Article

On Eccentric Topological Indices Based on Edges of Zero Divisor Graphs

Ali N. A. Koam ¹ , Ali Ahmad ^{2,*} and Azeem Haider ¹ ¹ Department of Mathematics, College of Science, Jazan University, New Campus, Jazan 2097, Saudi Arabia² College of Computer Science and Information Technology, Jazan University, Jazan 45142, Saudi Arabia

* Correspondence: ahmadsms@gmail.com

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Abstract: This article is devoted to the determination of edge-based eccentric topological indices of a zero divisor graph of some algebraic structures. In particular, we computed the first Zagreb eccentricity index, third Zagreb eccentricity index, geometric-arithmetic eccentricity index, atom-bond connectivity eccentricity index and a fourth type of eccentric harmonic index for zero divisor graphs associated with a class of finite commutative rings.

Keywords: eccentricity; topological index; zero divisor graphs; commutative ring

MSC: 05C12; 05C10; 05C90

1. Introduction

Algebraic structures have been studied for their close affiliation with representation theory and number theory, and they have been extensively studied in combinatorics [1,2]. In addition to the extensive theoretical research in these areas, finite rings and fields have received attention for their applications to cryptography and coding theory.

A foundational role of molecular descriptors is to consider molecules as real bodies and transform them into numbers, which enables mathematics to play a key role in chemistry, pharmaceutical sciences, environmental protection, quality control and health research. These molecular descriptors are graph invariants and are usually known as topological indices. In graph theory, by using topological indices we correlate various characteristics of a chemical structure in order to characterize it. These days there is an area of research devoted to computing topological indices for various structures.

In mathematical chemistry, a molecular or chemical graph is a presentation of the structural formula of a chemical compound in the form of graph structures. A chemical graph is a labeled graph with corresponding vertices representing the atoms of the compound and edges representing their chemical bonds. A topological index associated with a chemical graph is a numeric value which remains invariant under the graph automorphism and it also characterizes the topology of a chemical graph. Hence, these indices have a variety of applications in chemistry as well as in pharmaceutical science and in the structures of nanotubes [3,4]. For graph theory, topological indices are classified as “topological indices based on degrees” [5–8], “topological indices based on distances” and “topological indices based on eccentricities” of graphs. Some well-known topological indices based on the degrees of a graph are the Randi connectivity index, Zagreb indices, Harmonic index, atom bond connectivity, geometric arithmetic index and so forth, and the Wiener index, Hosoya index and Estrada index are distance based topological indices [9,10]. Similarly, eccentricity based topological indices include the Zagreb eccentricity index [11,12], geometric-arithmetic eccentricity index [13], connectivity of atomic bond eccentricity index [14] and the fourth type of eccentric harmonic index [15,16]. Topological

indices lay the criteria for the growth of chemical structures and how mathematical operations on the graphs, with the relief of topological indices, can extend multidisciplinary research.

2. Motivation behind Topological Indices

An association between the stability of linear alkanes and the branched alkanes is studied through the *ABC* index. This index is also used for the computation of strain energy for cyclo alkanes [17,18]. In order to correlate certain physico-chemical characteristics, the *GA* index has more efficient predictive power than does the Randic connectivity index [19,20]. The Zagreb indices (first and second) are powerful tools for the computation of total π -electron energy of the molecules within a particular approximate expression [21]. The topological indices that are based on degrees are more useful for examining the chemical characteristics of distinct molecular structures. These applications provide a motivation to study topological indices that are eccentricity based. The topological indices based on eccentricity are useful for the assessment of the toxicological, physico-chemical and pharmacological characteristics of a compound through the structure of its molecules. The study of the QSAR (quantitative structure activity relationship) is known for this sort of analysis [22]. In order to obtain further details on the applications of Topological indices see References [23,24].

3. Methods

In order to obtain our intended outcomes we used combinatorial tools such as vertex partition, edge partition, eccentricity, eccentricity counting for each degree and other graph theoretical methods with some analytic techniques [23,25]. Moreover, we use C++ for the verification of degree and their corresponding eccentricity.

4. Definitions and Notations

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. We associate with each graph G a numerical value, called the topological index (descriptor). This index remains invariant under graph automorphisms. The number of edges adjacent to a vertex v is called the degree of v which is denoted by $d(v)$ and the corresponding maximum and minimum degree of a vertex in a graph G is denoted by $\Delta(G)$ and $\delta(G)$ respectively. The $d(u, v)$ denotes the distance between any two vertices u and v which is defined as the number of edges in the shortest path between the vertices u and v . The eccentricity of a vertex v in a graph G is defined as:

$$\varepsilon_v = \max\{d(v, u) : u \in V(G)\}. \quad (1)$$

We study general topological invariant $T(G)$ based on the eccentricity of vertices in graph G .

$$T(G) = \sum_{uv \in E(G)} \phi(\varepsilon_u, \varepsilon_v), \quad (2)$$

where $\phi(\varepsilon_u, \varepsilon_v) = \phi(\varepsilon_v, \varepsilon_u)$ is a real function of ε_u and ε_v .

- If $\phi(\varepsilon_u, \varepsilon_v) = (\varepsilon_u + \varepsilon_v)^\beta$, where $\beta \neq 0$ is a real number, then $\phi(G)$ is the first Zagreb eccentricity index if $\beta = 1$ [11,12].
- If $\phi(\varepsilon_u, \varepsilon_v) = (\varepsilon_u \times \varepsilon_v)^\alpha$, where $\alpha \neq 0$ is a real number, then $\phi(G)$ is the third Zagreb eccentricity index if $\alpha = 1$ [11,12].
- If $\phi(\varepsilon_u, \varepsilon_v) = \frac{2\sqrt{\varepsilon_u \times \varepsilon_v}}{\varepsilon_u + \varepsilon_v}$, we obtain the geometric-arithmetic eccentricity index $GA_4(G)$ [13].
- If $\phi(\varepsilon_u, \varepsilon_v) = \sqrt{\frac{\varepsilon_u + \varepsilon_v - 2}{\varepsilon_u \times \varepsilon_v}}$, we obtain the atom-bond connectivity eccentricity index $ABC_5(G)$ [14].
- If $\phi(\varepsilon_u, \varepsilon_v) = \frac{2}{\varepsilon_u + \varepsilon_v}$, we obtain the fourth type of eccentric harmonic index $H_4(G)$ [15,16].

A commutative ring R with identity a non-zero element $x \in R$ is called a zero divisor if there exists another non zero element $y \in R$ such that $x \cdot y = 0$ in R . The set of all zero divisors in R is denoted

by $Z(R)$. By considering a finite commutative ring R , we associate a zero divisor graph $G(R)$ such that if $x_1, x_2 \in Z(R) = V(G(R))$ and $(x_1, x_2) \in E(G(R))$ if and only if $x_1 \cdot x_2 = 0$. The notion of the zero divisor graph was introduced by Beck [2] and later for any commutative ring R , Anderson and Livingston, in Reference [26], proved that the zero divisor graph $G(R)$ is a connected graph. Due to broad interest in parameters associated with graphs and rings, several authors investigated the associated identities of zero-divisor for commutative rings, see for example References [23,27,28].

5. Main Results

For positive integers $a, b \in \mathbb{Z}^+$, we consider $G(R)$ to be a zero divisor graph of commutative ring $R = \mathbb{Z}_a \times \mathbb{Z}_b$ whose vertex set is $V(G(R))$ and edge set is $E(G(R))$. Then

$$V(G(R)) = \{(x, y) \in R : x \in Z(\mathbb{Z}_a) \cup \{0\} \text{ or } y \in Z(\mathbb{Z}_b) \cup \{0\}\} \setminus \{(0, 0)\}$$

and

$$E(G(R)) = \{((x_1, y_1), (x_2, y_2)) \in V(G(R)) \times V(G(R)) : (x_1 x_2, y_1 y_2) = (0, 0) \text{ in } R\}.$$

This section is devoted on the discussion of eccentric topological indices based on the edges of zero divisor graphs for rings $\mathbb{Z}_a \times \mathbb{Z}_b$ with $a = p_1 p_2, p^2$ and $b = q^2$, where p_1, p_2, p and q are prime numbers.

5.1. Case 1: $\mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{q^2}$

Let Γ_R be a zero divisor graph of commutative ring $R = \mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{q^2}$, where p_1, p_2, q are prime numbers with $p_1 \neq p_2$, in this subsection we discuss the eccentric topological indices for Γ_R .

Theorem 1. Let Γ_R be the zero divisor graph of commutative ring $R = \mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{q^2}$, where p_1, p_2, q are prime numbers with $p_1 \neq p_2$. Then $|V(\Gamma_R)| = q^2(p_2 + p_1 - 1) + q(p_2 - 1)(p_1 - 1) - 1$ and $|E(\Gamma_R)| = q^2(5p_1 p_2 - 3p_1 - 3p_2 + \frac{3}{2}) + q(-4p_1 p_2 + 2p_1 + 2p_2 - \frac{3}{2}) + 1$.

Proof. Let $(u, v) \in V(\Gamma_R)$. Then we have following cases:

1. If $u = 0$ and for any $v \in \mathbb{Z}_{q^2}$ with $v \neq kq$, $0 \leq k \leq q - 1$, then each $(0, v)$ of this type is connected to the vertices of type $(u', 0)$ for all $u' \in \mathbb{Z}_{p_1 p_2} \setminus \{0\}$. Hence the degree of each such vertex is $p_1 p_2 - 1$. Similarly if $u \in \mathbb{Z}_{p_1 p_2} \setminus \{0, t_1 p_1, t_2 p_2\}$ with $1 \leq t_1 \leq p_2 - 1, 1 \leq t_2 \leq p_1 - 1$ and $v = 0$, then the degree of each such vertex is $q^2 - 1$.
2. If $u = 0$ and $v = kq$, $1 \leq k \leq q - 1$, then each such vertex $(0, v)$ is connected to the vertices of types $(u_1, 0), (0, v'), (u_2, v'), (u_3, v')$ and (u_4, v') for every $u_1 \in \mathbb{Z}_{p_1 p_2} \setminus \{0\}, u_2 = t_1 p_1, 1 \leq t_1 \leq p_2 - 1, u_3 = t_2 p_2, 1 \leq t_2 \leq p_1 - 1, u_4 \neq t_1 p_1, 0 \leq t_1 \leq p_2 - 1, u_4 \neq t_2 p_2, 0 \leq t_2 \leq p_1 - 1$ and $v' = tq, 1 \leq k \leq q - 1$ with $t \neq k$. Hence the degree of each such vertex is $p_1 p_2 - 1 + q - 2 + (p_1 - 1)(q - 1) + (p_2 - 1)(q - 1) + (p_2 - 1)(p_1 - 1)(q - 1) = p_1 p_2 - 2 + (p_2 + p_1 - 1)(q - 1) + (p_2 - 1)(p_1 - 1)(q - 1) = p_1 p_2 q - 2$.
3. If $u = t_1 p_1, 1 \leq t_1 \leq p_2 - 1$ and $v = 0$, then each such vertex $(u, 0)$ is connected to the vertices of type $(0, v_1), (u', 0), (u', v_2)$ and (u', v_3) for every $v_1 \in \mathbb{Z}_{q^2} \setminus \{0\}, u' = t_2 p_2, 1 \leq t_2 \leq p_1 - 1, v_2 \neq kq, 0 \leq k \leq q - 1$ and $v_3 = kq, 1 \leq k \leq q - 1$. Therefore the degree of each such vertex is $q^2 - 1 + p_1 - 1 + (p_1 - 1)(q^2 - q) + (p_1 - 1)(q - 1) = p_1 q^2 - 1$. Similarly if $u = t_2 p_2, 1 \leq t_2 \leq p_1 - 1$ and $v = 0$, then the degree of each such vertices is $p_2 q^2 - 1$.
4. If $u = t_1 p_1, 1 \leq t_1 \leq p_2 - 1$ and $v \neq kq, 0 \leq k \leq q - 1$, then each such vertex (u, v) is connected to vertex $(u', 0)$ for every $u' = t_2 p_2, 1 \leq t_2 \leq p_1 - 1$. Hence the degree of each such vertex is $p_1 - 1$. Similarly if $u = t_2 p_2, 1 \leq t_2 \leq p_1 - 1$ and $v \neq kq, 0 \leq k \leq q - 1$, then the degree of each such vertex is $p_2 - 1$.
5. If $u = t_1 p_1, 1 \leq t_1 \leq p_2 - 1$ and $v = kq, 1 \leq k \leq q - 1$, then each such vertex (u, v) is connected to the vertices of type $(u', 0), (0, v')$ and (u', v') for every $u' = t_2 p_2, 1 \leq t_2 \leq p_1 - 1, v' = kq, 1 \leq k \leq q - 1$. Therefore the degree of each such vertex is $p_1 - 1 + q - 1 + (p_1 - 1)(q - 1) = p_1 q - 1$.

Similarly, if $u = t_2 p_2, 1 \leq t_2 \leq p_1 - 1$ and $v = kq, 1 \leq k \leq q - 1$, then the degree of each such vertex is $p_2 q - 1$.

6. If $u \neq t_1 p_1$ and $u \neq t_2 p_2, 0 \leq t_1 \leq p_2 - 1, 0 \leq t_2 \leq p_1 - 1$ and $v = kq, 1 \leq k \leq q - 1$, then each such vertex (u, v) is connected to the vertex $(0, v')$ for every $v' = kq, 1 \leq k \leq q - 1$. Hence the degree of each such vertex (u, v) is $q - 1$.

The total number of vertices of Γ_R from the above cases is:

$$\begin{aligned} & q^2 - q + (p_2 - 1)(p_1 - 1) + q - 1 + p_2 - 1 + p_1 - 1 + (p_2 - 1)(q^2 - q) + (p_1 - 1)(q^2 - q) + (p_2 - 1) \\ & (q - 1) + (p_1 - 1)(q - 1) + (p_2 - 1)(p_1 - 1)(q - 1) \\ & = (q^2 - 1)(p_2 + p_1 - 2) + p_2 p_1 + q^2 - 2 + (p_2 - 1)(p_1 - 1)(q - 1) \\ & = q^2(p_2 + p_1 - 1) + q(p_2 - 1)(p_1 - 1) - 1. \end{aligned}$$

Hence, $|V(\Gamma_R)| = q^2(p_2 + p_1 - 1) + q(p_2 - 1)(p_1 - 1) - 1$.

The total number of edges of Γ_R is obtained by using hand shaking lemma as:

$$|E(\Gamma_R)| = \frac{\sum_{v \in V(\Gamma_R)} \deg(v)}{2} = \frac{1}{2} \{ (q^2 - q)(p_1 p_2 - 1) + (p_1 - 1)(p_2 - 1)(q^2 - 1) + (q - 1)(p_1 p_2 q - 2) + (p_1 - 1)(p_2 q^2 - 1) + (p_2 - 1)(p_1 q^2 - 1) + (p_2 - 1)(q^2 - q)(p_1 - 1) + (p_1 - 1)(q^2 - q)(p_2 - 1) + (p_2 - 1)(q - 1)(p_1 q - 1) + (p_1 - 1)(q - 1)(p_2 q - 1) + (q - 1)^2(p_1 - 1)(p_2 - 1) \}.$$

After simplification we get, $|E(\Gamma_R)| = q^2(5p_1 p_2 - 3p_1 - 3p_2 + \frac{3}{2}) + q(-4p_1 p_2 + 2p_1 + 2p_2 - \frac{3}{2}) + 1$. $\square$

From the proof of Theorem 1 and our convenience, we partition the vertex set of Γ_R as follows:

$$\begin{aligned} \vee_1 &= \{(0, v) : v \in \mathbb{Z}_{q^2}, v \neq kq, 0 \leq k \leq q - 1\} \\ \vee_2 &= \{(0, v) : v = kq, 1 \leq k \leq q - 1\} \\ \vee_3 &= \{(u, 0) : u = t_1 p_1, 1 \leq t_1 \leq p_2 - 1\} \\ \vee_4 &= \{(u, 0) : u = t_2 p_2, 1 \leq t_2 \leq p_1 - 1\} \\ \vee_5 &= \{(u, 0) : u \neq t_1 p_1, u \neq t_2 p_2, 1 \leq t_1 \leq p_2 - 1, 1 \leq t_2 \leq p_1 - 1\} \\ \vee_6 &= \{(u, v) : u = t_1 p_1, v \neq kq, 1 \leq t_1 \leq p_2 - 1, 0 \leq k \leq q - 1\} \\ \vee_7 &= \{(u, v) : u = t_2 p_2, v \neq kq, 1 \leq t_2 \leq p_1 - 1, 0 \leq k \leq q - 1\} \\ \vee_8 &= \{(u, v) : u = t_1 p_1, v = kq, 1 \leq t_1 \leq p_2 - 1, 1 \leq k \leq q - 1\} \\ \vee_9 &= \{(u, v) : u = t_2 p_2, v = kq, 1 \leq t_2 \leq p_1 - 1, 1 \leq k \leq q - 1\} \\ \vee_{10} &= \{(u, v) : u \neq t_1 p_1, u \neq t_2 p_2, v = kq, 1 \leq t_1 \leq p_2 - 1, 1 \leq t_2 \leq p_1 - 1, 1 \leq k \leq q - 1\} \end{aligned}$$

This shows that $V(\Gamma_R) = \bigsqcup_{i=1}^{10} \vee_i$. It is observed that $|\vee_1| = q^2 - q, |\vee_2| = q - 1, |\vee_3| = p_2 - 1, |\vee_4| = p_1 - 1, |\vee_5| = (p_1 - 1)(p_2 - 1), |\vee_6| = (p_2 - 1)(q^2 - q), |\vee_7| = (p_1 - 1)(q^2 - q), |\vee_8| = (p_2 - 1)(q - 1), |\vee_9| = (p_1 - 1)(q - 1)$ and $|\vee_{10}| = (p_1 - 1)(p_2 - 1)(q - 1)$. Let the degree of a vertex u in set A be denoted by $d_A(u)$ and the distance between two vertices sets X and Y be $d(X, Y) = d(Y, X)$. Then following theorem determines the eccentricity of the vertices in graph Γ_R .

Theorem 2. If p_1, p_2 and q are prime numbers such that $p_1 \neq p_2$ and the associated zero divisor graph Γ_R to the commutative ring $R = \mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{q^2}$, then the eccentricity of the vertices of corresponding graph Γ_R is either 2 or 3.

Proof. It is easy to see from the proof of Theorem 1 that, $d(\vee_1, \vee_3) = d(\vee_1, \vee_4) = d(\vee_1, \vee_5) = d(\vee_2, \vee_2) = d(\vee_2, \vee_3) = d(\vee_2, \vee_4) = d(\vee_2, \vee_5) = d(\vee_2, \vee_8) = d(\vee_2, \vee_9) = d(\vee_2, \vee_{10}) = d(\vee_3, \vee_4) = d(\vee_3, \vee_7) = d(\vee_3, \vee_9) = d(\vee_4, \vee_6) = d(\vee_4, \vee_8) = d(\vee_8, \vee_9) = 1$.

$$\begin{aligned}
d(v_1, v_2) &= d(v_1, v_3) + d(v_3, v_2) = 1 + 1 = 2 \\
d(v_1, v_6) &= d(v_1, v_4) + d(v_4, v_6) = 1 + 1 = 2 \\
d(v_1, v_7) &= d(v_1, v_3) + d(v_3, v_7) = 1 + 1 = 2 \\
d(v_1, v_8) &= d(v_1, v_4) + d(v_4, v_8) = 1 + 1 = 2 \\
d(v_1, v_9) &= d(v_1, v_3) + d(v_3, v_9) = 1 + 1 = 2 \\
d(v_1, v_{10}) &= d(v_1, v_2) + d(v_2, v_{10}) = 2 + 1 = 3 \\
d(v_1, v_1) &= d(v_1, v_3) + d(v_3, v_1) = 1 + 1 = 2
\end{aligned}$$

Hence the eccentricity of the corresponding vertices of set v_1 is 3 i.e., $e(v_1) = 3$. Similarly, we can see that $\max\{d(v_s, v_t)\} = 2$ for $s = 2, 3, 4$ and $1 \leq t \leq 10$. Therefore, $e(v_2) = e(v_3) = e(v_4) = 2$.

As $d(v_5, v_6) = d(v_5, v_1) + d(v_1, v_6) = 1 + 2 = 3$ and $\max\{d(v_5, v_t)\} = 3$ for $1 \leq t \leq 10$. This shows $e(v_5) = 3$. Similarly, it is easy to calculate $\max\{d(v_s, v_t)\} = 3$ for $s = 6, 7, 8, 9, 10$ and $1 \leq t \leq 10$. This implies $e(v_{10}) = e(v_9) = e(v_8) = e(v_7) = e(v_6) = 3$. $\square$

Lemma 1. For any three prime numbers p_1, p_2 and q such that $p_1 \neq p_2$ and the associated zero divisor graph Γ_R of commutative ring $R = \mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{q^2}$ we have,

$$\begin{aligned}
T(\Gamma_R) &= \left\{ \phi(2, 2) + q\phi(2, 3) \right\} (q-1)(p_2 + p_1 - 2) + \left\{ \phi(2, 2) + (2q^2 + q - 3)\phi(2, 3) + (2q^2 - 3q + 1)\phi(3, 3) \right\} \\
&\quad (p_2 - 1)(p_1 - 1) + \left\{ (q-2)\phi(2, 2) + 2(q-1)(p_1 p_2 - 1)\phi(2, 3) \right\} \frac{(q-1)}{2}.
\end{aligned}$$

Proof. The graph Γ_R contains $q^2(p_2 + p_1 - 1) + q(p_2 - 1)(p_1 - 1) - 1$ vertices and $q^2(5p_1 p_2 - 3p_1 - 3p_2 + \frac{3}{2}) + q(-4p_1 p_2 + 2p_1 + 2p_2 - \frac{3}{2}) + 1$ edges. Let us divide the edges of Γ_R into partition sets according to the eccentricity of its end vertices.

$$\Xi_{r,s} = \{uv \in E(\Gamma_R) : e_u = r, e_v = s\}.$$

This means that the set $\Xi_{r,s}$ contains the edges incident with one vertex of eccentricity r and the other vertex of eccentricity s . From the Theorem 1 and Theorem 2, we have $|\Xi_{2,2}| = (p_2 + p_1 - 2)(q-1) + (p_2 - 1)(p_1 - 1) + \frac{(q-1)(q-2)}{2}$, $|\Xi_{2,3}| = 2(q^2 - 1)(p_2 - 1)(p_1 - 1) + (q-1)^2(p_1 p_2 - 1) + (q^2 - q)(p_2 + p_1 - 2) + (q-1)(p_2 - 1)(p_1 - 1)$, $|\Xi_{3,3}| = (p_2 - 1)(p_1 - 1)(2q^2 - 3q + 1)$ and $E(\Gamma_R) = \Xi_{2,2} \cup \Xi_{2,3} \cup \Xi_{3,3}$. Then

$$\begin{aligned}
T(G_1) &= \sum_{uv \in E(\Gamma_R)} \phi(e_u, e_v) \\
&= \sum_{uv \in \Xi_{2,2}} \phi(2, 2) + \sum_{uv \in \Xi_{2,3}} \phi(2, 3) + \sum_{uv \in \Xi_{3,3}} \phi(3, 3) \\
&= (p_2 + p_1 - 2)(q-1)\phi(2, 2) + (p_2 - 1)(p_1 - 1)\phi(2, 2) + \frac{(q-1)(q-2)}{2}\phi(2, 2) \\
&\quad + \left\{ 2(p_2 - 1)(p_1 - 1)(q^2 - 1) + (q-1)^2(p_1 p_2 - 1) + (q^2 - q)(p_2 + p_1 - 2) \right\} \phi(2, 3) \\
&\quad + (q-1)(p_2 - 1)(p_1 - 1)\phi(2, 3) + (p_2 - 1)(p_1 - 1)(2q^2 - 3q + 1)\phi(3, 3) \\
&= \left\{ \phi(2, 2) + q\phi(2, 3) \right\} (q-1)(p_2 + p_1 - 2) + \left\{ (q-2)\phi(2, 2) + 2(q-1)(p_1 p_2 - 1)\phi(2, 3) \right\} \frac{(q-1)}{2} \\
&\quad + \left\{ \phi(2, 2) + (2q^2 + q - 3)\phi(2, 3) + (2q^2 - 3q + 1)\phi(3, 3) \right\} (p_2 - 1)(p_1 - 1).
\end{aligned}$$

$\square$

Now we present values of the eccentric topological indices of the graph Γ_R which are edge-based.

Theorem 3. If p_1, p_2 and q are prime numbers such that $p_1 \neq p_2$ and the associated zero divisor graph Γ_R to the commutative ring $R = \mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{q^2}$, then

the First Zagreb eccentricity index of Γ_R is:

$$M_1^*(\Gamma_R) = (5q + 4)(q - 1)(p_2 + p_1 - 2) + (10p_1 p_2 q - 10p_1 p_2 - 6q + 2) \frac{q-1}{2} + (22q^2 - 13q - 5)(p_2 - 1)(p_1 - 1),$$

the third Zagreb eccentricity index is:

$$M_3^*(\Gamma_R) = (6q + 4)(q - 1)(p_2 + p_1 - 2) + 2(q - 1)(3p_1 p_2 q - 3p_1 p_2 - 2q + 1) + (30q^2 - 21q - 5)(p_2 - 1)(p_1 - 1),$$

the geometric-arithmetic eccentricity index is:

$$GA_4(\Gamma_R) = \left(\frac{2\sqrt{6}}{5}q + 1\right)(q - 1)(p_2 + p_1 - 2) + \frac{(q-1)(q-2)}{2} + \frac{2\sqrt{6}}{5}(q - 1)^2(p_1 p_2 - 1) + \left(\frac{4\sqrt{6}+10}{5}q^2 + \frac{2\sqrt{6}-15}{5}q + \frac{10-6\sqrt{6}}{5}\right)(p_2 - 1)(p_1 - 1),$$

the atom-bond connectivity eccentricity index is:

$$ABC_5(\Gamma_R) = \frac{\sqrt{2}}{2}(q^2 - 1)(p_2 + p_1 - 2) + \frac{\sqrt{2}(q-1)}{4}(2p_1 p_2 q - 2p_1 p_2 - q) + \left(\frac{3\sqrt{2}+4}{3}q^2 + \frac{\sqrt{2}-4}{2}q + \frac{2-3\sqrt{2}}{3}\right)(p_2 - 1)(p_1 - 1),$$

the fourth type of eccentric harmonic index is:

$$H_4(\Gamma_R) = \frac{4q+5}{10}(q - 1)(p_2 + p_1 - 2) + \frac{(q-1)}{2}\left(\frac{4}{5}p_1 p_2 q - \frac{4}{5}p_1 p_2 - \frac{3}{10}q - \frac{1}{5}\right) + \left(\frac{22}{15}q^2 - \frac{3}{5}q - \frac{11}{30}\right)(p_2 - 1)(p_1 - 1).$$

Proof. For the first Zagreb eccentricity index $M_1^*(\Gamma_R)$ of Γ_R we obtain $\phi(\epsilon_u, \epsilon_v) = \epsilon_u + \epsilon_v$. So $\phi(2, 2) = 4$, $\phi(2, 3) = 5$ and $\phi(3, 3) = 6$. Thus by Lemma 1,

$$\begin{aligned} M_1^*(\Gamma_R) &= (4 + 5q)(q - 1)(p_2 + p_1 - 2) + (4(q - 2) + 10(q - 1)(p_1 p_2 - 1)) \frac{q-1}{2} \\ &\quad + \left(4 + 5(2q^2 + q - 3) + 6(2q^2 - 3q + 1)\right)(p_2 - 1)(p_1 - 1) \\ &= (5q + 4)(q - 1)(p_2 + p_1 - 2) + (10p_1 p_2 q - 10p_1 p_2 - 6q + 2) \frac{q-1}{2} \\ &\quad + (22q^2 - 13q - 5)(p_2 - 1)(p_1 - 1). \end{aligned}$$

For the third Zagreb eccentricity index $M_3^*(\Gamma_R)$ of Γ_R we obtain $\phi(\epsilon_u, \epsilon_v) = \epsilon_u \times \epsilon_v$. So $\phi(2, 2) = 4$, $\phi(2, 3) = 6$ and $\phi(3, 3) = 9$ So by Lemma 1,

$$\begin{aligned} M_3^*(\Gamma_R) &= (4 + 6q)(q - 1)(p_2 + p_1 - 2) + (4(q - 2) + 12(q - 1)(p_1 p_2 - 1)) \frac{q-1}{2} \\ &\quad + \left(4 + 6(2q^2 + q - 3) + 9(2q^2 - 3q + 1)\right)(p_2 - 1)(p_1 - 1) \\ &= (6q + 4)(q - 1)(p_2 + p_1 - 2) + 2(q - 1)(3p_1 p_2 q - 3p_1 p_2 - 2q + 1) \\ &\quad + (30q^2 - 21q - 5)(p_2 - 1)(p_1 - 1). \end{aligned}$$

For the geometric-arithmetic eccentricity index $GA_4(\Gamma_R)$ of Γ_R , we get $\phi(\epsilon_u, \epsilon_v) = \frac{2\sqrt{\epsilon_u \times \epsilon_v}}{\epsilon_u + \epsilon_v}$. So $\phi(2, 2) = 1$, $\phi(2, 3) = \frac{2\sqrt{6}}{5}$ and $\phi(3, 3) = 1$. Therefore,

$$\begin{aligned} GA_4(\Gamma_R) &= \left(1 + \frac{2\sqrt{6}}{5}q\right) (q-1)(p_2 + p_1 - 2) + \left((q-2) + 2(q-1)(p_1 p_2 - 1)\frac{2\sqrt{6}}{5}\right) \frac{q-1}{2} \\ &\quad + \left(1 + \frac{2\sqrt{6}}{5}(2q^2 + q - 3) + (2q^2 - 3q + 1)\right) (p_2 - 1)(p_1 - 1) \\ &= \left(\frac{2\sqrt{6}}{5}q + 1\right) (q-1)(p_2 + p_1 - 2) + \frac{(q-1)(q-2)}{2} + \frac{2\sqrt{6}}{5}(q-1)^2(p_1 p_2 - 1) \\ &\quad + \left(\frac{4\sqrt{6} + 10}{5}q^2 + \frac{2\sqrt{6} - 15}{5}q + \frac{10 - 6\sqrt{6}}{5}\right) (p_2 - 1)(p_1 - 1). \end{aligned}$$

For the atom-bond connectivity eccentricity index $ABC_5(\Gamma_R)$ of Γ_R , we obtain $\phi(\epsilon_u, \epsilon_v) = \sqrt{\frac{\epsilon_u + \epsilon_v - 2}{\epsilon_u \times \epsilon_v}}$. Thus $\phi(2, 2) = \phi(2, 3) = \frac{1}{\sqrt{2}}$ and $\phi(3, 3) = \frac{2}{3}$. Therefore,

$$\begin{aligned} ABC_5(\Gamma_R) &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}q\right) (q-1)(p_2 + p_1 - 2) + \left(\frac{1}{\sqrt{2}}(q-2) + 2(q-1)(p_1 p_2 - 1)\frac{1}{\sqrt{2}}\right) \frac{q-1}{2} \\ &\quad + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(2q^2 + q - 3) + \frac{2}{3}(2q^2 - 3q + 1)\right) (p_2 - 1)(p_1 - 1) \\ &= \frac{\sqrt{2}}{2}(q^2 - 1)(p_2 + p_1 - 2) + \frac{\sqrt{2}(q-1)}{4}(2p_1 p_2 q - 2p_1 p_2 - q) \\ &\quad + \left(\frac{3\sqrt{2} + 4}{3}q^2 + \frac{\sqrt{2} - 4}{2}q + \frac{2 - 3\sqrt{2}}{3}\right) (p_2 - 1)(p_1 - 1). \end{aligned}$$

For the fourth type of eccentric harmonic index $H_4(\Gamma_R)$ of Γ_R , we get $\phi(\epsilon_u, \epsilon_v) = \frac{2}{\epsilon_u + \epsilon_v}$, therefore $\phi(2, 2) = \frac{1}{2}$, $\phi(2, 3) = \frac{2}{5}$ and $\phi(3, 3) = \frac{1}{3}$. Thus,

$$\begin{aligned} H_4(\Gamma_R) &= \left(\frac{1}{2} + \frac{2}{5}q\right) (q-1)(p_2 + p_1 - 2) + \left(\frac{1}{2}(q-2) + 2(q-1)(p_1 p_2 - 1)\frac{2}{5}\right) \frac{q-1}{2} \\ &\quad + \left(\frac{1}{2} + \frac{2}{5}(2q^2 + q - 3) + \frac{1}{3}(2q^2 - 3q + 1)\right) (p_2 - 1)(p_1 - 1) \\ &= \frac{4q + 5}{10}(q-1)(p_2 + p_1 - 2) + \frac{(q-1)}{2}\left(\frac{4}{5}p_1 p_2 q - \frac{4}{5}p_1 p_2 - \frac{3}{10}q - \frac{1}{5}\right) \\ &\quad + \left(\frac{22}{15}q^2 - \frac{3}{5}q - \frac{11}{30}\right) (p_2 - 1)(p_1 - 1). \end{aligned}$$

□

5.2. Case 2: $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$

In this subsection, we discuss the eccentric topological indices for zero divisor graph Y_R defined on commutative ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, where p, q are prime numbers.

Theorem 4. Let p and q be prime numbers and Y_R be the zero divisor graph of commutative ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. Then $|V(Y_R)| = p^2 q + p q^2 - p q - 1$ and $|E(Y_R)| = \frac{(p-1)(q-1)(5pq-2)+2p^2q^2-p^2q-pq^2-2p-2q+4}{2}$.

Proof. For any two prime numbers p and q , we consider the ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ with usual operations. The associated zero divisor graph Y_R on ring R is defined as: Any $(u, v) \in V(Y_R)$ if and only if the elements $u = kp$, $0 \leq k \leq p-1$ or $v = tq$, $0 \leq t \leq q-1$ with $(u, v) \neq (0, 0)$. Therefore, a non zero

element (u, v) is not a zero divisor in the ring R if and only if u is not a multiple of p and v is not a multiple of q . As the number of elements in $\mathbb{Z}_{p^2}$ which are not a multiple of p are $p^2 - p$ and the number of elements in $\mathbb{Z}_{q^2}$ which are not a multiple of q are $q^2 - q$. Also, the element $(0, 0)$ is not a zero divisor in R . Thus the total number of non zero divisors in the ring R is $(p^2 - p)(q^2 - q) + 1$. This implies

$$|V(Y_R)| = p^2q^2 - ((p^2 - p)(q^2 - q) + 1) = p^2q + pq^2 - pq - 1.$$

For any $(u, v) \in V(Y_R)$, we discuss the degree of each vertex as follows:

1. If $u = 0$ and v is not a multiple of q , then vertex $(0, v)$ of this form is connected to each vertex $(u', 0)$, where $u' \in \mathbb{Z}_{p^2} \setminus \{0\}$. Hence each vertex $(0, v)$ has degree $p^2 - 1$. By symmetry each vertex $(u, 0)$ such that u is not a multiple of p has degree $q^2 - 1$.
2. If $u = 0$ and $v \in \{q, 2q, \dots, (q-1)q\}$, then vertex $(0, v)$ of this form is connected to each vertex (u', v') , where $u' \in \mathbb{Z}_{p^2}$ and $v' \in \{0, q, 2q, \dots, (q-1)q\}$ with $(u', v') \notin \{(0, 0), (0, v)\}$. Hence the each vertex of this type $(0, v)$ has degree $p^2q - 2$. By symmetry the degree of each vertex $(u, 0)$ such that $u \in \{p, 2p, \dots, (p-1)p\}$ is $pq^2 - 2$.
3. If u is not a multiple of p and $v \in \{q, 2q, \dots, (q-1)q\}$, then each vertex (u, v) is connected to each vertex $(0, v')$, $v' \in \{q, 2q, \dots, (q-1)q\}$. Hence each vertex (u, v) of this type has degree $q - 1$. By symmetry the degree of each vertex (u, v) if $u \in \{p, 2p, \dots, (p-1)p\}$ and v is not a multiple of q is $p - 1$.
4. If $u \in \{p, 2p, \dots, (p-1)p\}$ and $v \in \{q, 2q, \dots, (q-1)q\}$, then each vertex (u, v) of this form is connected to each vertex (u', v') , where $u' \in \{0, p, 2p, \dots, (p-1)p\}$ and $v' \in \{0, q, 2q, \dots, (q-1)q\}$ with $(u', v') \notin \{(0, 0), (0, v)\}$. Hence the degree of each vertex (u, v) is $pq - 2$.

From the above discussion, it is easy to see that the zero divisor graph Y_R contains $q^2 - q$ vertices of degree $p^2 - 1$, $p^2 - p$ vertices of degree $q^2 - 1$, $p - 1$ vertices of degree $pq^2 - 2$, $q - 1$ vertices of degree $qp^2 - 2$, $q(p - 1)(q - 1)$ vertices of degree $p - 1$, $p(p - 1)(q - 1)$ vertices of degree $q - 1$ and $(p - 1)(q - 1)$ vertices of degree $pq - 2$. By using the hand shaking lemma the number of edges of Y_R are $|E(Y_R)| = \frac{1}{2}((p^2 - 1)(q^2 - q) + (q^2 - 1)(p^2 - p) + (pq^2 - 2)(p - 1) + (qp^2 - 2)(q - 1) + (p - 1)q(p - 1)(q - 1) + (q - 1)p(p - 1)(q - 1) + (pq - 2)(p - 1)(q - 1)) = \frac{(p-1)(q-1)(5pq-2)+2p^2q^2-p^2q-pq^2-2p-2q+4}{2}$. $\square$

Let $\mathfrak{S}_i$ denotes the set of containing the vertices of degree i . From the proof of Theorem 4, we get $|\mathfrak{S}_{p^2-1}| = q^2 - q$, $|\mathfrak{S}_{q^2-1}| = p^2 - p$, $|\mathfrak{S}_{pq^2-2}| = p - 1$, $|\mathfrak{S}_{qp^2-2}| = q - 1$, $|\mathfrak{S}_{p-1}| = q(p - 1)(q - 1)$, $|\mathfrak{S}_{q-1}| = p(p - 1)(q - 1)$ and $|\mathfrak{S}_{pq-2}| = (p - 1)(q - 1)$. The following theorem determines the eccentricity of the vertices in graph of Y_R .

Theorem 5. Let p and q be two prime numbers and Y_R be the zero divisor graph of commutative ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. Then the eccentricity of a vertex of Y_R is either 2 or 3.

Proof. Similar proof of Theorem 2, we get $\epsilon(\mathfrak{S}_{pq^2-2}) = \epsilon(\mathfrak{S}_{qp^2-2}) = \epsilon(\mathfrak{S}_{p-1}) = 2$ and $\epsilon(\mathfrak{S}_{p^2-1}) = \epsilon(\mathfrak{S}_{q^2-1}) = \epsilon(\mathfrak{S}_{p-1}) = \epsilon(\mathfrak{S}_{q-1}) = 3$. $\square$

Lemma 2. Let p and q be two prime numbers and Y_R be the zero divisor graph of commutative ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. Then

$$T(Y_R) = \frac{p^2q + pq^2 - 4pq + 2}{2} \phi(2, 2) + pq(p - 1)(q - 1) \left(\frac{\phi(2, 2)}{2} + 2\phi(2, 3) + \phi(3, 3) \right).$$

Proof. The graph Y_R contains $p^2q + pq^2 - pq - 1$ vertices and $\frac{(p-1)(q-1)(5pq-2)+2p^2q^2-p^2q-pq^2-2p-2q+4}{2}$ edges. Let us divide the edges of Y_R into partition sets with respect to the degree of endpoints of each edge as:

$$\Xi_{r,s} = \{uv \in E(Y_R) : \epsilon_u = r, \epsilon_v = s\}.$$

This means that the set $\Xi_{r,s}$ contains the edges incident with one vertex of eccentricity r and the other vertex of eccentricity s . From the Theorem 4 and Theorem 5, we get $|\Xi_{2,2}| = \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + \frac{(p-1)(q-1)(pq+p+q-2)}{2}$, $|\Xi_{2,3}| = 2pq(p-1)(q-1)$, $|\Xi_{3,3}| = pq(p-1)(q-1)$ and $E(Y_R) = \Xi_{2,2} \cup \Xi_{2,3} \cup \Xi_{3,3}$. Then

$$\begin{aligned} T(Y_R) &= \sum_{uv \in E(Y_R)} \phi(\epsilon_u, \epsilon_v) \\ &= \sum_{uv \in \Xi_{2,2}} \phi(2,2) + \sum_{uv \in \Xi_{2,3}} \phi(2,3) + \sum_{uv \in \Xi_{3,3}} \phi(3,3) \\ &= \left\{ \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + \frac{(p-1)(q-1)(pq+p+q-2)}{2} \right\} \phi(2,2) \\ &\quad + 2pq(p-1)(q-1)\phi(2,3) + pq(p-1)(q-1)\phi(3,3) \\ &= \frac{p^2q + pq^2 - 4pq + 2}{2} \phi(2,2) + pq(p-1)(q-1) \left(\frac{\phi(2,2)}{2} + 2\phi(2,3) + \phi(3,3) \right). \end{aligned}$$

□

Now we present values of eccentric topological indices based on edges of graph Y_R .

Theorem 6. Let p and q be two prime numbers and Y_R be the zero divisor graph of commutative ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. Then the First Zagreb eccentricity index of Y_R is:

$$M_1^*(Y_R) = 2(p^2q + pq^2 - 4pq + 2) + 18pq(p-1)(q-1)$$

the third Zagreb eccentricity index is:

$$M_3^*(Y_R) = 2(p^2q + pq^2 - 4pq + 2) + 23pq(p-1)(q-1)$$

the geometric-arithmetic eccentricity index is:

$$GA_4(Y_R) = \frac{p^2q + pq^2 - 4pq + 2}{2} + pq(p-1)(q-1) \left(\frac{15 + 8\sqrt{6}}{10} \right)$$

the atom-bond connectivity eccentricity index is:

$$ABC_5(Y_R) = \frac{p^2q + pq^2 - 4pq + 2}{2\sqrt{2}} + pq(p-1)(q-1) \left(\frac{15 + 4\sqrt{2}}{6\sqrt{2}} \right)$$

the fourth type of eccentric harmonic index is:

$$H_4(Y_R) = \frac{p^2q + pq^2 - 4pq + 2}{4} + \frac{83}{60}pq(p-1)(q-1).$$

Proof. For the first Zagreb eccentricity index $M_1^*(Y_R)$ of Y_R we obtain $\phi(\epsilon_u, \epsilon_v) = \epsilon_u + \epsilon_v$. So $\phi(2,2) = 4$, $\phi(2,3) = 5$ and $\phi(3,3) = 6$. Thus by Lemma 2,

$$\begin{aligned} M_1^*(Y_R) &= \frac{p^2q + pq^2 - 4pq + 2}{2} \phi(2,2) + pq(p-1)(q-1) \left(\frac{\phi(2,2)}{2} + 2\phi(2,3) + \phi(3,3) \right) \\ &= 2(p^2q + pq^2 - 4pq + 2) + 18pq(p-1)(q-1). \end{aligned}$$

For the third Zagreb eccentricity index $M_3^*(Y_R)$ of Y_R we obtain $\phi(\epsilon_u, \epsilon_v) = \epsilon_u \times \epsilon_v$. So $\phi(2,2) = 4$, $\phi(2,3) = 6$ and $\phi(3,3) = 9$. So by Lemma 2,

$$M_3^*(Y_R) = \frac{p^2q + pq^2 - 4pq + 2}{2} \phi(2,2) + pq(p-1)(q-1) \left(\frac{\phi(2,2)}{2} + 2\phi(2,3) + \phi(3,3) \right) \\ = 2(p^2q + pq^2 - 4pq + 2) + 23pq(p-1)(q-1).$$

For the geometric-arithmetic eccentricity index $GA_4(Y_R)$ of Y_R , we get $\phi(\epsilon_u, \epsilon_v) = \frac{2\sqrt{\epsilon_u \times \epsilon_v}}{\epsilon_u + \epsilon_v}$. So $\phi(2,2) = 1$, $\phi(2,3) = \frac{2\sqrt{6}}{5}$ and $\phi(3,3) = 1$. Therefore,

$$GA_4(Y_R) = \frac{p^2q + pq^2 - 4pq + 2}{2} \phi(2,2) + pq(p-1)(q-1) \left(\frac{\phi(2,2)}{2} + 2\phi(2,3) + \phi(3,3) \right) \\ = \frac{p^2q + pq^2 - 4pq + 2}{2} + pq(p-1)(q-1) \left(\frac{15 + 8\sqrt{6}}{10} \right).$$

For the atom-bond connectivity eccentricity index $ABC_5(Y_R)$ of Y_R , we obtain $\phi(\epsilon_u, \epsilon_v) = \sqrt{\frac{\epsilon_u + \epsilon_v - 2}{\epsilon_u \times \epsilon_v}}$. Thus $\phi(2,2) = \phi(2,3) = \frac{1}{\sqrt{2}}$ and $\phi(3,3) = \frac{2}{3}$. Therefore,

$$ABC_5(Y_R) = \frac{p^2q + pq^2 - 4pq + 2}{2} \phi(2,2) + pq(p-1)(q-1) \left(\frac{\phi(2,2)}{2} + 2\phi(2,3) + \phi(3,3) \right) \\ = \frac{p^2q + pq^2 - 4pq + 2}{2\sqrt{2}} + pq(p-1)(q-1) \left(\frac{15 + 4\sqrt{2}}{6\sqrt{2}} \right)$$

For the fourth type of eccentric harmonic index $H_4(Y_R)$ of Y_R , we get $\phi(\epsilon_u, \epsilon_v) = \frac{2}{\epsilon_u + \epsilon_v}$, therefore $\phi(2,2) = \frac{1}{2}$, $\phi(2,3) = \frac{2}{5}$ and $\phi(3,3) = \frac{1}{3}$. Thus,

$$H_4(Y_R) = \frac{p^2q + pq^2 - 4pq + 2}{2} \phi(2,2) + pq(p-1)(q-1) \left(\frac{\phi(2,2)}{2} + 2\phi(2,3) + \phi(3,3) \right) \\ = \frac{p^2q + pq^2 - 4pq + 2}{4} + \frac{83}{60} pq(p-1)(q-1).$$

□

6. Conclusions

This article emphasizes the computation of first Zagreb eccentricity index, third Zagreb eccentricity index, geometric-arithmetic eccentricity index, atom-bond connectivity eccentricity index and the fourth type of eccentric harmonic index for zero divisor graphs associated with algebraic structures that are commutative rings $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_{q^2}$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. This study will be useful for the further comprehension of the attributes of various physical structures like carbohydrates, silicone structures, polymers, hexagonal chains and cylindrical fullerenes. Likewise, they can be used for creating a productive physical structure in mechanics as well as for different computer network problems.

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