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Note on Type 2 Degenerate q-Bernoulli Polynomials

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Abstract: The purpose of this paper is to introduce and study type 2 degenerate q-Bernoulli polynomials and numbers by virtue of the bosonic p-adic q-integrals. The obtained results are, among other things, several expressions for those polynomials, identities involving those numbers, identities regarding Carlitz's q-Bernoulli numbers, identities concerning degenerate q-Bernoulli numbers, and the representations of the fully degenerate type 2 Bernoulli numbers in terms of moments of certain random variables, created from random variables with Laplace distributions. It is expected that, as was done in the case of type 2 degenerate Bernoulli polynomials and numbers, we will be able to find some identities of symmetry for those polynomials and numbers.

Keywords: type 2 degenerate q-Bernoulli polynomials; p-adic q-integral

MSC: 11S80; 11B83

1. Introduction

There are various ways of studying special polynomials and numbers, including generating functions, combinatorial methods, umbral calculus techniques, matrix theory, probability theory, p-adic analysis, differential equations, and so on.

In [1], it was shown that odd integer power sums (alternating odd integer power sums) can be represented in terms of some values of the type 2 Bernoulli polynomials (the type 2 Euler polynomials). In addition, some identities of symmetry, involving the type 2 Bernoulli polynomials, odd integer power sums, the type 2 Euler polynomials, and alternating odd integer power sums, were obtained by introducing appropriate quotients of bosonic and fermionic p-adic integrals on $\mathbb{Z}_p$. Furthermore, in [1], it was shown that the moments of two random variables, constructed from random variables with Laplace distributions, are closely connected with the type 2 Bernoulli numbers and the type 2 Euler numbers.

In recent years, studying degenerate versions of various special polynomials and numbers, which began with the paper by Carlitz in [2], has attracted the interest of many mathematicians. For example, in [3], the degenerate type 2 Bernoulli and Euler polynomials, and their corresponding numbers were introduced and some properties of them, which include distribution relations, Witt type formulas, and analogues for the interpretation of integer power sums in terms of Bernoulli polynomials, were investigated by means of both types of p-adic integrals.

As a q-analogue of the Volkenborn integrals for uniformly differentiable functions, the bosonic p-adic q-integrals were introduced in [4] by Kim. These integrals, together with the fermionic p-adic integrals and the fermionic p-adic q-integrals, have proven to be very useful tools in studying many problems arising from number theory and combinatorics. For instance, in [5], the type 2 q-Bernoulli

(q-Euler) polynomials were introduced by virtue of the bosonic (fermionic) p-adic q-integrals. Then, it was noted, among other things, that the odd q-integer (alternating odd q-integer) power sums are expressed in terms of the type 2 q-Bernoulli (q-Euler) polynomials.

In this short paper, we would like to introduce the type 2 degenerate q-Bernoulli polynomials and the corresponding numbers by making use of the bosonic p-adic q-integrals, as a degenerate version of and also as a q-analogue of the type 2 Bernoulli polynomials, and derive several basic results for them. The obtained results are several expressions for those polynomials, identities involving those numbers, identities regarding Carlitz's q-Bernoulli numbers, identities concerning degenerate q-Bernoulli numbers, and the representations of the fully degenerate type 2 Bernoulli numbers ($q = 1$ and $x = 1$ cases of the type 2 degenerate q-Bernoulli polynomials) in terms of moments of certain random variables, created from random variables with Laplace distributions.

The motivation for introducing the type 2 degenerate q-Bernoulli polynomials and numbers is to study their number-theoretic and combinatorial properties, and their applications in mathematics and other sciences in general. One novelty of this paper is that they arise naturally by means of the bosonic p-adic q-integrals so that it is possible to easily find some identities of symmetry for those polynomials and numbers, as it was done, for example, in [1]. In the rest of this section, we recall what is needed in the latter part of the paper.

Throughout this paper, p is a fixed odd prime number. We use the standard notations $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ to denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm on $\mathbb{C}_p$ is normalized as $|p|_p = \frac{1}{p}$.

As is well known, the Bernoulli numbers are given by the recurrence relation

$$B_0 = 1, \quad (B+1)^n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1}$$

where, as usual, B^n are to be replaced by B_n (see [2,6,7]).

Additionally, the Bernoulli polynomials of degree n are given by

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \tag{2}$$

(see [3,8,9]).

Let q be an indeterminate in $\mathbb{C}_p$. For $q \in \mathbb{C}_p$, we assume that $|1-q|_p < p^{-\frac{1}{p-1}}$.

In [7], Carlitz considered the q-Bernoulli numbers which are given by the recurrence relation:

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{3}$$

where β_q^n are to be replaced by $\beta_{n,q}$, as usual.

In addition, he defined the q-Bernoulli polynomials as

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}, \quad (n \geq 0), \tag{4}$$

where $[x]_q = \frac{1-q^x}{1-q}$, (see [7]).

Recently, the type 2 Bernoulli polynomials have been defined as

$$\frac{t}{e^t - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \tag{5}$$

(see [1,3,8]).

When $x = 0$, $b_n = b_n(0)$ are called the type 2 Bernoulli numbers.

From (5), we note that

$$\sum_{l=0}^{n-1}(2l+1)^k = \frac{1}{k+1}(b_{k+1}(2n) - b_{k+1}), \quad (k \geq 0). \tag{6}$$

Let f be a uniformly differentiable function on $\mathbb{Z}_p$. Then, Kim defined the p-adic q-integral of f on $\mathbb{Z}_p$ as

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x)d\mu_q(x) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x+p^N\mathbb{Z}_p), \end{aligned} \tag{7}$$

(see [4]). Here, we note that $\mu_q(x+p^N\mathbb{Z}_p) = \frac{q^x}{[p^N]_q}$ is a distribution but not a measure. The details on the existence of the p-adic q-integrals for uniformly differentiable functions f on $\mathbb{Z}_p$ can be found in [4,10].

From (7), we note that

$$qI_q(f_1) = I_q(f) + (q-1)f(0) + \frac{q-1}{\log q}f'(0), \tag{8}$$

where $f_1(x) = f(x+1)$.

By virtue of (8) and induction, we get

$$q^n I_q(f_n) = I_q(f) + (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l), \tag{9}$$

where $f_n(x) = f(x+n), \quad (n \geq 1)$.

The degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \tag{10}$$

(see [11]), where $\lambda \in \mathbb{C}_p$ with $|\lambda|_p < p^{-\frac{1}{p-1}}$.

For brevity, we also set

$$e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{11}$$

Carlitz defined the degenerate Bernoulli polynomials as

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}, \tag{12}$$

where $B_{n,\lambda} = B_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

From (12), we note that

$$\sum_{l=0}^{n-1}(l)_{k,\lambda} = \frac{1}{k+1}(B_{k+1,\lambda}(n) - B_{k+1,\lambda}), \quad (n \geq 0), \tag{13}$$

where $(l)_{0,\lambda} = 1$, $(l)_{k,\lambda} = l(l-\lambda)\cdots(l-(k-1)\lambda), \quad (k \geq 1)$.

In the special case of $\lambda = 1$, the falling factorial sequence (also called the Pochammer symbol) is given by

$$(l)_0 = 1, (l)_k = l(l-1)\cdots(l-(k-1)), \quad (k \geq 1). \tag{14}$$

In this paper, we study type 2 degenerate q-Bernoulli polynomials and investigate some identities and properties for these polynomials.

2. Type 2 Degenerate q-Bernoulli Polynomials

Throughout this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{C}_p$. Now, we define the *type 2 degenerate q-Bernoulli polynomials* by

$$\sum_{n=0}^{\infty} b_{n,q}(x \mid \lambda)\frac{t^n}{n!} = \frac{1}{2}\int_{\mathbb{Z}_p} e_{\lambda}^{[x+2y]_q}(t)d\mu_q(y). \tag{15}$$

By (15), we get

$$\frac{1}{2}\int_{\mathbb{Z}_p}\left([x+2y]_q\right)_{n,\lambda} d\mu_q(y) = b_{n,q}(x \mid \lambda), \quad (n \geq 0). \tag{16}$$

When $x = 1$, $b_{n,q}(\lambda) = b_{n,q}(1 \mid \lambda)$ are called the type 2 degenerate q-Bernoulli numbers.
We observe here that

$$\begin{aligned}&\lim_{q\to 1}\lim_{\lambda\to 0}\frac{1}{2}\int_{\mathbb{Z}_p}\left([x+2y]_q\right)_{n,\lambda} d\mu_q(y)\\ =&\lim_{q\to 1}\lim_{\lambda\to 0}\frac{1}{2}b_{n,q}(x \mid \lambda) = b_n(x-1), \quad (n \geq 0).\end{aligned} \tag{17}$$

The degenerate Stirling numbers of the first kind appear as the coefficients in the expansion

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{1,\lambda}(n,l)x^l, \quad (n \geq 0), \tag{18}$$

(see [12]).

Thus, by (18), we get

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{Z}_p}\left([x+2y]_q\right)_{n,\lambda} d\mu_q(y)\\ =&\frac{1}{2}\sum_{l=0}^{n} S_{1,\lambda}(n,l)\int_{\mathbb{Z}_p}[x+2y]_q^l d\mu_q(y)\\ =&\sum_{l=0}^{n} S_{1,\lambda}(n,l)b_{l,q}(x),\end{aligned} \tag{19}$$

where $b_{l,q}(x)$ is the type 2 q-Bernoulli polynomials given by $\frac{1}{2}\int_{\mathbb{Z}_p}[x+2y]_q^n d\mu_q(y) = b_{n,q}(x)$, $(n \geq 0)$, (see [5]).

Therefore, by (16) and (19), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$b_{n,q}(x \mid \lambda) = \sum_{l=0}^{n} S_{1,\lambda}(n,l)b_{l,q}(x). \tag{20}$$

Now, we observe that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{Z}_p}\left([x+2y]_q\right)_{n,\lambda}d\mu_q(y)\\
=&\frac{1}{2}\sum_{l=0}^{n}S_{1,\lambda}(n,l)\int_{\mathbb{Z}_p}[x+2y]_q^l d\mu_q(y)\\
=&\frac{1}{2}\sum_{l=0}^{n}S_{1,\lambda}(n,l)\frac{1}{(1-q)^l}\sum_{m=0}^{l}\binom{l}{m}(-q^x)^m\frac{2m+1}{[2m+1]_q}.
\end{aligned}\tag{21}$$

Therefore, by (21), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$b_{n,q}(x\mid\lambda)=\frac{1}{2}\sum_{l=0}^{n}\frac{S_{1,\lambda}(n,l)}{(1-q)^l}\sum_{m=0}^{l}\binom{l}{m}(-q^x)^m\frac{2m+1}{[2m+1]_q}.\tag{22}$$

From (16), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty}b_{n,q}(x\mid\lambda)\frac{t^n}{n!}&=\frac{1}{2}\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\left([x+2y]_q\right)_{n,\lambda}d\mu_q(y)\frac{t^n}{n!}\\
&=\frac{1}{2}\int_{\mathbb{Z}_p}\left(1+\lambda t\right)^{\frac{[x+2y]_q}{\lambda}}d\mu_q(y)\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{k}{m}q^{mx}[x]_q^{k-m}S_1(n,k)\lambda^{n-k}b_{m,q}\right)\frac{t^n}{n!},
\end{aligned}\tag{23}$$

where $S_1(n,k)$ are the Stirling numbers of the first kind and $b_{n,q}$ are the type 2 q-Bernoulli numbers.

Therefore, by (23), we get the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$b_{n,q}(x\mid\lambda)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{k}{m}q^{mx}[x]_q^{k-m}S_1(n,k)\lambda^{n-k}b_{m,q}.\tag{24}$$

In particular,

$$b_{n,q}(\lambda)=\sum_{k=0}^{n}q^{kx}S_1(n,k)\lambda^{n-k}b_{k,q}.$$

In [4], Kim expressed Carlitz's q-Bernoulli polynomials in terms of the following p-adic q-integrals on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p}[x+y]_q^n d\mu_q(y)=\beta_{n,q}(x),\quad (n\geq 0).\tag{25}$$

From (9) and (25), we have

$$\begin{aligned}
q^n\beta_{m,q}(n) &= \int_{\mathbb{Z}_p} q^n[x+n]_q^m d\mu_q(x) \\
&= \int_{\mathbb{Z}_p} [x]_q^m d\mu_q(x) + (q-1)\sum_{l=0}^{n-1} q^l[l]_q^m + m\sum_{l=0}^{n-1}[l]_q^{m-1}q^{2l} \\
&= \beta_{m,q} + (q-1)\sum_{l=0}^{n-1} q^l[l]_q^m + m\sum_{l=0}^{n-1}[l]_q^{m-1}q^{2l} \\
&= \beta_{m,q} + (m+1)\sum_{l=0}^{n-1} q^{2l}[l]_q^{m-1} - \sum_{l=0}^{n-1} q^l[l]_q^{m-1},
\end{aligned} \tag{26}$$

where n is a positive integer.

Therefore, we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$q^n\beta_{m,q}(n) - \beta_{m,q} = (m+1)\sum_{l=0}^{n-1} q^{2l}[l]_q^{m-1} - \sum_{l=0}^{n-1} q^l[l]_q^{m-1}. \tag{27}$$

Let us take $f(x) = \Big([x]_q\Big)_{m,\lambda}$, $(m \geq 1)$. From (9), we have

$$\begin{aligned}
&q^n\int_{\mathbb{Z}_p}\Big([x+n]_q\Big)_{m,\lambda} d\mu_q(x) \\
=&\int_{\mathbb{Z}_p}\Big([x]_q\Big)_{m,\lambda} d\mu_q(x) + (q-1)\sum_{l=0}^{n-1} q^l\Big([l]_q\Big)_{m,\lambda} \\
&+\sum_{l=0}^{n-1}\left(\sum_{k=0}^{m-1}\frac{1}{[l]_q-k\lambda}\right)\Big([l]_q\Big)_{m,\lambda}q^{2l}.
\end{aligned} \tag{28}$$

In [13], the degenerate q-Bernoulli polynomials are defined by Kim as

$$\int_{\mathbb{Z}_p} e_\lambda^{[x+y]_q}(t)d\mu_q(y) = \sum_{n=0}^{\infty}\beta_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{29}$$

In particular, the degenerate q-Bernoulli numbers are given by $\beta_{n,\lambda,q} = \beta_{n,\lambda,q}(0)$.

From (29), we have

$$\int_{\mathbb{Z}_p}\Big([x+y]_q\Big)_{n,\lambda} d\mu_q(y) = \beta_{n,\lambda,q}(x), \ \ (n \geq 0). \tag{30}$$

By (28) and (30), this completes the proof for the next theorem.

Theorem 5. *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
&q^n\beta_{m,\lambda,q}(n) - \beta_{m,\lambda,q} \\
=&(q-1)\sum_{l=0}^{n-1} q^l\Big([l]_q\Big)_{m,\lambda} + \sum_{l=0}^{n-1}\left(\sum_{k=0}^{m-1}\frac{1}{[l]_q-k\lambda}\right)\Big([l]_q\Big)_{m,\lambda}q^{2l}.
\end{aligned} \tag{31}$$

Let us take $f(x)=\left([2x+1]_q\right)_{m,\lambda}$, $(m\geq 1)$. From (9), we have

$$\begin{aligned} & q^n\int_{\mathbb{Z}_p}\left([2x+2n+1]_q\right)_{m,\lambda}d\mu_q(x)\\ =&\int_{\mathbb{Z}_p}\left([2x+1]_q\right)_{m,\lambda}d\mu_q(x)+(q-1)\sum_{l=0}^{n-1}q^l\left([2l+1]_q\right)_{m,\lambda}\\ &+2\sum_{l=0}^{n-1}\left(\sum_{k=0}^{m-1}\frac{1}{[2l+1]_q-k\lambda}\right)\left([2l+1]_q\right)_{m,\lambda}q^{3l+1}. \end{aligned} \tag{32}$$

From (16) and (32), we have

$$\begin{aligned} & q^n b_{m,q}(2n+1\mid\lambda)-b_{m,q}(\lambda)\\ =&\frac{q-1}{2}\sum_{l=0}^{n-1}q^l\left([2l+1]_q\right)_{m,\lambda}+\sum_{l=0}^{n-1}\left(\sum_{k=0}^{m-1}\frac{1}{[2l+1]_q-k\lambda}\right)\left([2l+1]_q\right)_{m,\lambda}q^{3l+1}. \end{aligned} \tag{33}$$

Therefore, by (33), we obtain the following theorem.

Theorem 6. *For $m,n\in\mathbb{N}$, we have*

$$\begin{aligned} &\frac{q-1}{2}\sum_{l=0}^{n-1}q^l\left([2l+1]_q\right)_{m,\lambda}+\sum_{l=0}^{n-1}\left(\sum_{k=0}^{m-1}\frac{1}{[2l+1]_q-k\lambda}\right)\left([2l+1]_q\right)_{m,\lambda}q^{3l+1}\\ =&q^n b_{m,q}(2n+1\mid\lambda)-b_{m,q}(\lambda). \end{aligned} \tag{34}$$

From (7), we can derive the following integral equation:

$$\begin{aligned} \int_{\mathbb{Z}_p}f(x)d\mu_q(x) &=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}f(x)q^x\\ &=\lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{x=0}^{dp^N-1}f(x)q^x\\ &=\lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{a=0}^{d-1}\sum_{x=0}^{p^N-1}f(a+dx)q^{a+dx}\\ &=\sum_{a=0}^{d-1}q^a\frac{1}{[d]_q}\lim_{N\to\infty}\frac{1}{[p^N]_{q^d}}\sum_{x=0}^{p^N-1}f(a+dx)q^{dx}\\ &=\frac{1}{[d]_q}\sum_{a=0}^{d-1}q^a\int_{\mathbb{Z}_p}f(a+dx)d\mu_{q^d}(x), \end{aligned} \tag{35}$$

where d is a positive integer.

Lemma 1. *For $d\in\mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p}f(x)d\mu_q(x)=\frac{1}{[d]_q}\sum_{a=0}^{d-1}q^a\int_{\mathbb{Z}_p}f(a+dx)d\mu_{q^d}(x). \tag{36}$$

We obtain the following theorem from Lemma 1.

Theorem 7. *For $n, d \in \mathbb{N}$, we have*

$$b_{n,q}(\lambda) = [d]_q^{n-1} \sum_{a=0}^{d-1} q^a b_{n,q^d}\left(\frac{2a+1}{d} \mid \frac{\lambda}{[d]_q}\right). \tag{37}$$

Proof. Let us apply Lemma 1 with $f(x) = \Big([2x+1]_q\Big)_{n,\lambda}$, $(n \in \mathbb{N})$. Then, by virtue of (16), we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} \Big([2x+1]_q\Big)_{n,\lambda} d\mu_q(x) &= \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a \int_{\mathbb{Z}_p} \Big([2(a+dx)+1]_q\Big)_{n,\lambda} d\mu_{q^d}(x) \\
=& \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a [d]_q^n \int_{\mathbb{Z}_p} \left(\left[\frac{2a+1}{d} + 2x\right]_{q^d}\right)_{n,\frac{\lambda}{[d]_q}} d\mu_{q^d}(x) \\
=& [d]_q^{n-1} \sum_{a=0}^{d-1} q^a \int_{\mathbb{Z}_p} \left(\left[\frac{2a+1}{d} + 2x\right]_{q^d}\right)_{n,\frac{\lambda}{[d]_q}} d\mu_{q^d}(x) \\
=& 2[d]_q^{n-1} \sum_{a=0}^{d-1} q^a b_{n,q^d}\left(\frac{2a+1}{d} \mid \frac{\lambda}{[d]_q}\right).
\end{aligned}$$

□

3. Further Remarks

Assume that $X_1, X_2, X_3, \cdots$ are independent random variables, each of which has the Laplace distribution with parameters 0 and 1. Namely, each of them has the probability density function given by $\frac{1}{2}\exp(-|x|)$.

Let Z be the random variable given by $Z = \sum_{k=1}^{\infty} \frac{X_k}{2k\pi}$. In addition, let b_n be the type 2 Bernoulli numbers defined by

$$\frac{t}{e^t - e^{-t}} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}. \tag{38}$$

Then, it was shown in [1] that

$$\begin{aligned}
\sum_{n=0}^{\infty} E[Z^n] \frac{(it)^n}{n!} &= \frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-1} b_n \frac{t^n}{n!}.
\end{aligned} \tag{39}$$

Thereby, it was obtained that

$$i^n E[Z^n] = \left(\frac{1}{2}\right)^{n-1} b_n. \tag{40}$$

Before proceeding further, we recall that the Volkenborn integral (also called the p-adic invariant integral) for a uniformly differentiable function f on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(y) d\mu(y) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{y=0}^{p^N - 1} f(y). \tag{41}$$

Then, it is well known (see [14]) that this integral satisfies the following integral equation:

$$\int_{\mathbb{Z}_p} f(y+1) d\mu(y) = \int_{\mathbb{Z}_p} f(y) d\mu(y) + f'(0). \tag{42}$$

When $q = 1$ and $x = 1$, by virtue of (41), (15) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} b_n(\lambda)\frac{t^n}{n!} &= \frac{1}{2}\int_{\mathbb{Z}_p} e_\lambda^{1+2y}(t)d\mu(y)\\ &= \frac{\frac{1}{\lambda}\log(1+\lambda t)}{e_\lambda(t) - e_\lambda^{-1}(t)}.\end{aligned} \tag{43}$$

Here, $b_n(\lambda)$ may be called the fully degenerate type 2 Bernoulli numbers, even though they were defined slightly differently in [3]. Replacing t with $\frac{2}{\lambda}\log(1+\lambda t)$ in (39) and by making use of (43), we have

$$\begin{aligned}&\sum_{m=0}^{\infty} E[Z^m](2i)^m \frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^m\\ &\quad= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_{1,\lambda}(n,m)(2i)^m E[Z^m]\right)\frac{t^n}{n!}\\ &\quad= 2\sum_{n=0}^{\infty} b_n(\lambda)\frac{t^n}{n!}.\end{aligned} \tag{44}$$

Here, $S_{1,\lambda}(n,k)$ are the degenerate Stirling numbers of the first kind (see [12]) either given by

$$\frac{1}{m}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^m = \sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!}, \tag{45}$$

or given by

$$(x)_{n,\lambda} = \sum_{m=0}^{n} S_{1,\lambda}(n,m)x^m = \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}x^m. \tag{46}$$

Thus, by (44), we have shown that

$$2b_n(\lambda) = \sum_{m=0}^{n} S_{1,\lambda}(n,m)(2i)^m E[Z^m].$$

Here, we remark that we only considered $q = 1$ and $x = 1$ cases of (15), namely the fully degenerate type 2 Bernoulli numbers. This is because we do not see how to express type 2 degenerate q-Bernoulli polynomials or type 2 degenerate q-Bernoulli numbers in terms of the moments of some suitable random variables, constructed from random variables with Laplace distributions. We leave this as an open problem to the interested reader.

4. Conclusions

Studies on various special polynomials and numbers have been preformed using several different methods, such as generating functions, combinatorial methods, umbral calculus techniques, matrix theory, probability theory, p-adic analysis, differential equations, and so on.

One way of introducing new special polynomials and numbers is to study various degenerate versions of some known special polynomials and numbers, which began with Carlitz's paper in [2]. Actually, degenerate versions were investigated not only for some polynomials but also for a transcendental function, namely the gamma function. For this, we refer the reader to [11]. Another way of introducing new special polynomials and numbers is to study various q-analogues of some known special polynomials and numbers. The bosonic p-adic q-integrals, together with the fermionic p-adic q-integrals, turned out to be very powerful and fruitful tools for naturally constructing such q-analogues. They were introduced by Kim in [4] and have been widely used ever since their invention.

In this paper, the type 2 degenerate q-Bernoulli polynomials and the corresponding numbers were introduced and investigated as a degenerate version of and also as a q-analogue of type 2 Bernoulli polynomials by making use of the bosonic p-adic q-integrals [1,3,5]. Here, as an introductory paper on the subject, only very basic results were obtained. The obtained results are several expressions for those polynomials, identities involving those numbers, identities regarding Carlitz's q-Bernoulli numbers, identities concerning degenerate q-Bernoulli numbers, and the representations of the fully degenerate type 2 Bernoulli numbers ($q = 1$ and $x = 1$ cases of the type 2 degenerate q-Bernoulli polynomials) in terms of moments of certain random variables, created from random variables with Laplace distributions. We are planning to study more detailed results relating to these polynomials and numbers in a forthcoming paper.

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