) if it satisfies the following condition:

1. $a * b = b * a, (\forall) a, b \in [0, 1];$
2. $a * 1 = a, (\forall) a \in [0, 1];$
3. $(a * b) * c = a * (b * c), (\forall) a, b, c \in [0, 1];$
4. If $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$, then $a * b \leq c * d$.

Remark 1. Three basic examples of continuous *t*-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$ (the minimum *t*-norm), $a \cdot b = ab$ (usual multiplication in $[0, 1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz *t*-norm). Our basic reference for fuzzy metric spaces and related structures is [17], while for *t*-norms, is [18].

Definition 2 ([18]). A *t*-norm $*$ is strictly monotonic if

$$(\forall) x \in (0, 1), y < z \Rightarrow x * y < x * z.$$

A *t*-norm is strict if it is continuous and strictly monotonic.

Remark 2. We note that the usual multiplication $\cdot$ is a strict *t*-norm but the minimum *t*-norm $\wedge$ is continuous but not strictly monotonic. This remark leads us to the following more general definition.

Definition 3. A *t*-norm is called almost strictly monotonic if

$$(\forall) x, y \in (0, 1) \Rightarrow x * y > 0.$$

A *t*-norm is called almost strict if it is continuous and almost strictly monotonic.

Remark 3. The usual multiplication $\cdot$ and the minimum *t*-norm $\wedge$ are almost strict.

Definition 4 ([19]). The triple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous *t*-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ satisfying the following conditions:

- (M1) $M(x, y, 0) = 0, (\forall) x, y \in X;$
- (M2) $(\forall) x, y \in X, x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0;$
- (M3) $M(x, y, t) = M(y, x, t), (\forall) x, y \in X, (\forall) t > 0;$
- (M4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), (\forall) x, y, z \in X, (\forall) t, s > 0;$
- (M5) $(\forall) x, y \in X, M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous and $\lim_{t \rightarrow \infty} M(x, y, t) = 1.$

Definition 5 ([12]). Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be F -bounded if

$$(\exists)\alpha \in (0, 1), (\exists)t > 0 \text{ such that } M(x, y, t) > 1 - \alpha, (\forall)x, y \in A.$$

Definition 6 ([9]). Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t -norm. A fuzzy set N in $X \times [0, \infty)$ is called a fuzzy norm on X if it satisfies:

- (N1) $N(x, 0) = 0, (\forall)x \in X$;
- (N2) $[N(x, t) = 1, (\forall)t > 0]$ if and only if $x = 0$;
- (N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;
- (N4) $N(x + y, t + s) \geq N(x, t) * N(y, s), (\forall)x, y \in X, (\forall)t, s \geq 0$;
- (N5) $(\forall)x \in X, N(x, \cdot)$ is left continuous and $\lim_{t \rightarrow \infty} N(x, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNL space).

Example 1 ([20]). Let $(X, \|\cdot\|)$ be a normed linear space. Let $N : X \times [0, \infty) \rightarrow [0, 1]$ defined by

$$N(x, t) = \begin{cases} \frac{t}{t + \|x\|} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Then $(X, N, \wedge)$ is a FNL space.

Theorem 1 ([9]). Let $(X, N, *)$ be a FNL space.

- We define $M : X \times X \times [0, \infty) \rightarrow [0, 1]$ by $M(x, y, t) = N(x - y, t)$. Then M is a fuzzy metric on X .
- For $x \in X, \alpha \in (0, 1), t > 0$ we define the open ball

$$B(x, \alpha, t) := \{y \in X : N(x - y, t) > 1 - \alpha\}.$$

Then

$$\mathcal{T}_N := \{T \subset X : x \in T \text{ iff } (\exists)t > 0, (\exists)\alpha \in (0, 1) : B(x, \alpha, t) \subseteq T\}$$

is a topology on X and $(X, \mathcal{T}_N)$ is a metrizable topological vector space.

Recall [21] that considering $(X_1, N_1, *)$, $(X_2, N_2, *)$ two FNL spaces, the application

$$N : X_1 \times X_2 \times [0, \infty) \longrightarrow [0, 1]$$

$$N((x_1, x_2), t) = N_1(x_1, t) * N_2(x_2, t), (\forall)(x_1, x_2) \in X_1 \times X_2, (\forall)t > 0$$

is a fuzzy norm on the Cartesian product $X_1 \times X_2$, named the fuzzy product norm.

We denote by pr_i the projection function from $X_1 \times X_2$ onto X_i , defined by $pr_i(x_1, x_2) = x_i$ for $i \in \{1, 2\}$.

The next result deals with the Cartesian product of fuzzy normed linear spaces.

Theorem 2. Let $(X_1, N_1, *)$, $(X_2, N_2, *)$ be FNL spaces with the topologies $\mathcal{T}_{N_1}$ and $\mathcal{T}_{N_2}$, respectively. If N is the fuzzy product norm, then $\mathcal{T}_N$ is the product topology on $X_1 \times X_2$.

Proof. We first prove that $(\forall)r \in (0, 1), (\forall)t > 0, (\exists)r_1, r_2 \in (0, 1), (\exists)t_1 > 0, t_2 > 0$ such that $B(0, r_1, t_1) \times B(0, r_2, t_2) \subset B((0, 0), r, t)$. Consider $r \in (0, 1), t > 0$. From Lemma 3.6 [22], $(\exists)r_1, r_2 \in (0, 1)$ such that $1 - \frac{r}{2} = (1 - r_1) * (1 - r_2)$. By taking $t_1 = t_2 = t > 0, (\forall)(x_1, x_2) \in B(0, r_1, t_1) \times B(0, r_2, t_2)$, we obtain $N((x_1, x_2), t) = N_1(x_1, t) * N_2(x_2, t) \geq (1 - r_1) * (1 - r_2) = 1 - \frac{r}{2} > 1 - r$.

Conversely, we have to prove that $(\forall)r_1, r_2 \in (0, 1), (\forall)t_1 > 0, t_2 > 0 (\exists)r \in (0, 1), (\exists)t > 0$ such that $B((0, 0), r, t) \subset B(0, r_1, t_1) \times B(0, r_2, t_2)$. Consider $r_1, r_2 \in (0, 1), t_1 > 0, t_2 > 0$. Then for

$r = \min(r_1, r_2) \in (0, 1)$ and $t = \min(t_1, t_2) > 0$, $(\forall)(x_1, x_2) \in B((0, 0), r, t)$, we get $x_1 \in B(0, r_1, t_1)$ and $x_2 \in B(0, r_2, t_2)$. Indeed from $N_1(x_1, t) * N_2(x_2, t) > (1 - r) \geq 1 - r_i, i \in \{1, 2\}$ it results $N_1(x_1, t) > 1 - r_1$ (otherwise $N_1(x_1, t) * N_2(x_2, t) \leq (1 - r_1) * 1 = 1 - r_1$) and analogously $N_2(x_2, t) > 1 - r_2$, hence $N_1(x_1, t_1) \geq N_1(x_1, t) > 1 - r_1$ and $N_2(x_2, t_2) \geq N_2(x_2, t) > 1 - r_2$. $\square$

Definition 7 ([5]). Let $(X, N, *)$ be a FNL space and (x_n) be a sequence in X . The sequence (x_n) is said to be convergent if $(\exists)x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1, (\forall)t > 0$. In this case, x is called the limit of the sequence (x_n) and we denote $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$.

Definition 8 ([5]). Let $(X, N, *)$ be a FNL space. A subset B of X is called the closure of the subset A of X if for any $x \in B$, $(\exists)(x_n) \subset A$ such that $x_n \rightarrow x$. We denote the set B by $\overline{A}$.

A subset A of X is called closed if $A = \overline{A}$.

Remark 4. As any FNL space is a fuzzy metric space, the notion of F-bounded set can be used in the context of FNL spaces. More precisely, a subset A of a FNL space X will be called F-bounded if

$$(\exists)\alpha \in (0, 1), (\exists)t > 0 \text{ such that } N(x - y, t) > 1 - \alpha, (\forall)x, y \in A.$$

We will denote by $FB(X)$ the family of all F-bounded subset of X .

Definition 9 ([5]). A subset A of a FNL space X is said to be bounded if

$$(\exists)\alpha \in (0, 1), (\exists)t > 0 \text{ such that } N(x, t) > 1 - \alpha, (\forall)x \in A.$$

We will denote by $B(X)$ the family of all bounded subset of X .

Definition 10 ([11]). A subset A of a FNL space X is called fuzzy bounded if

$$(\forall)\alpha \in (0, 1), (\exists)t_\alpha > 0 \text{ such that } N(x, t_\alpha) > 1 - \alpha, (\forall)x \in A.$$

We will denote by $fB(X)$ the family of all fuzzy bounded subset of X .

Definition 11 ([11]). A subset A of a FNL space X is called fuzzy totally bounded if

$$(\forall)\alpha \in (0, 1), (\exists)\{x_1, x_2, \dots, x_n\} \subset X : A \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, \alpha)).$$

We will denote by $ftB(X)$ the family of all fuzzy totally bounded subsets of X .

3. Fuzzy Bounded Sets

One might think by looking at the concepts of boundedness presented above, that there is still one concept missing, namely the one in which the boundedness of a set A is defined as follows:

$$(\forall)\alpha \in (0, 1), (\exists)t_\alpha > 0 \text{ such that } N(x - y, t_\alpha) > 1 - \alpha, (\forall)x, y \in A.$$

In fact it is not missing because it coincides with one above, as the following theorem shows.

Theorem 3. Let $(X, N, *)$ be a FNL space. A subset A of X is fuzzy bounded if and only if

$$(\forall)\alpha \in (0, 1), (\exists)t_\alpha > 0 \text{ such that } N(x - y, t_\alpha) > 1 - \alpha, (\forall)x, y \in A.$$

Proof. “ $\Rightarrow$ ” Let

$\alpha \in (0, 1)$. Then there exists $\beta \in (0, 1)$ such that $(1 - \beta) * (1 - \beta) > 1 - \alpha$. Since A is fuzzy bounded, for $\beta \in (0, 1)$ there exists $t_\beta > 0$ such that $N(x, t_\beta) > 1 - \beta, (\forall)x \in A$.

Let $x, y \in A$ and $t_\alpha = 2t_\beta$. We have that

$$N(x - y, t_\alpha) \geq N(x, t_\beta) * N(y, t_\beta) \geq (1 - \beta) * (1 - \beta) > 1 - \alpha.$$

" $\Leftarrow$ " Let $\alpha \in (0, 1)$. Using Lemma 3.6 [22] we obtain that there exist $\gamma, \delta \in (0, 1)$ such that $1 - \frac{\alpha}{2} = (1 - \gamma) * \delta$.

Let $x_0 \in A$ be fixed. As $\lim_{t \rightarrow \infty} N(x_0, t) = 1$, we have that there exists $t_1 > 0$ such that $N(x_0, t_1) > \delta$. From our hypothesis, for $\gamma \in (0, 1)$ there exists $t_2 > 0$ such that $N(x - x_0, t_2) > 1 - \gamma, (\forall)x \in A$. Let $t = t_1 + t_2$. Then, for all $x \in A$, we have

$$N(x, t) \geq N(x - x_0, t_2) * N(x_0, t_1) \geq (1 - \gamma) * \delta = 1 - \frac{\alpha}{2} > 1 - \alpha.$$

□

Remark 5. One can observe that a subset A of a topological linear space X is called bounded if for each neighbourhood V of 0_X , there exists a positive number k such that $A \subseteq kV$.

Theorem 4. Let $(X, N, *)$ be a FNL space. A subset A of X is fuzzy bounded if and only if A is bounded in topology $\mathcal{T}_N$.

Proof. " $\Rightarrow$ " Let V be a neighbourhood of 0_X . Then there exist $\alpha \in (0, 1), t > 0$ such that $B(0, \alpha, t) \subseteq V$. Since A is fuzzy bounded, for $\alpha \in (0, 1), (\exists)t_\alpha > 0$ such that $N(x, t_\alpha) > 1 - \alpha, (\forall)x \in A$. Let $k = \frac{t_\alpha}{t}$. We have that $N(x, tk) = N(x, t_\alpha) > 1 - \alpha, (\forall)x \in A$. Thus

$$A \subset B(0, \alpha, tk) = kB(0, \alpha, t) \subseteq kV.$$

" $\Leftarrow$ " Let $\alpha \in (0, 1)$. Since $B(0, \alpha, 1)$ is a neighbourhood of 0_X , there exists $k > 0$ such that $A \subseteq kB(0, \alpha, 1) = B(0, \alpha, k)$. Thus $N(x, k) > 1 - \alpha, (\forall)x \in A$. Hence A is fuzzy bounded. □

Remark 6. Previous result was mentioned by Sadeqi and Kia (see [11]) in the context of FNL spaces of type $(X, N, \wedge)$.

Corollary 1 ([23]). Let $(X, N, *)$ be a FNL space. Then:

1. If A, B are fuzzy bounded, then $A \cup B$ and $A + B$ are fuzzy bounded;
2. If A is fuzzy bounded, then $\overline{A}$ is fuzzy bounded.

Corollary 2. Let $(X_i, N_i, *), i \in \{1, 2\}$, be two FNL spaces. Then $A \in fB(X_1 \times X_2)$ if and only if $pr_i(A) \in fB(X_i), i \in \{1, 2\}$.

Proposition 1. Let $(X, N, *)$ be a FNL space and $\{A_n\}_{n=1}^\infty$ be fuzzy bounded subsets of X . Then there exist $\{t_n\}_{n=1}^\infty, t_n > 0, (\forall)n \in \mathbb{N}^*$ such that $\bigcup_{n=1}^\infty t_n A_n$ is a fuzzy bounded subset of X .

Proof. Let ρ be the metric of X . Let $\alpha \in (0, 1)$ and $t > 0$. Since $B(0, \alpha, t)$ is a neighbourhood of 0_X , there exists $\lambda > 0$ such that

$$S(\lambda) = \{x \in X : \rho(x, 0) < \lambda\} \subset B(0, \alpha, t).$$

As $\{A_n\}_{n=1}^{\infty}$ are fuzzy bounded subsets of X , there exists $s_n > 0$ such that $A_n \subseteq s_n S(1)$. Let $t_n = \frac{1}{s_n}$. Then

$$\bigcup_{n=1}^{\infty} t_n A_n \subseteq S(1) = \lambda^{-1} S(\lambda) \subseteq \lambda^{-1} B(0, \alpha, t) = B\left(0, \alpha, \frac{t}{\lambda}\right).$$

□

4. Bounded Sets

Proposition 2. Let $(X, N, *)$ be a FNL space and A_1, A_2 be two bounded subsets of X . Then $A_1 \cup A_2$ is bounded.

Proof. Since A_1, A_2 are bounded subsets of X , there exist $\alpha_1, \alpha_2 \in (0, 1), t_1, t_2 > 0$ such that $N(x, t_1) > 1 - \alpha_1, (\forall) x \in A_1$ and $N(x, t_2) > 1 - \alpha_2, (\forall) x \in A_2$. Let $t = \max\{t_1, t_2\}$ and $\alpha = \max\{\alpha_1, \alpha_2\}$. Let $x \in A_1 \cup A_2$. If $x \in A_1$, then $N(x, t) \geq N(x, t_1) > 1 - \alpha_1 \geq 1 - \alpha$. Similarly, if $x \in A_2$, we obtain that $N(x, t) \geq N(x, t_2) > 1 - \alpha_2 \geq 1 - \alpha$. Thus $N(x, t) > 1 - \alpha, (\forall) x \in A_1 \cup A_2$. □

Proposition 3. Let $(X, N, *)$ be a FNL space, where $*$ is almost strict. If A_1, A_2 are two bounded subsets of X , then $A_1 + A_2$ is a bounded subset of X .

Proof. Since A_1, A_2 are bounded subsets of X , there exist $\alpha_1, \alpha_2 \in (0, 1), t_1, t_2 > 0$ such that $N(x, t_1) > 1 - \alpha_1, (\forall) x \in A_1$ and $N(x, t_2) > 1 - \alpha_2, (\forall) x \in A_2$. Let $\alpha \in (0, 1)$ such that $\alpha > 1 - (1 - \alpha_1) * (1 - \alpha_2)$ and $t = t_1 + t_2$. Let $z \in A_1 + A_2$. Then there exist $x \in A_1, y \in A_2$ such that $z = x + y$. We have that

$$N(z, t) = N(x + y, t_1 + t_2) \geq N(x, t_1) * N(y, t_2) \geq (1 - \alpha_1) * (1 - \alpha_2) > 1 - \alpha.$$

□

Proposition 4. Let $(X, N, *)$ be a FNL space and $A \in B(X)$. Then $\overline{A} \in B(X)$.

Proof. As A is bounded we have that there exist $\alpha_0 \in (0, 1), t_0 > 0$ such that

$$N(x, t_0) > 1 - \alpha_0, (\forall) x \in A.$$

Let $\alpha_1 \in (0, 1)$ such that $(1 - \alpha_0) * (1 - \alpha_1) > 0$ and $\alpha \in (0, 1)$ such that $1 - \alpha < (1 - \alpha_0) * (1 - \alpha_1)$. Let $t_1 > 0$ and $x \in \overline{A}$. Thus $(\exists)(x_n) \subset A$ such that $x_n \rightarrow x$. Hence $\lim_{n \rightarrow \infty} N(x_n - x, t_1) = 1$. Thus there exists $n_0 \in \mathbb{N}$ such that $N(x_n - x, t_1) > 1 - \alpha_1, (\forall) n \geq n_0$. Therefore, for $n \geq n_0$, we have that

$$N(x, t_0 + t_1) = N(x - x_n + x_n, t_0 + t_1) \geq N(x - x_n, t_1) * N(x_n, t_0) \geq (1 - \alpha_1) * (1 - \alpha_0) > 1 - \alpha.$$

Hence $\overline{A}$ is bounded. □

Proposition 5. Let $(X_i, N_i, *), i \in \{1, 2\}$, be two FNL spaces where “*” is almost strict and let N be the fuzzy product norm. Then $A \in B(X_1 \times X_2)$ if and only if $pr_i(A) \in B(X_i), i \in \{1, 2\}$.

Proof. Let $A \in B(X_1 \times X_2)$. Then there exist $\alpha \in (0, 1), t > 0$ such that $N((x_1, x_2), t) > 1 - \alpha, (\forall) (x_1, x_2) \in A$. Following the proof of Theorem 2, it results that $N_1(x_1, t) > 1 - \alpha, (\forall) x_1 \in pr_1(A)$ and $N_2(x_2, t) > 1 - \alpha, (\forall) x_2 \in pr_2(A)$. Therefore $pr_i(A) \in B(X_i), i \in \{1, 2\}$.

Conversely, since $pr_i(A) \in B(X_i), i \in \{1, 2\}$, there exist $\alpha_1, \alpha_2 \in (0, 1), t_1 > 0, t_2 > 0$ such that $N_1(x_1, t_1) > 1 - \alpha_1, N_2(x_2, t_2) > 1 - \alpha_2, (\forall) x_1 \in pr_1(A), (\forall) x_2 \in pr_2(A)$. Since “*” is almost strict there exist $\alpha \in (0, 1)$ such that $(1 - \alpha_0) * (1 - \alpha_1) > 1 - \alpha$. Consider $t = \max\{t_1, t_2\}$. Then $N((x_1, x_2), t) = N_1(x_1, t) * N_2(x_2, t) \geq N_1(x_1, t_1) * N_2(x_2, t_2) \geq (1 - \alpha_0) * (1 - \alpha_1) > 1 - \alpha$, hence $A \subset pr_1(A) \times pr_2(A) \in B(X_1 \times X_2)$. It follows $A \in B(X_1 \times X_2)$. □

5. F-Bounded Sets

Proposition 6. Let $(X, N, *)$ be a FNL space and $A \in FB(X)$. Then $\overline{A} \in FB(X)$.

Proof. Since $A \in FB(X)$, there exist $\alpha_0 \in (0, 1)$, $t_0 > 0$ such that $N(x - y, t_0) > 1 - \alpha_0$, $(\forall)x, y \in A$. Let $x, y \in \overline{A}$. Then there exist $(x_n), (y_n) \subset A$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Let $\beta \in (0, 1)$, $\beta > \alpha_0$ and $s = 3t_0$. We have that

$$N(x - y, s) \geq N(x - x_n, t_0) * N(x_n - y_n, t_0) * N(y_n - y, t_0) \geq N(x - x_n, t_0) * (1 - \alpha_0) * N(y_n - y, t_0).$$

For $n \rightarrow \infty$ we obtain that

$$N(x - y, s) \geq 1 - \alpha_0 > 1 - \beta.$$

Thus $\overline{A} \in FB(X)$. $\square$

Proposition 7. Let $(X, N, *)$ be a FNL space. If $A \subseteq X$ satisfies

$$(\exists)\alpha_0 \in (0, 1) : \sup\{t \geq 0 : N(x - y, t) \leq 1 - t\} < \alpha_0, (\forall)x, y \in A,$$

then A is F-bounded.

Proof. For $x, y \in A$, let $d(x, y) = \sup\{t \geq 0 : N(x - y, t) \leq 1 - t\}$. By our hypothesis $(\exists)\alpha_0 \in (0, 1)$ such that $d(x, y) < \alpha_0$, $(\forall)x, y \in A$. Thus $N(x - y, \alpha_0) > 1 - \alpha_0$, $(\forall)x, y \in A$. Hence A is F-bounded. $\square$

6. Fuzzy Totally Bounded Sets

Theorem 5. Let $(X, N, *)$ be a FNL space. The following statements are equivalent:

1. A is fuzzy totally bounded;
2. $(\forall)\alpha \in (0, 1), (\exists)\{x_1, x_2, \dots, x_n\} \subset A : A \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, \alpha))$;
3. $(\forall)\alpha \in (0, 1), (\forall)t > 0, (\exists)\{x_1, x_2, \dots, x_n\} \subset A : A \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, t))$;
4. $(\forall)\alpha \in (0, 1), (\forall)t > 0, (\exists)\{x_1, x_2, \dots, x_n\} \subset X : A \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, t))$.

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in (0, 1)$. Then there exists $n \in \mathbb{N}, n \geq 2$ such that

$$\left(1 - \frac{\alpha}{n}\right) * \left(1 - \frac{\alpha}{n}\right) > 1 - \alpha.$$

Indeed, if we suppose that

$$\left(1 - \frac{\alpha}{n}\right) * \left(1 - \frac{\alpha}{n}\right) \leq 1 - \alpha, (\forall)n \in \mathbb{N}, n \geq 2,$$

by passing to the limit, for $n \rightarrow \infty$, we obtain that $1 * 1 \leq 1 - \alpha$, which is a contradiction.

As A is fuzzy totally bounded,

$$(\exists)\{x_1, x_2, \dots, x_m\} \subset X : A \subset \bigcup_{i=1}^m \left(x_i + B\left(0, \frac{\alpha}{n}, \frac{\alpha}{n}\right)\right).$$

Let $y_i \in A \cap (x_i + B(0, \frac{\alpha}{n}, \frac{\alpha}{n}))$, $i = \overline{1, m}$. We show that $A \subset \bigcup_{i=1}^m (y_i + B(0, \alpha, \alpha))$. Let $x \in A$. Then there exists $k \in \{1, \dots, m\}$ such that $x \in x_k + B(0, \frac{\alpha}{n}, \frac{\alpha}{n})$, namely $N(x - x_k, \frac{\alpha}{n}) > 1 - \frac{\alpha}{n}$. We have that

$$\begin{aligned} N(x - y_k, \alpha) &\geq N\left(x - y_k, \frac{\alpha}{n} + \frac{\alpha}{n}\right) \geq N\left(x - x_k, \frac{\alpha}{n}\right) * N\left(x_k - y_k, \frac{\alpha}{n}\right) \geq \\ &\geq \left(1 - \frac{\alpha}{n}\right) * \left(1 - \frac{\alpha}{n}\right) > 1 - \alpha. \end{aligned}$$

Thus $x \in y_k + B(0, \alpha, \alpha)$.

(2) $\Rightarrow$ (3). Let $\alpha \in (0, 1)$ and $t > 0$. Let $\beta = \min\{\alpha, t\}$. By our hypothesis,

$$(\exists)\{x_1, x_2, \dots, x_n\} \subset A : A \subset \bigcup_{i=1}^n (x_i + B(0, \beta, \beta)) \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, t)).$$

(3) $\Rightarrow$ (4). It is obviously.

(4) $\Rightarrow$ (1). Let $\alpha \in (0, 1)$. For $t = \alpha$, by our hypothesis, $(\exists)\{x_1, x_2, \dots, x_n\} \subset X$ such that $A \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, \alpha))$. $\square$

Proposition 8. Let $(X, N, *)$ be a FNL space. If A, B are fuzzy totally bounded subsets of X , then $A + B$ and $A \cup B$ are fuzzy totally bounded.

Proof. Let $\alpha \in (0, 1)$ and $t > 0$. Then there exist $\alpha_1, \alpha_2 \in (0, 1)$ such that $(1 - \alpha_1) * (1 - \alpha_2) > 1 - \alpha$. Let $t_1 = t_2 = \frac{t}{2}$. Then

$$B(0, \alpha_1, t_1) + B(0, \alpha_2, t_2) \subset B(0, \alpha, t).$$

Indeed, if $x \in B(0, \alpha_1, t_1)$ and $y \in B(0, \alpha_2, t_2)$, then $N(x, \frac{t}{2}) > 1 - \alpha_1$ and $N(y, \frac{t}{2}) > 1 - \alpha_2$. Thus

$$N(x + y, t) \geq N\left(x, \frac{t}{2}\right) * N\left(y, \frac{t}{2}\right) \geq (1 - \alpha_1) * (1 - \alpha_2) > 1 - \alpha.$$

Hence $x + y \in B(0, \alpha, t)$. If A, B are fuzzy totally bounded, then there exist $\{x_1, x_2, \dots, x_n\} \subset A$ and $\{y_1, y_2, \dots, y_m\} \subset B$ such that $A \subset \bigcup_{i=1}^n (x_i + B(0, \alpha_1, t_1))$ and $B \subset \bigcup_{k=1}^m (y_k + B(0, \alpha_2, t_2))$. Therefore

$$A + B \subset \bigcup_{i=1}^n \bigcup_{k=1}^m (x_i + y_k + B(0, \alpha_1, t_1) + B(0, \alpha_2, t_2)) \subset \bigcup_{i=1}^n \bigcup_{k=1}^m (x_i + y_k + B(0, \alpha, t)).$$

Hence $A + B$ is fuzzy totally bounded.

Let now $\alpha \in (0, 1)$. As A, B are fuzzy totally bounded, there exist $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\} \subset X$ such that $A \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, \alpha))$ and $B \subset \bigcup_{k=1}^m (y_k + B(0, \alpha, \alpha))$. Thus

$$A \cup B \subset \left(\bigcup_{i=1}^n (x_i + B(0, \alpha, \alpha)) \right) \cup \left(\bigcup_{k=1}^m (y_k + B(0, \alpha, \alpha)) \right).$$

Hence $A \cup B$ is fuzzy totally bounded. $\square$

Lemma 1. Let $\alpha, \beta \in (0, 1)$ such that $\beta < \alpha$. Then $\overline{B(0, \beta, \beta)} \subset B(0, \alpha, \alpha)$.

Proof. If $x \in \overline{B(0, \beta, \beta)}$, then there exists $(x_n) \subset B(0, \beta, \beta)$ such that $x_n \rightarrow x$, namely $N(x_n, \beta) > 1 - \beta$ and $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1, (\forall) t > 0$.

Let $\gamma \in (0, 1)$ such that $(1 - \beta) * (1 - \gamma) > 1 - \alpha$. The existence of γ results by the continuity of the mapping $g : [0, 1] \rightarrow [0, 1], g(y) = (1 - \beta) * y$. Indeed, for $\alpha_1 \in (0, 1) : \beta < \alpha_1 < \alpha$, as $g(0) = 0, g(1) = 1 - \beta$ and $0 < 1 - \alpha_1 < 1 - \beta$, there exists $\gamma \in (0, 1)$ such that $g(1 - \gamma) = 1 - \alpha_1$, namely $(1 - \beta) * (1 - \gamma) = 1 - \alpha_1 > 1 - \alpha$.

Finally, for $t > 0$ such that $\alpha = \beta + t$, as $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$ there exists $n_0 \in \mathbb{N}^*$ such that $N(x - x_n, t) > 1 - \gamma, (\forall) n \geq n_0$. Thus

$$N(x, \alpha) = N(x - x_n + x_n, \beta + t) \geq N(x - x_n, t) * N(x_n, \beta) \geq (1 - \gamma) * (1 - \beta) > 1 - \alpha.$$

Hence $x \in B(0, \alpha, \alpha)$. $\square$

Proposition 9. Let $(X, N, *)$ be a FNL space. If A is fuzzy totally bounded, then $\overline{A}$ is fuzzy totally bounded.

Proof. Let $\alpha \in (0, 1)$. Let $\beta < \alpha$. As A is fuzzy totally bounded, $(\exists)\{x_1, x_2, \dots, x_n\} \subset X$ such that $A \subset \bigcup_{i=1}^n (x_i + B(0, \beta, \beta))$. Thus, by Lemma 1, it follows

$$\overline{A} \subset \bigcup_{i=1}^n (x_i + \overline{B(0, \beta, \beta)}) \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, \alpha)).$$

Hence $\overline{A}$ is fuzzy totally bounded. $\square$

Proposition 10. Let $(X, \|\cdot\|)$ be a normed linear space and let $(X, N, *)$ be the FNL space defined by

$$N(x, t) = \begin{cases} \frac{t}{t + \|x\|} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases},$$

where $*$ is an arbitrary t -norm. Then $\mathcal{T}_N$ coincides with the norm topology on X . Moreover:

1. A set M is bounded in $(X, \|\cdot\|)$ if and only if M is fuzzy bounded in $(X, N, *)$;
2. A set M is totally bounded in $(X, \|\cdot\|)$ if and only if M is fuzzy totally bounded in $(X, N, *)$.

Proof. Let $\epsilon > 0$ and $B(\epsilon) := \{x \in X : \|x\| < \epsilon\}$. We show that there exist $\alpha_0 \in (0, 1)$ and $t_0 > 0$ such that $B(0, \alpha_0, t_0) \subseteq B(\epsilon)$. Let $\alpha_0 = \frac{1}{2}$ and $t_0 = \frac{\epsilon}{2} > 0$. Let $x \in B(0, \alpha_0, t_0)$. Then

$$N(x, t_0) > 1 - \alpha_0 \Rightarrow \frac{\frac{\epsilon}{2}}{\frac{\epsilon}{2} + \|x\|} > \frac{1}{2} \Rightarrow \|x\| < \frac{\epsilon}{2}.$$

Thus $x \in B(\epsilon)$.

Conversely, let $\alpha \in (0, 1), t_0 > 0$. We show that there exists $\epsilon > 0$ such that $B(\epsilon) \subseteq B(0, \alpha, t)$. Let $\epsilon = \frac{\alpha t}{1 - \alpha} > 0$ and $x \in B(\epsilon)$. Thus $\|x\| < \frac{\alpha t}{1 - \alpha}$. Hence

$$N(x, t) = \frac{t}{t + \|x\|} > \frac{t}{t + \frac{\alpha t}{1 - \alpha}} = 1 - \alpha.$$

Therefore $x \in B(0, \alpha, t)$. $\square$

Theorem 6. Let $(X, N, *)$ be a FNL space and $K \subset X$ be a compact set in $(X, \mathcal{T}_N)$. Then K is fuzzy totally bounded.

Proof. Let $\alpha \in (0, 1)$ and $t > 0$. As $K \subset \bigcup_{x \in K} (x + B(0, \alpha, t))$ and K is compact, $(\exists)\{x_1, x_2, \dots, x_n\} \subset K$ such that $K \subset \bigcup_{i=1}^n (x_i + B(0, \alpha, t))$. By Theorem 5 we obtain that K is fuzzy totally bounded. $\square$

Theorem 7. Let $(X_i, N_i, *)$, $i \in \{1, 2\}$ and let N be the fuzzy product norm. Then $A \in ftB(X_1 \times X_2)$ if and only if $pr_i(A) \in ftB(X_i)$, $i \in \{1, 2\}$.

Proof. Let $A \in ftB(X_1 \times X_2)$, $\alpha_i \in (0, 1)$, $t_i > 0$, $i \in \{1, 2\}$. Since pr_i is continuous from $(X_1 \times X_2, \mathcal{T}_N)$ onto $(X_i, \mathcal{T}_{N_i})$, $i \in \{1, 2\}$, it results the inverse image of $B(0, \alpha_i, t_i)$ through pr_i , $pr_i^{-1}(B(0, \alpha_i, t_i))$ is a neighbourhood of $(0, 0)$ in $(X_1 \times X_2, \mathcal{T}_N)$. Therefore, there exist $\alpha \in (0, 1)$ and $t > 0$ such that $B(0, \alpha, t) \subset pr_i^{-1}(B(0, \alpha_i, t_i))$, $i \in \{1, 2\}$. By Theorem 5 it follows that there exist $y_1, y_2, \dots, y_n \in X_1 \times X_2$ such that $A \subset \bigcup_{j=1}^n (y_j + B(0, \alpha, t))$ whence $pr_i(A) \subset pr_i(\bigcup_{j=1}^n (y_j + pr_i^{-1}(B(0, \alpha_i, t_i)))) \subset \bigcup_{j=1}^n (pr_i(y_j) + B(0, \alpha_i, t_i))$, $i \in \{1, 2\}$. Therefore $pr_i(A) \in ftB(X_i)$, $i \in \{1, 2\}$. Conversely, suppose that $pr_i(A) \in ftB(X_i)$, $i \in \{1, 2\}$. Let $\alpha \in (0, 1)$ and $t > 0$. By Theorem 2, it results that there exist $\alpha_i \in (0, 1)$, $t_i > 0$, $i \in \{1, 2\}$ such that $B(0, \alpha_1, t_1) \times B(0, \alpha_2, t_2) \subset B((0, 0), \alpha, t)$. Since $pr_i(A)$, $i \in \{1, 2\}$ is fuzzy totally bounded, it follows that there exist $x_1^i, \dots, x_{n_i}^i \in pr_i(A)$, $i \in \{1, 2\}$ such that $pr_i(A) \subset \bigcup_{k=1}^{n_i} (x_k^i + B(0, \alpha_i, t_i))$, $i \in \{1, 2\}$. Thus $A \subset pr_1(A) \times pr_2(A) \subset \bigcup_{k=1}^{n_1} \bigcup_{l=1}^{n_2} ((x_k^1, x_l^2) + B(0, \alpha_1, t_1) \times B(0, \alpha_2, t_2)) \subset \bigcup_{k=1}^{n_1} \bigcup_{l=1}^{n_2} ((x_k^1, x_l^2) + B(0, \alpha, t))$, hence A is fuzzy totally bounded. $\square$

7. A Comparative Study among Different Types of Boundedness

Theorem 8. Let $(X, N, *)$ be a FNL space. We have that $ftB(X) \subseteq fB(X) \subseteq FB(X) \subseteq B(X)$.

Proof. We prove first that $ftB(X) \subseteq fB(X)$. Let $\alpha \in (0, 1)$. Then there exist $\beta, \gamma \in (0, 1)$ such that $(1 - \beta) * (1 - \gamma) > 1 - \alpha$. As A is fuzzy totally bounded, $(\exists) \{x_1, x_2, \dots, x_n\} \subset X : A \subset \bigcup_{i=1}^n (x_i + B(0, \beta, \beta))$. As $\lim_{t \rightarrow \infty} N(x_i, t) = 1$, we have that there exist $t_i > 0$ such that $N(x_i, t_i) > 1 - \gamma$. Let $t_0 = \max_{i=1, \dots, n} \{t_i\}$. Then $N(x_i, t_0) > 1 - \gamma$, $(\forall) i = \overline{1, n}$. Let $t_\alpha = 2 \max\{t_0, \beta\}$. Let $x \in A$. We show that $N(x, t_\alpha) > 1 - \alpha$. Indeed, as $x \in A$, there exists $k \in \{1, \dots, n\}$ such that $x \in x_k + B(0, \beta, \beta)$, i.e., $N(x - x_k, \beta) > 1 - \beta$. Thus

$$N(x, t_\alpha) \geq N\left(x - x_k, \frac{t_\alpha}{2}\right) * N\left(x_k, \frac{t_\alpha}{2}\right) \geq N(x - x_k, \beta) * N(x_k, t_0) \geq (1 - \beta) * (1 - \gamma) > 1 - \alpha.$$

Now we prove that $fB(X) \subseteq FB(X)$. Let A be a fuzzy bounded subset of X . Let $\alpha_1, \alpha_2 \in (0, 1)$ such that $(1 - \alpha_1) * (1 - \alpha_2) \in (0, 1)$. As A is fuzzy bounded, we have that there exist $t_1, t_2 > 0$ such that $N(x, t_1) > 1 - \alpha_1$, $(\forall) x \in A$ and $N(x, t_2) > 1 - \alpha_2$, $(\forall) x \in A$. Let $\alpha \in (0, 1)$ such that $1 - \alpha < (1 - \alpha_1) * (1 - \alpha_2)$ and $t = t_1 + t_2$. Then, for all $x, y \in A$, we have

$$N(x - y, t) \geq N(x, t_1) * N(y, t_2) \geq (1 - \alpha_1) * (1 - \alpha_2) > 1 - \alpha.$$

Finally, we prove that $FB(X) \subseteq B(X)$. Let A be a F-bounded subset of X . Then there exist $\alpha_0 \in (0, 1)$ and $t_0 > 0$ such that $N(x - y, t_0) > 1 - \alpha_0$, $(\forall) x, y \in A$. Let $y_0 \in A$ be fixed. Then there exists $t'_0 > 0$ such that $(1 - \alpha_0) * N(y_0, t'_0) > 0$. Let $\alpha_1 > 1 - (1 - \alpha_0) * N(y_0, t'_0)$ and $t_1 = t_0 + t'_0$. Then, for all $x \in A$, we have

$$N(x, t_1) \geq N(x - y_0, t_0) * N(y_0, t'_0) \geq (1 - \alpha_0) * N(y_0, t'_0) > 1 - \alpha_1.$$

$\square$

Theorem 9. Let $(X, N, *)$ be a FNL space, where $*$ is almost strict. Then $FB(X) = B(X)$.

Proof. By previous theorem we have that $FB(X) \subseteq B(X)$. For inverse inclusion, let A be a bounded subset of X . Then there exist $\alpha \in (0, 1)$ and $t > 0$ such that $N(x, t) > 1 - \alpha, (\forall) x \in A$. As $*$ is almost strict, $(\exists) \beta \in (0, 1)$ such that $\beta > 1 - (1 - \alpha) * (1 - \alpha)$. Let $x, y \in A$ and $s = 2t$. Then $N(x - y, s) \geq N(x, t) \wedge N(y, t) > (1 - \alpha) * (1 - \alpha) > 1 - \beta$. Thus $A \in FB(X)$. $\square$

Corollary 3. Let $(X, N, *)$ be a FNL space, where $*$ is almost strict and let A_1, A_2 be F-bounded subsets of X . Then $A_1 \cup A_2$ and $A_1 + A_2$ are F-bounded subsets of X .

Corollary 4. Let $(X_i, N_i, *), i \in \{1, 2\}$, be two FNL spaces where $*$ is almost strict and let N be the fuzzy product norm. Then $A \in FB(X_1 \times X_2)$ if and only if $pr_i(A) \in FB(X_i), i \in \{1, 2\}$.

Theorem 10. The inclusion $fB(X) \subseteq FB(X)$ is strict.

Proof. Let $A = (0, 2)$ and $X = \mathcal{C}(A) := \{f : A \rightarrow \mathbb{R} : f \text{ continuous}\}$. Let $\{q_\alpha\}_{\alpha \in (0, 1)}$ defined by $q_\alpha(f) = \sup_{x \in K_\alpha} |f(x)|, (\forall) f \in X$, where $K_\alpha = [1 - \alpha, 1 + \alpha]$. It is easy to show that $\{q_\alpha\}_{\alpha \in (0, 1)}$ is a sufficient and ascending family of semi-norms on the linear space X . Let $N : X \times [0, \infty) \rightarrow [0, 1]$, defined by

$$N(f, t) = \begin{cases} \sup\{\alpha \in (0, 1) : q_\alpha(f) < t\} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \text{ or } \{\alpha \in (0, 1) : q_\alpha(f) < t\} = \emptyset \end{cases}.$$

Then, by Theorem 8 of [9], we have that $(X, N, \wedge)$ is a FNL space.

It is obvious that $M = B\left(0, \frac{1}{2}, 1\right)$ is bounded in X and using previous theorem it is F-bounded. We will prove that M is not fuzzy bounded. Let $\alpha = \frac{1}{4}$. We show that $(\forall) t > 0, (\exists) f_t \in M$ such that $N(f_t, t) \leq 1 - \alpha = \frac{3}{4}$.

$$\text{Let } t > 0 \text{ and } f_t : (0, 2) \rightarrow \mathbb{R} \text{ defined by } f_t(x) = \begin{cases} 4tx & \text{if } x \in \left(0, \frac{1}{4}\right] \\ 2t - 4tx & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right] \\ 0 & \text{if } x \in \left(\frac{1}{2}, 2\right] \end{cases}.$$

It is obvious that f_t is continuous. We show that $f_t \in M$. But

$$f_t \in M \Leftrightarrow N(f_t, 1) > \frac{1}{2} \Leftrightarrow \sup\{\alpha \in (0, 1) : q_\alpha(f_t) < 1\} > \frac{1}{2}.$$

Let $\epsilon \in (0, 1/2)$ such that $f_t\left(\frac{1}{2} - \epsilon\right) < 1$. As

$$q_{\frac{1}{2} + \epsilon}(f_t) = \sup_{x \in [\frac{1}{2} - \epsilon, \frac{3}{2} + \epsilon]} |f_t(x)| = f_t\left(\frac{1}{2} - \epsilon\right) < 1,$$

we have that

$$\sup\{\alpha \in (0, 1) : q_\alpha(f_t) < 1\} \geq \frac{1}{2} + \epsilon > \frac{1}{2}.$$

Finally, we prove that $N(f_t, t) = \frac{3}{4}$. Indeed, for $\alpha \geq \frac{3}{4}$, as $\left[\frac{1}{4}, \frac{7}{4}\right] \subseteq K_\alpha$, we have that $q_\alpha(f_t) = t$. Hence $\{\alpha \in (0, 1) : q_\alpha(f_t) < t\} \subset (0, \frac{3}{4})$. On the other hand, for $\alpha \in (0, \frac{3}{4})$, as $K_\alpha \subseteq \left(\frac{1}{4}, \frac{7}{4}\right)$ we have that $q_\alpha(f_t) < t$. Hence $\{\alpha \in (0, 1) : q_\alpha(f_t) < t\} = (0, \frac{3}{4})$. Thus $N(f_t, t) = \frac{3}{4}$. $\square$

Proposition 11. Let $(X, || \cdot ||)$ be a normed linear space and $N : X \times [0, \infty) \rightarrow [0, 1]$ defined by

$$N(x, t) = \begin{cases} \frac{t}{t + ||x||} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

A subset A of the FNL space $(X, N, *)$ is bounded if and only if A is fuzzy bounded.

Proof. We have that $fB(X) \subseteq B(X)$. It remains to prove that $B(X) \subseteq fB(X)$. Let A be a bounded set. Then

$$(\exists)\alpha \in (0, 1), (\exists)t > 0 : N(x, t) > 1 - \alpha, (\forall)x \in A.$$

Thus $\frac{t}{t+||x||} > 1 - \alpha, (\forall)x \in A$. Therefore $||x|| < \frac{t\alpha}{1-\alpha}, (\forall)x \in A$. Hence A is bounded in $(X, || \cdot ||)$. This means that A is fuzzy bounded. $\square$

Proposition 12. The inclusion $ftB(X) \subseteq fB(X)$ is strict.

Proof. Let $\mathcal{C}([0, 1]) := \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}$. $\mathcal{C}([0, 1])$ with the norm $||f|| = \sup_{x \in [0, 1]} |f(x)|$ became a normed linear space. If $*$ is an arbitrary t-norm and

$$N(x, t) = \begin{cases} \frac{t}{t+||x||} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases},$$

then $(\mathcal{C}([0, 1]), N, *)$ is a FNL space.

Let $M := \{f \in \mathcal{C}([0, 1]) : ||f|| < 1\}$. As M is bounded in $(\mathcal{C}([0, 1]), || \cdot ||)$, by Proposition 10 we obtain that M is fuzzy bounded in $(\mathcal{C}([0, 1]), N, *)$. If we suppose that M is fuzzy totally bounded in $(\mathcal{C}([0, 1]), N, *)$, then M is totally bounded in $(\mathcal{C}([0, 1]), || \cdot ||)$. Thus in $\mathcal{C}([0, 1])$ we have a totally bounded set which is a neighborhood of the origin. Hence $\mathcal{C}([0, 1])$ is finitely dimensional, which is absurd. Hence M is not fuzzy totally bounded. $\square$

8. Conclusions and Further Works

In this present paper we have made a comparative study among different types of boundedness in fuzzy normed linear spaces introduced by various authors. We have established the implications between them and have illustrated by examples that these concepts are not similar.

The present study will be followed by a detailed analysis of various boundedness type for linear operators between FNL spaces and the relationship among them and with the notion of fuzzy continuity. In this approach, we are motivated and inspired by the work of Lafuerza-Guillén, Rodríguez-Lallena and Sempí in the context of probabilistic normed spaces (see [24]) and by the results already obtained by Bag and Samanta [20], by Sadeqi and Kia [11] and by Saadati and Vaezpour [10] in the context of FNL spaces.

Author Contributions: The individual contributions of the authors are as follows: “conceptualization, S.N. and T.B.; methodology, F.P.; validation, T.B., F.P. and S.N.; formal analysis, F.P.; investigation, T.B.; resources, T.B.; writing—original draft preparation, S.N. and F.P.; writing—review and editing, S.N. and F.P.; project administration, T.B.; funding acquisition, T.B. and F.P.”

Funding: This research received no funding.

Conflicts of Interest: The authors declare no conflict of interest.

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