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Article

Schur-Power Convexity of a Completely Symmetric Function Dual

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Received: 24 May 2019; Accepted: 8 July 2019; Published: 10 July 2019

Abstract: In this paper, by applying the decision theorem of the Schur-power convex function, the Schur-power convexity of a class of complete symmetric functions are studied. As applications, some new inequalities are established.

Keywords: Schur-power convexity; Schur-convexity; Schur-geometric convexity; Schur-harmonic convexity; completely symmetric function; dual form

MSC: Primary 05E05; 26B25; 26D15

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1. Introduction and Preliminaries

Convexity is a natural notion and plays an important and fundamental role in mathematics, physics, chemistry, biology, economics, engineering, and other sciences. To solve practical problems, several interesting concepts of generalized convexity or generalized concavity have been introduced and studied. Recent important investigations and developments in convex analysis have focused on the study of Schur-convexity, and Schur-geometric and Schur-harmonic convexity of various symmetric functions; see, e.g., [1–20] and references therein. It is worth mentioning that discovering and judging Schur-convexity of various symmetric functions is an important topic in the study of the majorization theory. A lot of achievements in this field have been investigated by several authors; for more details, see the first author's monographs [21,22].

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the set of positive integers and real numbers, respectively. Let X be a nonempty set. Denote $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$. For a positive integer n, the set X^n for the Cartesian product is the collection of all n-tuples of elements of X. Therefore, we can write $\mathbb{R}^n$, $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ as follows:

$$X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}},$$

where $X \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-\}$.

Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$ in $\mathbb{R}^n$. A set $D \subset \mathbb{R}^n$ is said to be *convex* if $\boldsymbol{x}, \boldsymbol{y} \in D$ and $0 \le \alpha \le 1$ imply

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \cdots, \alpha x_n + (1-\alpha)y_n) \in D.$$

Let $D \subset \mathbb{R}^n$ be a convex set. A function $f : D \to \mathbb{R}$ is said to be *convex* on D if

$$f\left(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}\right) \le \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in D$, and all $\alpha \in [0,1]$. The function f is said to be *concave* on D if and only if $-f$ is convex on D.

For the reader's convenience and explicit later use, we now recall some basic definitions and notation that will be needed in this paper.

Definition 1 (see [23,24]). (i) *A set $\Omega \subset \mathbb{R}^n$ is called symmetric, if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.*

(ii) *A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.*

Definition 2 (see [23,24]). *Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.*

(i) *$\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$.*

(ii) *Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.*

Definition 3 (see [23,24]). *Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.*

(i) *$\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \cdots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.*

(ii) *Let $\Omega \subset \mathbb{R}^n$, the function φ: $\Omega \to \mathbb{R}$ is said to be Schur-convex on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is a Schur-convex function on Ω.*

The following useful characterizations of Schur-convex and Schur-concave functions were established in [23,24].

Lemma 1 (see [23,24]). *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω° is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω°. Then φ is a Schur-convex (or Schur-concave, respectively) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \ (or \leq 0, \ respectively) \tag{1}$$

holds for any $\boldsymbol{x} \in \Omega^\circ$.

In 1923, Professor Issai Schur made the first systematic study of the functions preserving the ordering of majorization. In Schur's honor, such functions are said to be "Schur-convex". It is known that Schur-convexity can be applied extensively in analytic inequalities, combinatorial optimization, quantum physics, information theory, and other related fields (see, e.g., [23]).

Definition 4 (see [25,26]). *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$.*

(i) *A set $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x},\boldsymbol{y} \in \Omega$ and $\alpha,\beta \in [0,1]$ such that $\alpha + \beta = 1$.*

(ii) *Let $\Omega \subset \mathbb{R}^n_+$. The function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex on Ω if $(\log x_1, \log x_2, \ldots, \log x_n) \prec (\log y_1, \log y_2, \ldots, \log y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur-geometrically concave on Ω if and only if $-\varphi$ is Schur-geometrically convex on Ω.*

Lemma 2. (*Schur-geometrically convex function decision theorem*) *[25,26] Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric and geometrically convex set with a nonempty interior Ω°. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. If φ is symmetric on Ω and*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (or \leq 0, respectively) \tag{2}$$

holds for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$, then φ is a Schur-geometrically convex (or Schur-geometrically concave, respectively) function.

The Schur-geometric convexity was first proposed and studied by Zhang [25] in 2004 and was widely investigated and improved by many authors, see [27–29] and references therein. We also note that some authors use the term "Schur multiplicative convexity".

In 2009, Chu [1–3] introduced the notion of Schur-harmonically convex function and established some interesting inequalities for Schur-harmonically convex functions.

Definition 5 (see [1]). *Let $\Omega \subset \mathbb{R}^n_+$ or $\Omega \subset \mathbb{R}^n_-$.*

(i) *A set Ω is said to be harmonically convex if $\dfrac{\boldsymbol{xy}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ and $\dfrac{1}{\boldsymbol{x}} = (\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots, \dfrac{1}{x_n})$.*

(ii) *A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-harmonically convex on Ω if $\dfrac{1}{\boldsymbol{x}} \prec \dfrac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur-harmonically concave function on Ω if and only if $-\varphi$ is a Schur-harmonically convex function.*

Lemma 3. *(Schur-harmonically convex function decision theorem) [1] Let $\Omega \subset \mathbb{R}^n_+$ or $\Omega \subset \mathbb{R}^n_-$ be a symmetric and harmonically convex set with inner points and let $\varphi : \Omega \to \mathbb{R}$ be a continuously symmetric function which is differentiable on Ω°. Then φ is Schur-harmonically convex (or Schur-harmonically concave, respectively) on Ω if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right) \geq 0 \quad (or\ \leq 0, respectively), \quad \boldsymbol{x} \in \Omega^\circ. \tag{3}$$

In 2010, Yang [30] defined and introduced the concepts of the Schur-f-convex function and Schur-power convex function which are the generalization and unification of the concepts of Schur-convexity, Schur-geometric convexity, and Schur-harmonic convexity. He established useful characterizations of Schur m-power convex functions and presented their important properties; see [30].

Definition 6 (see [30]). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \neq 0; \\ \ln x, & m = 0. \end{cases} \tag{4}$$

Then a function $\varphi : \Omega \subset \mathbb{R}^n_+ \to \mathbb{R}$ is said to be Schur m-power convex on Ω if

$$(f(x_1), f(x_2), \ldots, f(x_n)) \prec (f(y_1), f(y_2), \ldots, f(y_n))$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \Omega$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$.

If $-\varphi$ is Schur m-power convex, then we say that φ is Schur m-power concave.

Lemma 4 (see [30]). *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric set with nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m} \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad if\ m \neq 0 \tag{5}$$

and

$$(\log x_1 - \log x_2)\left[x_1 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad if\ m = 0 \tag{6}$$

for all $\boldsymbol{x} \in \Omega^\circ$.

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, recall that the complete symmetric function $c_n(\boldsymbol{x}, r)$ is defined by

$$c_n(\boldsymbol{x}, r) = \sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \tag{7}$$

where $c_0(\boldsymbol{x}, r) = 1,\ r \in \{1, 2, \ldots, n\},\ \ i_1, i_2, \ldots, i_n$ are non-negative integers.

The collection of complete symmetric functions is an important class of symmetric functions which has been investigated by many mathematicians and there are many interesting results in the literature.

In 2006, Guan [5] discussed the Schur-convexity of $c_n(\boldsymbol{x}, r)$ and proved the following result.

Proposition 1. *$c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex on $\mathbb{R}^n_+$.*

Subsequently, Chu et al. [2] established the following proposition.

Proposition 2. *$c_n(\boldsymbol{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}^n_+$.*

In 2016, Shi et al. [19] further studied the Schur-convexity of $c_n(\boldsymbol{x}, r)$ on $\mathbb{R}^n_-$ and presented the following important result.

Proposition 3 (see [19])**.** *If r is even integer (or odd integer, respectively), then $c_n(\boldsymbol{x}, r)$ is decreasing and Schur-convex (or increasing and Schur-concave, respectively) on $\mathbb{R}^n_-$.*

Recall that the dual form of the complete symmetric function $c_n(\boldsymbol{x}, r)$ is defined by

$$c_n^*(\boldsymbol{x}, r) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j x_j, \tag{8}$$

where $c_0^*(\boldsymbol{x}, r) = 1,\ r \in \{1, 2, \ldots, n\},\ \ i_1, i_2, \ldots, i_n$ are non-negative integers.

In 2013, Zhang and Shi [18] established the following two interesting propositions.

Proposition 4 (see [18])**.** *For $r = 1, 2, \ldots, n$, $c_n^*(\boldsymbol{x}, r)$ is increasing and Schur-concave on $\mathbb{R}^n_+$.*

Proposition 5 (see [18])**.** *For $r = 1, 2, \ldots, n$, $c_n^*(\boldsymbol{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}^n_+$.*

Notice that

$$c_n^*(-\boldsymbol{x}, r) = (-1)^r c_n^*(\boldsymbol{x}, r).$$

It is not difficult to prove the following result.

Proposition 6. *If r is even integer (or odd integer, respectively), then $c_n^*(\boldsymbol{x}, r)$ is decreasing and Schur-concave (or increasing and Schur-convex, respectively) on $\mathbb{R}^n_-$.*

In 2014, Sun et al. [6] studied the Schur-convexity, Schur-geometric and harmonic convexities of the following composite function of $c_n(\boldsymbol{x}, r)$:

$$c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} \left(\frac{x_j}{1-x_j}\right)^{i_j}. \tag{9}$$

By using Lemmas 1–3, they proved the following Theorems 1–3, respectively.

Theorem 1. *For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in (0,1)^n \cup (1, +\infty)^n$ and $r \in \mathbb{N}$,*

(*i*) $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is increasing and Schur-convex on* $(0,1)^n$*;*
(*ii*) *if r is even integer (or odd integer, respectively), then* $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-convex (or Schur-concave, respectively) on* $(1,+\infty)^n$*, and is decreasing (or increasing, respectively).*

Theorem 2. *For* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in (0,1)^n \cup (1,+\infty)^n$ *and* $r \in \mathbb{N}$*,*

(*i*) $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-geometrically convex on* $(0,1)^n$*;*
(*ii*) *if r is even integer (or odd integer, respectively), then* $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-geometrically convex (or Schur-geometrically concave, respectively) on* $(1,+\infty)^n$*.*

Theorem 3. *For* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in (0,1)^n \cup (1,+\infty)^n$ *and* $r \in \mathbb{N}$*,*

(*i*) $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-harmonically convex on* $(0,1)^n$*;*
(*ii*) *if r is even integer (or odd integer, respectively), then* $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-harmonically convex (or Schur-harmonically concave, respectively) on* $(1,+\infty)^n$*.*

In 2016, Shi et al. [19] applied the properties of Schur-convex, Schur-geometrically convex, and Schur-harmonically convex functions respectively to give simple proofs of Theorems 1–3.

Recall that the dual form of the function $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is defined by

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{x_j}{1-x_j}\right). \tag{10}$$

A function associated with this function is

$$c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{x_j}{x_j-1}\right). \tag{11}$$

In this work, we will establish some important results for the Schur-power convexity of symmetric functions $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)$ and $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$. As their applications, some new inequalities are obtained in Section 3.

2. Main Results

The following lemmas are very crucial for our main results.

Lemma 5. *Let* $m \geq -1$*. For* $x_1, x_2 \in (1,+\infty)$ *and* $x_1 > x_2$*, we have*

$$x_1(x_1-1)x_2^{1-m} \geq x_2(x_2-1)x_1^{1-m}; \tag{12}$$

$$(x_1-1)^2 x_2^{1-m} \geq (x_2-1)^2 x_1^{1-m}; \tag{13}$$

$$\frac{(x_1-1)^2 x_2^{2-m}}{x_2-1} \geq \frac{(x_2-1)^2 x_1^{2-m}}{x_1-1}. \tag{14}$$

Proof. Since

$$((t-1)t^m)' = t^{m-1}[t(m+1)-m] \geq t^{m-1}[(m+1)-m] \geq 0, \ for \ t > 1,$$

we have

$$(x_1-1)x_1^m \geq (x_2-1)x_2^m, \ for \ x_1 > x_2.$$

This inequality is equivalent to inequality (12). Since

$$\left(\frac{(t-1)^2}{t^{(1-m)}}\right)' = \frac{(t-1)[t(m+1)+1-m]}{t^m t^{[2(1-m)]}}$$

$$\geq \frac{(t-1)[(m+1)+1-m]}{t^m t^{[2(1-m)]}} \geq 0, \ for \ t > 1,$$

we obtain

$$\frac{(x_1-1)^2}{x_1^{(1-m)}} \geq \frac{(x_2-1)^2}{x_2^{(1-m)}}, \ for \ x_1 > x_2.$$

This inequality is equivalent to inequality (13). Since

$$\begin{aligned}\left(\frac{(t-1)^3}{t^{(2-m)}}\right)' &= \frac{(t-1)^2 t^{1-m}[t(m+1)+2-m]}{t^{[2(2-m)]}} \\ &\geq \frac{(t-1)^2 t^{1-m}[(m+1)+2-m]}{t^{[2(2-m)]}} \geq 0, \ for \ t > 1,\end{aligned}$$

we get

$$\frac{(x_1-1)^3}{m x_1^{(2-m)}} \geq \frac{(x_2-1)^3}{m x_2^{(2-m)}}, \ for \ x_1 > x_2.$$

This inequality is equivalent to inequality (14). □

Lemma 6. *Let $m \leq 0$. For $x_1, x_2 \in (0,1)$ and $x_1 > x_2$, we have*

$$x_1^{1-m} x_2(1-x_2) \leq x_2^{1-m} x_1(1-x_1); \tag{15}$$

$$x_1^{1-m}(1-x_2)^2 \leq x_2^{1-m}(1-x_1)^2; \tag{16}$$

$$\frac{(1-x_2)^2 x_1^{2-m}}{1-x_1} \leq \frac{(1-x_1)^2 x_2^{2-m}}{1-x_2}. \tag{17}$$

Proof. Since

$$((1-t)t^m)' = t^{m-1}[m(1-t)-t] = t^{m-1}[m-t(m+1)] \leq 0, \ for \ t \in (0,1),$$

we get

$$(1-x_1)x_1^m \leq (1-x_2)x_2^m, \ for \ x_1 > x_2.$$

This inequality is equivalent to inequality (15). Since

$$\left(\frac{(1-t)^2}{t^{(1-m)}}\right)' = \frac{(t-1)[m(1-t)-(t+1)]}{t^m t^{[2(1-m)]}} \leq 0, \ for \ t \in (0,1),$$

we obtain

$$\frac{(1-x_1)^2}{x_1^{(1-m)}} \geq \frac{(1-x_2)^2}{x_2^{(1-m)}}, \ for \ x_1 > x_2.$$

This inequality is equivalent to inequality (16). Since

$$\left(\frac{(1-t)^3}{t^{(2-m)}}\right)' = \frac{(1-t)^2 t^{1-m}[m(1-t)-2(1+t)]}{t^{[2(2-m)]}} \leq 0, \ for \ t \in (0,1),$$

we have

$$\frac{(1-x_1)^3}{m x_1^{(2-m)}} \geq \frac{(1-x_2)^3}{m x_2^{(2-m)}}, \ for \ x_1 > x_2.$$

This inequality is equivalent to inequality (17). □

Now, we establish the following new result for the Schur-power convexity of $c_n^*((x/(x-1)), r)$.

Theorem 4. *Let $r \in \mathbb{N}$. If $m \geq -1$, then $c_n^*\left(\frac{x}{x-1}, r\right)$ is decreasing and Schur m-power convex on $(1, +\infty)^n$.*

Proof. Let $q(t) = \frac{t}{t-1}$. Then

$$q'(t) = -\frac{1}{(t-1)^2}, \quad q''(t) = \frac{2}{(t-1)^3}. \tag{18}$$

From Proposition 4, we know that $c_n^*(x, r)$ is increasing on $\mathbb{R}_+^n$, but $q(t)$ is decreasing on $\mathbb{R}$, therefore, the function $c_n^*\left(\frac{x}{x-1}, r\right)$ is decreasing on $(1, +\infty)^n$.

For $r = 1$ and $r = 2$, it is easy to prove that $c_n^*\left(\frac{x}{x-1}, r\right)$ is Schur m-power convex on $(1, +\infty)^n$. Now consider the case of $r \geq 3$. By the symmetry of $c_n^*\left(\frac{x}{x-1}, r\right)$, without loss of generality, we may assume $x_1 > x_2$. So

$$c_n^*\left(\frac{x}{x-1}, r\right) = \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2=0}} \sum_{j=1}^n \frac{i_j x_j}{x_j - 1} \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1=0, i_2\neq 0}} \sum_{j=1}^n \frac{i_j x_j}{x_j - 1}$$
$$\times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2\neq 0}} \sum_{j=1}^n \frac{i_j x_j}{x_j - 1} \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1=0, i_2=0}} \sum_{j=1}^n \frac{i_j x_j}{x_j - 1}.$$

Then we have

$$\begin{aligned}
\frac{\partial c_n^*\left(\frac{x}{x-1}, r\right)}{\partial x_1} &= c_n^*\left(\frac{x}{x-1}, r\right) \\
&\times \left(\sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2=0}} \frac{-i_1}{(x_1-1)^2 \sum_{j=1}^n \frac{i_j x_j}{x_j-1}} + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2\neq 0}} \frac{-i_1}{(x_1-1)^2 \sum_{j=1}^n \frac{i_j x_j}{x_j-1}} \right) \\
&= c_n^*\left(\frac{x}{x-1}, r\right) \Big(\sum_{\substack{k+k_3+\ldots+k_n=r \\ k\neq 0}} \frac{-k}{(x_1-1)^2 (\frac{k x_1}{x_1-1} + \sum_{j=3}^n \frac{k_j x_j}{x_j-1})} \\
&+ \sum_{\substack{k+m+i_3+\cdots+i_n=r \\ k\neq 0, m\neq 0}} \frac{-k}{(x_1-1)^2 (\frac{k x_1}{x_1-1} + \frac{m x_2}{x_2-1} + \sum_{j=3}^n \frac{k_j x_j}{x_j-1})} \Big).
\end{aligned} \tag{19}$$

By the same arguments, we get

$$\begin{aligned}
\frac{\partial c_n^*\left(\frac{x}{x-1}, r\right)}{\partial x_2} &= c_n^*\left(\frac{x}{x-1}, r\right) \Big(\sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \frac{-k}{(x_2-1)^2 (\frac{k x_2}{x_2-1} + \sum_{j=3}^n \frac{k_j x_j}{x_j-1})} \\
&+ \sum_{\substack{k+m+i_3+\cdots+i_n=r \\ k\neq 0, m\neq 0}} \frac{-k}{(x_2-1)^2 (\frac{k x_2}{x_2-1} + \frac{m x_1}{x_1-1} + \sum_{j=3}^n \frac{k_j x_j}{x_j-1})} \Big),
\end{aligned} \tag{20}$$

then, it follows from (19) and (20) that

$$x_1^{1-m} \frac{\partial c_n^*\left(\frac{x}{x-1}, r\right)}{\partial x_1} - x_2^{1-m} \frac{\partial c_n^*\left(\frac{x}{x-1}, r\right)}{\partial x_2} = c_n^*\left(\frac{x}{x-1}, r\right) (C_1 + C_2),$$

where

$$C_1 = \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \left(\frac{-kx_1^{1-m}}{(x_1-1)^2(\frac{kx_1}{x_1-1} + \sum\limits_{j=3}^{n} \frac{k_jx_j}{x_j-1})} - \frac{-kx_2^{1-m}}{(x_2-1)^2(\frac{kx_2}{x_2-1} + \sum\limits_{j=3}^{n} \frac{k_jx_j}{x_j-1})} \right)$$
$$= k \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \frac{\lambda_1}{(x_1-1)^2(\frac{kx_1}{x_1} + \sum\limits_{j=3}^{n} \frac{k_jx_j}{x_j-1})(x_2-1)^2(\frac{kx_2}{x_2-1} + \sum\limits_{j=3}^{n} \frac{k_jx_j}{x_j-1})}$$

with

$$\lambda_1 = k[x_1(x_1-1)x_2^{1-m} - x_2(x_2-1)x_1^{1-m}] + [(x_1-1)^2x_2^{1-m} - (x_2-1)^2x_1^{1-m}] \sum_{j=3}^{n} \frac{k_jx_j}{x_j-1}$$

and

$$C_2 = \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \Big(\frac{-kx_1^{1-m}}{(x_1-1)^2(\frac{kx_1}{x_1-1} + \frac{mx_2}{x_2-1} + \sum\limits_{j=3}^{n} \frac{k_jx_j}{x_j-1})} - \frac{-kx_2^{1-m}}{(x_2-1)^2(\frac{kx_2}{x_2-1} + \frac{mx_1}{x_1-1} + \sum\limits_{j=3}^{n} \frac{k_jx_j}{x_j-1})} \Big)$$
$$= k \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \frac{\lambda_2}{(x_1-1)^2(\frac{kx_1}{x_1-1} + \frac{mx_2}{x_2-1} + \sum\limits_{j=3}^{n} \frac{k_jx_j}{x_j-1})(x_2-1)^2(\frac{kx_2}{x_2-1} + \frac{mx_1}{x_1-1} + \sum\limits_{j=3}^{n} \frac{k_jx_j}{x_j-1})},$$

with

$$\lambda_2 = k[x_1(x_1-1)x_2^{1-m} - x_2(x_2-1)x_1^{1-m}] + m \left[\frac{(x_1-1)^2x_2^{2-m}}{x_2-1} - \frac{(x_2-1)^2x_1^{2-m}}{x_1-1} \right]$$
$$+ [(x_1-1)^2x_2^{1-m} - (x_2-1)^2x_1^{1-m}] \sum_{j=3}^{n} \frac{k_jx_j}{x_j-1}.$$

By Lemma 5, it is easy to see that $C_1 \geq 0$ and $C_2 \geq 0$ for $\boldsymbol{x} \in (1, +\infty)^n$, so

$$x_1^{1-m} \frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_1} - x_2^{1-m} \frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_2} \geq 0.$$

By Lemma 4, we prove that $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)$ is Schur m-Power convex on $(1, +\infty)^n$ for $m \geq -1$. The proof is completed. □

Next, we present some new results for the Schur-power convexity of $c_n^*((x/(1-x)), r)$.

Theorem 5. *Let $r \in \mathbb{N}$.*

(i) $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is increasing on $\mathbb{R}_+^n$ and Schur-convex on $[\frac{1}{2}, 1)^n$;
(ii) If $m \leq 0$, then $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-m-power convex on $(0, 1)^n$;
(iii) For $m \geq -1$, if r is even integer (or odd integer, respectively), then $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-m-power convex (or Schur-m-power concave, respectively) on $(1, +\infty)^n$.

Proof. (i) Let $p(t) = \frac{t}{1-t}$. Then

$$p'(t) = \frac{1}{(1-t)^2}, \quad p''(t) = \frac{2}{(1-t)^3}. \tag{21}$$

From Proposition 4, we know that $c_n^*(\boldsymbol{x}, r)$ is increasing on $\mathbb{R}_+^n$, but $p(t)$ is increasing on $\mathbb{R}$, therefore, the function $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is increasing on $\mathbb{R}_+^n$.

For the case of $r = 1$ and $r = 2$, it is easy to prove that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-convex on $[\frac{1}{2}, 1)^n$.

Now consider the case of $r \geq 3$. By the symmetry of $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$, without loss of generality, we may assume $x_1 > x_2$. So

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2=0}} \sum_{j=1}^{n} \frac{i_j x_j}{1-x_j} \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1=0, i_2\neq 0}} \sum_{j=1}^{n} \frac{i_j x_j}{1-x_j}$$
$$\times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2\neq 0}} \sum_{j=1}^{n} \frac{i_j x_j}{1-x_j} \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1=0, i_2=0}} \sum_{j=1}^{n} \frac{i_j x_j}{1-x_j}.$$

Then we obtain

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_1} = c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$$
$$\times \left(\sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2=0}} \frac{i_1}{(1-x_1)^2 \sum\limits_{j=1}^{n} \frac{i_j x_j}{1-x_j}} + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2\neq 0}} \frac{i_1}{(1-x_1)^2 \sum\limits_{j=1}^{n} \frac{i_j x_j}{1-x_j}} \right)$$
$$= c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) \Big(\sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \frac{k}{(1-x_1)^2 (\frac{kx_1}{1-x_1} + \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j})} \tag{22}$$
$$+ \sum_{\substack{k+m+i_3+\cdots+i_n=r \\ k\neq 0, m\neq 0}} \frac{k}{(1-x_1)^2 (\frac{kx_1}{1-x_1} + \frac{mx_2}{1-x_2} + \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j})} \Big). \tag{23}$$

By the same arguments,

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_2} = c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) \Big(\sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \frac{k}{(1-x_2)^2 (\frac{kx_2}{1-x_2} + \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j})}$$
$$+ \sum_{\substack{k+m+i_3+\cdots+i_n=r \\ k\neq 0, m\neq 0}} \frac{k}{(1-x_2)^2 (\frac{kx_2}{1-x_2} + \frac{mx_1}{1-x_1} + \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j})} \Big), \tag{24}$$

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_1} - \frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_2} = c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) (D_1 + D_2),$$

where

$$D_1 = \sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \left(\frac{k}{(1-x_1)^2 (\frac{kx_1}{1-x_1} + \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j})} - \frac{k}{(1-x_2)^2 (\frac{kx_2}{1-x_2} + \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j})} \right)$$
$$= k \sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \frac{k(x_1+x_2-1)(x_1-x_2) + (x_1-x_2)(2-x_1-x_2) \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j}}{(1-x_1)^2 (\frac{kx_1}{1-x_1} + \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j})(1-x_2)^2 (\frac{kx_2}{1-x_2} + \sum\limits_{j=3}^{n} \frac{k_j x_j}{1-x_j})}$$

and

$$D_2=\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\Big(\frac{k}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}-\frac{k}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}\Big)$$

$$=k\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\frac{\delta_1}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}$$

with

$$\delta_1=k(x_1+x_2-1)(x_1-x_2)+\left(\frac{(1-x_2)^2mx_1}{1-x_1}-\frac{(1-x_1)^2mx_2}{1-x_2}\right)+(x_1-x_2)(2-x_1-x_2)\sum_{j=3}^{n}\frac{k_jx_j}{1-x_j}.$$

Let $q(t)=\frac{(1-t)^3}{mt}$. Then $q'(t)=-\frac{m(1+2t)(1-t)^2}{m^2t^2}\leq 0$ which implies that $q(t)$ is descending on $\mathbb{R}_+$. So that $\frac{(1-x_1)^3}{mx_1}\leq\frac{(1-x_2)^3}{mx_2}$, namely $\frac{(1-x_2)^2mx_1}{1-x_1}-\frac{(1-x_1)^2mx_2}{1-x_2}\geq 0$. It is easy to see that $D_1\geq 0$ and $D_2\geq 0$ for $\boldsymbol{x}\in[\frac{1}{2},1)^n$, so

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_1}-\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_2}\geq 0.$$

By Lemma 1, we obtain $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ is Schur-convex on $[\frac{1}{2},1)^n$.

(ii) For $r=1$ and $r=2$, it is easy to prove that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ is Schur m-power convex on $(0,1)^n$.

Now consider the case of $r\geq 3$. By the symmetry of $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$, without loss of generality, we may assume $x_1>x_2$. From (22) and (24), we have

$$x_1^{1-m}\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_1}-x_2^{1-m}\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_2}=c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)(F_1+F_2),$$

where

$$F_1=\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}}\Big(\frac{kx_1^{1-m}}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}-\frac{kx_2^{1-m}}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}\Big)$$

$$=k\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}}\frac{\delta_1}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}$$

with

$$\delta_1=k[x_1^{1-m}x_2(1-x_2)-x_2^{1-m}x_1(1-x_1)]+[x_1^{1-m}(1-x_2)^2-x_2^{1-m}(1-x_1)^2]\sum_{j=3}^{n}\frac{k_jx_j}{1-x_j}$$

and

$$F_2=\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\Big(\frac{kx_1^{1-m}}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}-\frac{kx_2^{1-m}}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}\Big)$$

$$=k\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\frac{\delta_2}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}$$

with

$$\delta_2 = k[x_1^{1-m}x_2(1-x_2) - x_2^{1-m}x_1(1-x_1)] + m\left[\frac{(1-x_2)^2x_1^{2-m}}{1-x_1} - \frac{(1-x_1)^2x_2^{2-m}}{1-x_2}\right]$$
$$+ ([x_1^{1-m}(1-x_2)^2 - x_2^{1-m}(1-x_1)^2]\sum_{j=3}^{n}\frac{k_jx_j}{1-x_j}.$$

By Lemma 6, it is easy to see that $F_1 \geq 0$ and $F_2 \geq 0$ for $\boldsymbol{x} \in (0,1)^n$, and then

$$x_1^{1-m}\frac{\partial c_n^*\left(\frac{x}{1-x}, r\right)}{\partial x_1} - x_2^{1-m}\frac{\partial c_n^*\left(\frac{x}{1-x}, r\right)}{\partial x_2} \geq 0.$$

By Lemma 4, we show that $c_n^*\left(\frac{x}{1-x}, r\right)$ is Schur-m power convex on $(0,1)^n$.

(iii) Notice that

$$c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right) = (-1)^r c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right), \tag{25}$$

and combining with the Schur-power convexity of $c_n^*\left(\frac{x}{x-1}, r\right)$ on $(1,+\infty)^n$ (see Theorem 4), we can prove (iii). The proof is completed. □

According to the relationship between the Schur-power convex function and the Schur-convex function, the Schur-geometrically convex function, and the Schur-harmonically function, we can establish the following two corollaries immediately.

Corollary 1. *Let $r \in \mathbb{N}$. Then $c_n^*\left(\frac{x}{x-1}, r\right)$ is Schur-convex, Schur-geometrically convex, and Schur-harmonically convex on $(1,+\infty)^n$.*

Corollary 2. *Let $r \in \mathbb{N}$.*

(i) The function $c_n^*\left(\frac{x}{1-x}, r\right)$ is Schur-geometrically convex and Schur-harmonically convex on $(0,1)^n$.
(ii) If r is even integer (or odd integer, respectively), then $c_n^*\left(\frac{x}{1-x}, r\right)$ is Schur-convex, Schur-geometric convex, and Schur-harmonic convex (or Schur-concave, Schur-geometric concave, and Schur-harmonic concave, respectively) on $(1,+\infty)^n$.

Finally, an open problem arises naturally at the end of this section.

Problem 1. *For $\boldsymbol{x} \in \left(0, \frac{1}{2}\right)^n$, what is the Schur-convexity of $c_n^*\left(\frac{x}{1-x}, r\right)$? Is it Schur-convex or Schur-concave, or is it uncertain?*

3. Some Applications

It is not difficult to prove the following theorem by applying Corollary 2 and the majorizing relation

$$(A_n(\boldsymbol{x}), A_n(\boldsymbol{x}), \ldots, A_n(\boldsymbol{x})) \prec (x_1, x_2, \ldots, x_n).$$

Theorem 6. *If $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in [\frac{1}{2}, 1)^n$ and $r \in \mathbb{N}$, or r is even integer and $\boldsymbol{x} \in (1,+\infty)^n$, then*

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) \geq \left(\frac{rA_n(\boldsymbol{x})}{1-A_n(\boldsymbol{x})}\right)^{\binom{n+r-1}{r}}, \tag{26}$$

where $A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!((n+r-1)-r)!}$.

If r is odd and $\boldsymbol{x} \in (1,+\infty)^n$, then the inequality (26) is reversed.

By Corollary 2 and the majorizing relation

$$(\log G_n(\boldsymbol{x}), \log G_n(\boldsymbol{x}), \ldots, \log G_n(\boldsymbol{x})) \prec (\log x_1, \log x_2, \ldots, \log x_n),$$

we can establish the following result.

Theorem 7. *If $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in (0,1)^n$ and $r \in \mathbb{N}$ or r is even integer $\boldsymbol{x} \in (1, +\infty)^n$, then*

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) \geq \left(\frac{rG_n(\boldsymbol{x})}{1-G_n(\boldsymbol{x})}\right)^{\binom{n+r-1}{r}}, \tag{27}$$

where $G_n(\boldsymbol{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$ and $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!((n+r-1)-r)!}$.

If r is odd integer and $\boldsymbol{x} \in (1, +\infty)^n$, then the inequality (27) is reversed.

By using Corollary 2 and the majorizing relation

$$\left(\frac{1}{H_n(\boldsymbol{x})}, \frac{1}{H_n(\boldsymbol{x})}, \ldots, \frac{1}{H_n(\boldsymbol{x})}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right),$$

we obtain the following theorem.

Theorem 8. *If $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in (0,1)^n$ and $r \in \mathbb{N}$, or r is even integer and $\boldsymbol{x} \in (1, +\infty)^n$, then*

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) \geq \left(\frac{rH_n(\boldsymbol{x})}{1-H_n(\boldsymbol{x})}\right)^{\binom{n+r-1}{r}}, \tag{28}$$

where $H_n(\boldsymbol{x}) = \frac{n}{\sum_{i=1}^n x_i^{-1}}$ and $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!((n+r-1)-r)!}$.

If r is odd and $\boldsymbol{x} \in (1, +\infty)^n$, then the inequality (28) is reversed.

4. Conclusions

In this paper, we establish the following two important main results of this paper for the Schur-power convexity of symmetric functions $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)$ and $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$:

- (see Theorem 4) Let $r \in \mathbb{N}$. If $m \geq -1$, then $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)$ is decreasing and Schur m-power convex on $(1, +\infty)^n$.
- (see Theorem 5) Let $r \in \mathbb{N}$.
  - (i) $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is increasing on $\mathbb{R}_+^n$ and Schur-convex on $[\frac{1}{2}, 1)^n$;
  - (ii) If $m \leq 0$, then $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-m-power convex on $(0,1)^n$;
  - (iii) For $m \geq -1$, if r is even integer (or odd integer, respectively), then $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-m-power convex (or Schur-m-power concave, respectively) on $(1, +\infty)^n$.

As applications of our new results, some new inequalities are presented in Section 3.

Author Contributions: Both authors contributed equally to this work. Both authors read and approved the final manuscript.

Funding: The second author is supported by Grant No. MOST 107-2115-M-017-004-MY2 of the Ministry of Science and Technology of the Republic of China.

Acknowledgments: The authors wish to express their hearty thanks to the anonymous referees for their valuable suggestions and comments.

Conflicts of Interest: The authors declare no conflict of interest.

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