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On Some Formulas for Kaprekar Constants

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Abstract: Let $b \geq 2$ and $n \geq 2$ be integers. For a b-adic n-digit integer x, let A (resp. B) be the b-adic n-digit integer obtained by rearranging the numbers of all digits of x in descending (resp. ascending) order. Then, we define the *Kaprekar transformation* $T_{(b,n)}(x) := A - B$. If $T_{(b,n)}(x) = x$, then x is called a *b-adic n-digit Kaprekar constant*. Moreover, we say that a b-adic n-digit Kaprekar constant x is *regular* when the numbers of all digits of x are distinct. In this article, we obtain some formulas for regular and non-regular Kaprekar constants, respectively. As an application of these formulas, we then see that for any integer $b \geq 2$, the number of b-adic odd-digit regular Kaprekar constants is greater than or equal to the number of all non-trivial divisors of b. Kaprekar constants have the symmetric property that they are fixed points for recursive number theoretical functions $T_{(b,n)}$.

Keywords: Kaprekar constants; Kaprekar transformation; fixed points for recursive functions

MSC: 2010: 11A99; 11P99

1. Introduction

Let $\mathbb{Z}$ be the set of all rational integers. In this article, the symbol $[\alpha]$ with any rational number α stands for the greatest integer that is less than or equal to α.

For integers $b \geq 2$ and $n \geq 2$, we denote by $\mathbb{Z}(b,n)$ the set of all b-adic n-digit integers, i.e.,

$$\begin{aligned}\mathbb{Z}(b,n) &= \{x \in \mathbb{Z} \mid 0 \leq x \leq b^n - 1\} \\ &= \{a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 \mid 0 \leq a_0, a_1, \ldots, a_{n-1} \leq b-1\}.\end{aligned}$$

For any:

$$x = a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 \in \mathbb{Z}(b,n)$$

with $0 \leq a_0, a_1, \ldots, a_{n-1} \leq b-1$, we denote the b-adic expression of x by:

$$x = (a_{n-1} \cdots a_1 a_0)_b.$$

In the case where $b = 10$, we omit the subscript as:

$$x = a_{n-1} \cdots a_1 a_0$$

as usual if any confusion occurs with the product of $a_0, a_1, \ldots, a_{n-1}$.

Definition 1. *Let $c_{n-1} \geq \cdots \geq c_1 \geq c_0$ be the rearrangement of the numbers $a_0, a_1, \ldots, a_{n-1}$ of all digits of $x \in \mathbb{Z}(b,n)$ in descending order. We define the Kaprekar transformation as:*

$$T_{(b,n)} : \mathbb{Z}(b,n) \to \mathbb{Z}(b,n); \quad x \mapsto (c_{n-1} \cdots c_1 c_0)_b - (c_0 c_1 \cdots c_{n-1})_b.$$

Definition 2. (1) *For any $x \in \mathbb{Z}(b,n)$, we say that x is a b-adic n-digit Kaprekar constant if $T_{(b,n)}(x) = x$.*

(2) *We see immediately that zero is a b-adic n-digit Kaprekar constant for any $b \geq 2$ and $n \geq 2$, which we call the trivial Kaprekar constant. Then, we denote by $\nu(b,n)$ the number of all b-adic n-digit non-trivial Kaprekar constants. By Ref. [1] (Proposition 1.3), we see that:*

$$\nu(b,n) \leq {}_{b-1+[\frac{n}{2}]}C_{[\frac{n}{2}]} - 1,$$

where we put:

$${}_rC_s := \frac{r!}{s!(r-s)!} = \frac{r(r-1)\cdots(r-s+1)}{s\cdots 1}$$

for any integers $r > s > 0$.

(3) *We say that a b-adic n-digit non-trivial Kaprekar constant $x = (a_{n-1}\cdots a_1 a_0)_b$ is regular when $a_i \neq a_j$ for any $i \neq j$. We denote by $\nu_{\text{reg}}(b,n)$ (resp. $\nu_{\text{non-reg}}(b,n)$) the number of all b-adic n-digit regular (resp. non-regular) Kaprekar constants. By the definition, we see immediately that:*

$$\nu(b,n) = \nu_{\text{reg}}(b,n) + \nu_{\text{non-reg}}(b,n)$$

and if $b < n$, then $\nu_{\text{reg}}(b,n) = 0$ and $\nu(b,n) = \nu_{\text{non-reg}}(b,n)$.

Example 1. *Kaprekar [2,3], who was the initiator of this research, discovered that $\nu(10,4) = 1$, and the only non-trivial 10-adic four-digit Kaprekar constant is: 6174.*

Example 2. *Here is the list of all b-adic n-digit non-trivial Kaprekar constants for $2 \leq b \leq 15$ and $2 \leq n \leq 7$. Note that, in the list below, we omit the subscript b. Further, the symbol $-$ means that $\nu(b,n) = 0$, and non-trivial Kaprekar constants with the symbol $*$ are regular.*

n	2	3	4	5	6	7
$b = 2$	01*	011	0111 1001	01111 10101	011111 101101 110001	0111111 1011101 1101001
3	–	–	–	20211	–	2202101
4	–	132*	3021*	–	213312 310221 330201	3203211
5	13*	–	3032	–	–	–
6	–	253*	–	41532*	325523 420432 530421*	–
7	–	–	–	–	–	–
8	25*	374*	–	–	437734 640632	6417532*
9	–	–	–	62853*	–	–
10	–	495*	6174*	–	549945 631764	–
11	37*	–	–	–	–	–
12	–	5(11)6*	–	83(11)74*	65(11)(11)56	962(11)853*
13	–	–	–	–	951(10)74*	–
14	49*	6(13)7*	–	–	76(13)(13)67	–
15	–	–	92(11)6*	(10)4(14)95*	–	–

Then, we obtain the following list of the numbers $\nu = \nu(b,n)$*,* $\nu_{\mathrm{r}} = \nu_{\mathrm{reg}}(b,n)$ *and* $\nu_{\mathrm{nr}} = \nu_{\text{non-reg}}(b,n)$*:*

	$n=2$			$n=3$			$n=4$			$n=5$			$n=6$			$n=7$		
b	ν	ν_{r}	ν_{nr}	ν	ν_{r}	ν_{nr}	ν	ν_{r}	ν_{nr}	ν	ν_{r}	ν_{nr}	ν	ν_{r}	ν_{nr}	ν	ν_{r}	ν_{nr}
2	1	1	0	1	0	1	2	0	2	2	0	2	3	0	3	3	0	3
3	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	1
4	0	0	0	1	1	0	1	1	0	0	0	0	3	0	3	1	0	1
5	1	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
6	0	0	0	1	1	0	0	0	0	1	1	0	3	1	2	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	1	0	1	1	0	0	0	0	0	0	0	2	0	2	1	1	0
9	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
10	0	0	0	1	1	0	1	1	0	0	0	0	2	0	2	0	0	0
11	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	1	1	1	0	0	0	1	1	0	1	0	1	1	1	0
13	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
14	1	1	0	1	1	0	0	0	0	0	0	0	1	0	1	0	0	0
15	0	0	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0

Now, we have the following:

Questions: (1) Are there any formulas for $\nu(b,n)$, $\nu_{\mathrm{reg}}(b,n)$ and $\nu_{\text{non-reg}}(b,n)$ in terms of b and n?

(2) Are there any formulas for b-adic n-digit regular or non-regular Kaprekar constants in terms of b and n?

Known results: There are some known results that answer some parts of the questions above as follows:

(1) In the case where $n = 2$, by the results on the two-digit Kaprekar transformation given by Young [4] (cf. [1], Theorem 3.1), we see that for any integer $b \geq 2$, there exists a b-adic two-digit non-trivial Kaprekar constant if and only if $b+1$ is divisible by three.

Since there is no two-digit non-regular Kaprekar constant by definition, we see immediately that for any integer $b \geq 2$,

$$\nu_{\text{non-reg}}(b,2) = 0 \quad \text{and} \quad \nu(b,2) = \nu_{\mathrm{reg}}(b,2).$$

In this article, we shall prove in Theorem 4(1) and Corollary 3(1) that any two-digit regular Kaprekar constant is of the form:

$$(m(2m+1))_{3m+2}$$

with any integers $m \geq 0$ and:

$$\nu_{\mathrm{reg}}(b,2) = \begin{cases} 1 & \text{if } 3 \mid (b+1), \\ 0 & \text{otherwise.} \end{cases}$$

(2) In the case where $n = 3$, Eldridge and Sagong [5] proved that any three-digit non-trivial Kaprekar constant is of the form:

$$(m(2m+1)(m+1))_{2m+2}$$

with any integers $m \geq 0$ and that for any integer $b \geq 2$,

$$\nu(b,3) = \begin{cases} 1 & \text{if } b \text{ is even}, \\ 0 & \text{if } b \text{ is odd.} \end{cases}$$

In particular, we see immediately that:

$$\nu_{\text{reg}}(b,3) = \begin{cases} 1 & \text{if } b \geq 4 \text{ is even,} \\ 0 & \text{if } b = 2 \text{ or } b \geq 3 \text{ is odd,} \end{cases}$$

and:

$$\nu_{\text{non-reg}}(b,3) = \begin{cases} 1 & \text{if } b = 2, \\ 0 & \text{if } b \geq 3. \end{cases}$$

(3) In the case where $n = 4$, Hasse and Prichett [6] obtained a formula:

$$((3m+3)m(4m+3)(2m+2))_{5m+5}$$

for $(5m+5)$-adic four-digit non-trivial Kaprekar constants with any integer $m \geq 0$. This implies that if $b \geq 5$ and $5 \mid b$, then $\nu_{\text{reg}}(b,4) \geq 1$.

In this article, we shall prove in Theorem 4(2) and Corollary 3(2) that any four-digit regular Kaprekar constant is equal to $(3021)_4$ or given by the above formula obtained by Hasse and Prichett with $m \geq 1$ and that for any integer $b \geq 2$,

$$\nu_{\text{reg}}(b,4) = \begin{cases} 1 & \text{if } b = 4 \text{ or, } b \geq 10 \text{ and } 5 \mid b, \\ 0 & \text{otherwise.} \end{cases}$$

(4) In the case where $n = 5$, Prichett [7] obtained a formula:

$$((2m+2)m(3m+2)(2m+1)(m+1))_{3m+3}$$

for $(3m+3)$-adic five-digit non-trivial Kaprekar constants with any integers $m \geq 0$. This implies that if $b \geq 6$ and $3 \mid b$, then $\nu_{\text{reg}}(b,5) \geq 1$.

In this article, we shall prove in Theorem 3(1) and Corollary 3(3) that any five-digit regular Kaprekar constant is given by the above formula obtained by Prichett with $m \geq 1$ and that for any integer $b \geq 2$,

$$\nu_{\text{reg}}(b,5) = \begin{cases} 1 & \text{if } b \geq 6 \text{ and } 3 \mid b, \\ 0 & \text{otherwise.} \end{cases}$$

(5) In the case where $b = 2$, the first author [1] showed that for any $n \geq 2$, all two-adic n-digit non-trivial Kaprekar constants are of the form:

$$(\overbrace{1\cdots1}^{k-1}\;0\;\overbrace{1\cdots1}^{n-2k}\;\overbrace{0\cdots0}^{k-1}1)_2$$

with all integers $1 \leq k \leq \left[\dfrac{n}{2}\right]$ and $\nu(2,n) = \left[\dfrac{n}{2}\right]$. In particular, we see immediately that:

$$\nu_{\text{reg}}(2,n) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3 \end{cases}$$

and:

$$\nu_{\text{non-reg}}(2,n) = \begin{cases} 0 & \text{if } n = 2, \\ \left[\dfrac{n}{2}\right] & \text{if } n \geq 3. \end{cases}$$

(6) In the case where $b = 3$, the authors [8] showed that for any $n \geq 2$, all three-adic n-digit non-trivial Kaprekar constants are of the form:

$$(\overbrace{2\cdots 2}^{k}\overbrace{1\cdots 1}^{\ell-k-1}0\overbrace{2\cdots 2}^{\ell-k}\overbrace{1\cdots 1}^{\ell-k}\overbrace{0\cdots 0}^{k-1}1)_3.$$

with all pairs (k,ℓ) of integers satisfying $0 < k < \ell$ and $n = 3\ell - k$, and:

$$\nu(3,n) = \left[\frac{1}{6}\left(n - \frac{1+3(-1)^n}{2}\right)\right].$$

In particular, we see immediately that:

$$\nu_{\text{reg}}(3,n) = 0, \quad \nu_{\text{non-reg}}(3,n) = \nu(3,n).$$

We have the impression that the behavior of the values of $\nu(b,n)$, $\nu_{\text{reg}}(b,n)$ and $\nu_{\text{non-reg}}(b,n)$ in the list in Example 2 is not only complicated, but also suggestive of some general rules. It seems that it is very hard to obtain general results without observing any case-by-case results. The aim of this article is to see formulas for b-adic n-digit regular and non-regular Kaprekar constants and to study the properties of $\nu_{\text{reg}}(b,n)$ and $\nu_{\text{non-reg}}(b,n)$ towards answers to the questions above.

Firstly, we see formulas for Kaprekar constants in the following:

Theorem 1. *Let $m \geq 0$ and $n \geq 2$ be any integers. We put:*

$$b(m,n) = \begin{cases} 3m+2 & \text{if } n = 2, \\ 2^{\frac{n-4}{2}}(4m+3) + m + 2 & \text{if } n \text{ is even and } n \geq 4, \\ \dfrac{n+1}{2}(m+1) & \text{if } n \text{ is odd.} \end{cases}$$

(1) *We assume that n is even and define the $b(m,n)$-adic n-digit integer:*

$$K(m,n) = \begin{cases} (m(2m+1))_{b(m,2)} & \text{if } n = 2, \\ ((3m+3)m(4m+3)(2m+2))_{b(m,4)} & \text{if } n = 4, \\ (a_{n-1}a_{n-2}\cdots a_i \cdots a_{\frac{n}{2}+1}a_{\frac{n}{2}}a_{\frac{n}{2}-1}\cdots a_j \cdots a_1 a_0)_{b(m,n)} & \text{if } n \geq 6, \end{cases}$$

where we put:

$$\begin{aligned} &a_{n-1} = 2^{\frac{n-4}{2}}(4m+3) - m, \\ &a_i = (2^{\frac{n-4}{2}} - 2^{n-i-2})(4m+3) + m + 1 \qquad \text{for } n-2 \geq i \geq \frac{n}{2}+1, \\ &a_{\frac{n}{2}} = m, \\ &a_j = 2^{j-1}(4m+3) \qquad \text{for } \frac{n}{2} - 1 \geq j \geq 1, \\ &a_0 = 2m+2. \end{aligned}$$

Then, $K(m,n)$ is a non-trivial Kaprekar constant, which is regular if and only if $n = 2$ or $m \geq 1$.

(2) *We assume that n is odd and define the $b(m,n)$-adic n-digit integer:*

$$L(m,n) = \begin{cases} (m(2m+1)(m+1))_{b(m,3)} & \text{if } n = 3, \\ (b_{n-1}\cdots b_i \cdots b_{\frac{n+3}{2}}b_{\frac{n+1}{2}}b_{\frac{n-1}{2}}b_{\frac{n-3}{2}}\cdots b_j \cdots b_1 b_0)_{b(m,n)} & \text{if } n \geq 5, \end{cases}$$

where we put:

$$
\begin{aligned}
&b_i = \left(i - \frac{n-1}{2}\right)(m+1) \qquad \text{for } n-1 \geq i \geq \frac{n+3}{2},\\
&b_{\frac{n+1}{2}} = m,\\
&b_{\frac{n-1}{2}} = \frac{n+1}{2}(m+1) - 1,\\
&b_j = (j+1)(m+1) - 1 \left(= b_{\frac{n+1}{2}+j} - 1\right) \qquad \text{for } \frac{n-3}{2} \geq j \geq 1,\\
&b_0 = m+1.
\end{aligned}
$$

Then, $L(m,n)$ is a non-trivial Kaprekar constant, which is regular if and only if $m \geq 1$.

Remark 1. (1) *We can see that for any integer $n \geq 2$, the sequence:*

$$b(n) := \{b(m,n) \mid m = 0,1,2,\ldots\}$$

consisting of bases defined in Theorem 1 is the arithmetic progression with the common difference:

$$
\begin{cases}
3 & \text{if } n = 2,\\
2^{\frac{n}{2}} + 1 & \text{if } n \text{ is even and } n \geq 4,\\
\dfrac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
$$

and the first term:

$$
\begin{cases}
2 & \text{if } n = 2,\\
3 \times 2^{\frac{n-4}{2}} + 2 & \text{if } n \text{ is even and } n \geq 4,\\
\dfrac{n+1}{2} & \text{if } n \text{ is odd.}
\end{cases}
$$

(2) *As we have seen in the known results above, the regular Kaprekar constants $K(m,4)$, $L(m,3)$, and $L(m,5)$ have already been obtained by Hasse and Prichett [6], Eldridge and Sagong [5], and Prichett [7], respectively.*

Definition 3. (1) *We call the double series:*

$$
\begin{aligned}
K &:= \{K(m,n) \mid m = 1,2,3,\ldots, n = 2,4,6,\ldots\},\\
L &:= \{L(m,n) \mid m = 1,2,3,\ldots, n = 3,5,7,\ldots\}
\end{aligned}
$$

the systems of regular Kaprekar constants.

(2) *Let $n \geq 2$ be any integer. We call the sequence:*

$$
\begin{aligned}
K(n) &:= \{K(m,n) \mid m = 1,2,3,\ldots\} \quad \text{with even } n, \text{ or}\\
L(n) &:= \{L(m,n) \mid m = 1,2,3,\ldots\} \quad \text{with odd } n
\end{aligned}
$$

the progression of n-digit regular Kaprekar constants with arithmetic progression $b(n) \setminus \{b(0,n)\}$ of bases.

By Theorem 1, we see that the formulas for the numbers $a_{n-1},\ldots,a_0$ (resp. $b_{n-1},\ldots,b_0$) of digits of members in $K(n)$ (resp. $L(n)$) are given by polynomials in m of degree one. This implies that they can be regarded as arithmetic progressions indexed by $m = 1,2,3,\ldots$, as well as the arithmetic progression $b(n) \setminus \{b(0,n)\}$ of bases.

(3) *Let $m \geq 1$ be any integer. We call the sequences:*

$$K[m] := \{K(m,n) \mid n = 2,4,6,\ldots\}$$
$$(\text{resp. } L[m] := \{L(m,n) \mid n = 3,5,7,\ldots\})$$

the m-th chain of regular Kaprekar constants in the system K (resp. L) with ascending even (resp. odd) digits.

Example 3. (1) *Here are examples of some members $K(m,n)$ in the progressions $K(n)$ and the chains $K[m]$ of regular Kaprekar constants with $1 \leq m \leq 5$ and $n = 2,4,6$.*

	$K(2)$	$K(4)$	$K(6)$
$K[1]$	$(13)_5$	$(6174)_{10}$	$((13)91(14)74)_{17}$
$K[2]$	$(25)_8$	$(92(11)6)_{15}$	$((20)(14)2(22)(11)6)_{26}$
$K[3]$	$(37)_{11}$	$((12)3(15)8)_{20}$	$((27)(19)3(30)(15)8)_{35}$
$K[4]$	$(49)_{14}$	$((15)4(19)(10))_{25}$	$((34)(24)4(38)(19)(10))_{44}$
$K[5]$	$(5(11))_{17}$	$((18)5(23)(12))_{30}$	$((41)(29)5(46)(23)(12))_{53}$

(2) *Here are examples of some members $L(m,n)$ in the progressions $L(n)$ and the chains $L[m]$ of regular Kaprekar constants with $1 \leq m \leq 5$ and $n = 3,5,7$.*

	$L(3)$	$L(5)$	$L(7)$
$L[1]$	$(132)_4$	$(41532)_6$	$(6417532)_8$
$L[2]$	$(253)_6$	$(62853)_9$	$(962(11)853)_{12}$
$L[3]$	$(374)_8$	$(83(11)74)_{12}$	$((12)83(15)(11)74)_{16}$
$L[4]$	$(495)_{10}$	$((10)4(14)95)_{15}$	$((15)(10)4(19)(14)95)_{20}$
$L[5]$	$(5(11)6)_{12}$	$((12)5(17)(11)6)_{18}$	$((18)(12)5(23)(17)(11)6)_{24}$

Remark 2. *By the cases where $n = 4$ and $n = 6$ in the lists in Examples 2 and 3, we see that the progressions $K(n)$ and $L(n)$ of n-digit regular Kaprekar constants may not consist of all n-digit regular Kaprekar constants in general. Actually, for any $n \geq 2$, it seems that it is very hard to obtain formulas for all n-digit regular Kaprekar constants. In Section 2, we obtain some partial results on them with specified n.*

As a corollary of Theorem 1, we immediately obtain some results on the positivity of the numbers $\nu_{\text{reg}}(b,n)$ of all b-adic n-digit regular Kaprekar constants as in the following:

Corollary 1. (1) *Let $n \geq 2$ and $b \geq 2$ be any integers. If n and b satisfy one of the following conditions:*

(i) *$n = 2$ and $b = 3m + 2$ with $m \geq 1$,*

(ii) *n is even, $n \geq 4$ and $b = 2^{\frac{n-4}{2}}(4m+3) + m + 2$ with $m \geq 1$,*

(iii) *n is odd and $b = \dfrac{n+1}{2}(m+1)$ with $m \geq 1$,*

then:

$$\nu_{\text{reg}}(b,n) \geq 1.$$

(2) *If an integer $b \geq 4$ is not a prime number, then for any non-trivial divisor d of b,*

$$\nu_{\text{reg}}(b, 2d-1) \geq 1.$$

Therefore, the number of all b-adic odd-digit regular Kaprekar constants is greater than or equal to the number of all non-trivial divisors of b.

Secondly, we obtain formulas for non-regular Kaprekar constants by means of double series of regular Kaprekar constants obtained in Theorem 1 in the following:

Theorem 2. *Let the notation be as in Theorem 1.*

(1) *We assume that* $m \equiv 1 \pmod 3$ *and* $n \equiv 0 \pmod 4$, *and put:*

$$\beta_{m,n} = \frac{b(m,n)-1}{3}.$$

For any integer $r \geq 2$, *we denote by* $K(m,n,r)$ *the* $b(m,n)$*-adic* $(n+2r)$*-digit integer:*

$$\left((3m+3)\,\overbrace{\beta_{m,4}\cdots\beta_{m,4}}^{r}\,m(4m+3)\,\overbrace{(2\beta_{m,4})\cdots(2\beta_{m,4})}^{r}(2m+2) \right)_{b(m,4)}$$

in the case where $n = 4$, *and:*

$$\left(a_{n-1}\cdots a_{\frac{n}{2}+1}\,\overbrace{\beta_{m,n}\cdots\beta_{m,n}}^{r}\,a_{\frac{n}{2}}\,a_{\frac{n}{2}-1}\,\overbrace{(2\beta_{m,n})\cdots(2\beta_{m,n})}^{r}\,a_{\frac{n}{2}-2}\cdots a_0 \right)_{b(m,n)}$$

in the case where $n \geq 8$. *Then,* $K(m,n,r)$ *is a non-regular Kaprekar constant.*

(2) *We assume that* $m = 1$, $n \equiv 3 \pmod 6$ *and* $n \geq 9$. *For any integer* $r \geq 2$, *we denote by* $L(1,n,r)$ *the* $b(1,n)(= n+1)$*-adic* $(n+2r)$*-digit integer:*

$$\left(b_{n-1}\cdots b_{\frac{2n}{3}}\,\overbrace{\frac{n}{3}\cdots\frac{n}{3}}^{r}\,b_{\frac{2n}{3}-1}\cdots b_{\frac{n}{3}}\,\overbrace{\frac{2n}{3}\cdots\frac{2n}{3}}^{r}\,b_{\frac{n}{3}-1}\cdots b_0 \right)_{b(1,n)}$$

$$\left(= T_{(b(1,n),n)}\left(n\cdots\frac{2n+3}{3}\,\overbrace{\frac{2n}{3}\cdots\frac{2n}{3}}^{r}\,\frac{2n}{3}\cdots\frac{n+3}{3}\,\overbrace{\frac{n}{3}\cdots\frac{n}{3}}^{r}\,\frac{n}{3}\cdots 1 \right) \right).$$

Then, $L(1,n,r)$ *is a non-regular Kaprekar constant.*

Example 4. (1) *Here is an example of the non-regular constant* $K(m,n,r)$ *obtained in Theorem 2(1) in the case where* $m = 4$, $n = 8$, *and* $r = 2$.

(m,n)	$(4,8)$
$b(m,n)$	82
$K(m,n)$	$((72)(62)(43)4(76)(38)(19)(10))_{82}$
$\beta_{m,n},\ 2\beta_{m,n}$	27, 54
$K(m,n,r)$	$((72)(62)(43)(27)(27)4(76)(54)(54)(38)(19)(10))_{82}$

(2) *Here is an example of the non-regular constant* $L(1,n,r)$ *obtained in Theorem 2(2) in the case where* $n = 9$ *and* $r = 4$.

$(1,n)$	$(1,9)$
$b(1,n)$	10
$L(1,n)$	$(864197532)_{10}$
$\frac{n}{3},\ \frac{2n}{3}$	3, 6
$L(1,n,r)$	$(86433331976666532)_{10}$

As a corollary of Theorem 2, we immediately obtain the following result on the positivity of the numbers $\nu_{\text{reg}}(b, n)$ of all b-adic n-digit non-regular Kaprekar constants:

Corollary 2. *For any integers $m \geq 1$ and $n \geq 4$ satisfying:*

$$m \equiv 1 \pmod 3 \text{ and } n \equiv 0 \pmod 4$$

or:

$$m = 1,\ n \equiv 3 \pmod 6 \text{ and } n \geq 9,$$

and for any integer $r \geq 2$, we see that:

$$\nu_{\text{non-reg}}(b(m,n), n + 2r) \geq 1.$$

In Section 1, we shall prove Theorems 1 and 2 and Corollaries 1 and 2. In Section 2.1, we shall obtain some formulas for all n-digit regular Kaprekar constants in Theorem 3 for $n = 5, 7, 9, 11$ and Theorem 4 for $n = 2, 4, 6, 8$. Moreover, we shall see some conditional results on formulas for n-digit regular Kaprekar constants in Proposition 1 for $n = 13, 15, 17$. Then, we shall see in Section 2.2 some observations on the values of $\nu_{\text{reg}}(b, n)$. We think that this article is fit for the Special Issue "Number Theory and Symmetry," since Kaprekar constants have the symmetric property that they are fixed points for recursive number theoretical functions $T_{(b,n)}$.

2. Proofs of Theorems and Corollaries in the Introduction

In this section, we prove Theorem 1 and Corollary 1 on regular Kaprekar constants and Theorem 2 and Corollary 2 on non-regular Kaprekar constants, respectively.

2.1. A Proof of Theorem 1

(1) Let the notation be as in Part (1) of Theorem 1. Here, we omit proving the Parts (i)–(iii), since they can be checked by direct calculations.

(iv) In the case where $n \geq 8$ is even, let:

$$K(m,n) = (a_{n-1}a_{n-2}\cdots a_i \cdots a_{\frac{n}{2}+1}a_{\frac{n}{2}}a_{\frac{n}{2}-1}\cdots a_j \cdots a_1 a_0)_{b(m,n)}$$

be the $b(m,n)$-adic n-digit integer defined in the assertion of Theorem 1(1). Let $c_{n-1} \geq \cdots \geq c_1 \geq c_0$ be the rearrangement of the numbers $a_0, \ldots, a_{n-1}$ of all digits of $K(m,n)$ in descending order. Then, the relation between $a_0, \ldots, a_{n-1}$ and $c_0, \ldots, c_{n-1}$ is given as in the following:

Lemma 1. *In the situation above, we see that:*

$$\begin{aligned}
&c_{n-1} = a_{\frac{n}{2}-1},\ c_{n-2} = a_{n-1},\\
&c_i = a_{i+1},\ c_{n-i-1} = a_{n-i-2} \quad \text{for } n-3 \geq i \geq \frac{n}{2},\\
&c_1 = a_0,\ c_0 = a_{\frac{n}{2}}.
\end{aligned}$$

Proof. Since for any $n - 3 \geq i \geq \dfrac{n}{2}$,

$$\begin{aligned}
a_{i+1} &= (2^{\frac{n-4}{2}} - 2^{n-i-3})(4m+3) + m + 1,\\
a_{n-i-2} &= 2^{n-i-3}(4m+3),
\end{aligned}$$

we see easily that:

$$a_{n-2} > a_{n-3} > \cdots > a_{\frac{n}{2}+1}$$

and:
$$a_{\frac{n}{2}-2} > a_{\frac{n}{2}-3} > \cdots > a_1.$$

Moreover,
$$\begin{aligned}
a_{\frac{n}{2}-1} &= 2^{\frac{n-4}{2}}(4m+3) \\
&\geq 2^{\frac{n-4}{2}}(4m+3) - m = a_{n-1} \\
&> 2^{\frac{n-4}{2}}(4m+3) - (3m+2) = a_{n-2}, \\
a_{\frac{n}{2}+1} - a_{\frac{n}{2}-2} &= (2^{\frac{n-4}{2}} - 2^{\frac{n}{2}-3})(4m+3) + (m+1) - 2^{\frac{n}{2}-3}(4m+3) \\
&= m+1 > 0
\end{aligned}$$

and:
$$a_1 = 4m+3 > a_0 = 2m+2 > a_{\frac{n}{2}} = m.$$

Therefore, the lemma is proven. □

We put:
$$T_{(b(m,n),n)}(K(m,n)) = (a'_{n-1} \cdots a'_1 a'_0)_{b(m,n)}$$
with integers $0 \leq a'_0, a'_1, \ldots, a'_{n-1} \leq b(m,n) - 1$. By Ref. [1] (Theorem 1.1 (6)) and Lemma 1, we then see that:
$$\begin{aligned}
a'_{n-1} &= c_{n-1} - c_0 = a_{\frac{n}{2}-1} - a_{\frac{n}{2}} = 2^{\frac{n-4}{2}}(4m+3) - m = a_{n-1}, \\
a'_{n-2} &= c_{n-2} - c_1 = a_{n-1} - a_0 = 2^{\frac{n-4}{2}}(4m+3) - m - (2m+2) \\
&= (2^{\frac{n-4}{2}} - 1)(4m+3) + m + 1 = a_{n-2}, \\
a'_{\frac{n}{2}} &= c_{\frac{n}{2}} - c_{\frac{n}{2}-1} - 1 = a_{\frac{n}{2}+1} - a_{\frac{n}{2}-2} - 1 \\
&= (2^{\frac{n-4}{2}} - 2^{\frac{n}{2}-3})(4m+3) + (m+1) - 2^{\frac{n}{2}-3}(4m+3) - 1 \\
&= m = a_{\frac{n}{2}}, \\
a'_{\frac{n}{2}-1} &= b(m,n) - 1 - (c_{\frac{n}{2}} - c_{\frac{n}{2}-1}) \\
&= 2^{\frac{n-4}{2}}(4m+3) + m + 2 - 1 - (m+1) \\
&= 2^{\frac{n-4}{2}}(4m+3) = a_{\frac{n}{2}-1}, \\
a'_1 &= b(m,n) - 1 - (c_{n-2} - c_1) \\
&= 2^{\frac{n-4}{2}}(4m+3) + m + 2 - 1 - ((2^{\frac{n-4}{2}} - 1)(4m+3) + m + 1) \\
&= 4m+3 = a_1, \\
a'_0 &= b(m,n) - (c_{n-1} - c_0) \\
&= 2^{\frac{n-4}{2}}(4m+3) + m + 2 - (2^{\frac{n-4}{2}}(4m+3) - m) \\
&= 2m+2 = a_0.
\end{aligned}$$

Moreover, we see that for any $n-3 \geq i \geq \dfrac{n}{2}+1$,

$$\begin{aligned} a'_i &= c_i - c_{n-i-1} = a_{i+1} - a_{n-i-2} \\ &= (2^{\frac{n-4}{2}} - 2^{n-i-3})(4m+3) + m + 1 - 2^{n-i-3}(4m+3) \\ &= (2^{\frac{n-4}{2}} - 2^{n-i-2})(4m+3) + m + 1 = a_i, \\ a'_{n-i-1} &= b(m,n) - 1 - a'_i \\ &= 2^{\frac{n-4}{2}}(4m+3) + m + 2 - 1 - ((2^{\frac{n-4}{2}} - 2^{n-i-2})(4m+3) + m + 1) \\ &= 2^{n-i-2}(4m+3) = a_{n-i-1}. \end{aligned}$$

Therefore, we see that:

$$\begin{aligned} T_{(b(m,n),n)}(K(m,n)) &= (a'_{n-1} \cdots a'_1 a'_0)_{b(m,n)} \\ &= (a_{n-1} \cdots a_1 a_0)_{b(m,n)} \\ &= K(m,n), \end{aligned}$$

i.e., $K(m,n)$ is a non-trivial Kaprekar constant, which is regular if and only if $m \geq 1$, which implies that $a_{\frac{n}{2}-1} \neq a_{n-1}$.

(2) Let the notation be as in Part (2) of Theorem 1.

As we have seen in the known results (2) and (4) in the Introduction, the cases where $n = 3$ and $n = 5$ have already been proven by Eldridge and Sagong [5] and Prichett [7], respectively. Therefore, it suffices to prove Part (2) in the case where $n \geq 7$.

For any odd integer $n \geq 7$, let:

$$L(m,n) = (b_{n-1} \cdots b_i \cdots b_{\frac{n+3}{2}} b_{\frac{n+1}{2}} b_{\frac{n-1}{2}} b_{\frac{n-3}{2}} \cdots b_j \cdots b_1 b_0)_{b(m,n)}$$

be the $b(m,n)$-adic n-digit integer defined in the assertion of Theorem 1(2). Let $c_{n-1} \geq \cdots \geq c_1 \geq c_0$ be the rearrangement of the numbers $b_0, \ldots, b_{n-1}$ of all digits of $L(m,n)$ in descending order. Then, the relation between $b_0, b_1, \ldots, b_{n-1}$ and $c_0, c_1, \ldots, c_{n-1}$ is given as in the following:

Lemma 2. *In the situation above, we see that:*

$$\begin{aligned} &c_{n-1} = b_{\frac{n-1}{2}}, \\ &c_{2i-1} = b_{\frac{n-1}{2}+i},\ c_{2i-2} = b_{i-1} \quad \textit{for}\ \frac{n-1}{2} \geq i \geq 2, \\ &c_1 = b_0,\ c_0 = b_{\frac{n+1}{2}}. \end{aligned}$$

Proof. By the definition of the numbers of all digits of $L(m,n)$ in Theorem 1(2), we see immediately that:

$$\begin{aligned} &c_{n-1} = \frac{n+1}{2}(m+1) - 1 = b_{\frac{n-1}{2}}, \\ &c_{2i-1} = i(m+1) = b_{\frac{n-1}{2}+i},\ c_{2i-2} = i(m+1) - 1 = b_{i-1} \text{ for } \frac{n-1}{2} \geq i \geq 2, \\ &c_1 = m+1 = b_0,\ c_0 = m = b_{\frac{n+1}{2}}. \end{aligned}$$

Therefore, the lemma is proven. □

Then, we can prove Part (2) in the case where $n \geq 7$ by the same argument as in the proof of Theorem 1(1)(iv). Therefore, we omit the details of the calculations here.

2.2. A Proof of Corollary 1

(1) In Cases (i) and (ii), we have the $b(m,n)$-adic n-digit regular Kaprekar constant $K(m,n)$ by Theorem 1 (1). On the other hand, in Case (iii), we have the $b(m,n)$-adic n-digit regular Kaprekar constant $L(m,n)$ by Theorem 1(2). Therefore, we see that for any integers $b \geq 2$ and $n \geq 2$ satisfying Condition (i), (ii), or (iii),

$$\nu_{\text{reg}}(b,n) \geq 1,$$

and Part (1) is proven.

(2) For any integer $b \geq 4$ that is not a prime number, let d be any non-trivial divisor of b, i.e., d is a divisor of b satisfying $1 < d < b$. We put:

$$m_d = \frac{b}{d} - 1, \quad n_d = 2d - 1.$$

Since $m_d \geq 1$ is an integer and $n_d \geq 3$ is an odd integer satisfying $b(m_d, n_d) = b$, by Theorem 1(2), we have the b-adic n_d-digit regular Kaprekar constant $L(m_d, n_d)$. Therefore, we see that:

$$\nu_{\text{reg}}(b, n_d) \geq 1.$$

Moreover, since $n_d \neq n_{d'}$ for any non-trivial divisors $d \neq d'$ of b, we see that $L(m_d, n_d) \neq L(m_{d'}, n_{d'})$. Therefore, the number of all b-adic odd-digit regular Kaprekar constants is greater than or equal to the number of all non-trivial divisors of b, and Part (2) is proven.

2.3. A Proof of Theorem 2

(1) We assume that:

$$m \equiv 1 \pmod 3 \quad \text{and} \quad n \equiv 0 \pmod 4.$$

(a) In the case where $n = 4$, $b(m,4) = 5m + 5$, and:

$$\beta_{m,4} = \frac{b(m,4) - 1}{3} = \frac{5m + 4}{3}$$

which is an integer, since the assumption $m \equiv 1 \pmod 3$ implies that:

$$b(m,4) \equiv 2m - 1 \equiv 1 \pmod 3.$$

Then, for any $r \geq 2$, the $b(m,4)$-adic $(2r+4)$-digit integer obtained by rearranging of the numbers of all digits of $K(m,4,r)$ in descending order is:

$$\left((4m+3)\, \overbrace{(2\beta_{m,4}) \cdots (2\beta_{m,4})}^{r} (3m+3)(2m+2)\, \overbrace{\beta_{m,4} \cdots \beta_{m,4}}^{r}\, m \right)_{b(m,4)}.$$

By Ref. [1] (Theorem 1.1 (6)) and the case where $n = 4$ in Theorem 1(1), we then see that:

$$T_{(b(m,4),2r+4)}(K(m,4,r)) = K(m,4,r),$$

since $b(m,4) - 1 - \beta_{m,4} = 2\beta_{m,4}$. Therefore, $K(m,4,r)$ is a non-regular Kaprekar constant.

(b) In the case where $n \geq 8$, $b(m,n) = 2^{\frac{n-4}{2}}(4m+3) + m + 2$, and

$$\beta_{m,m} = \frac{b(m,n) - 1}{3} = \frac{1}{3}\left(2^{\frac{n-4}{2}}(4m+3) + m + 1\right)$$

which is an integer, since $n \equiv 0 \pmod 4$ implies that $\frac{n-4}{2}$ is even and $m \equiv 1 \pmod 3$ implies that:

$$b(m,n) \equiv (-1)^{\frac{n-4}{2}} m + m - 1 \equiv 1 \pmod 3.$$

Let the notation be as in Theorem 1(1). Since, $n \geq 8$, we see that:

$$\begin{aligned}
a_{\frac{n}{2}-2} - \beta_{m,n} &= 2^{\frac{n}{2}-3}(4m+3) - \frac{1}{3}\left(2^{\frac{n-4}{2}}(4m+3) + m + 1\right) \\
&= \left(\frac{2^{\frac{n}{2}}}{6} - \frac{1}{3}\right) m + \frac{2^{\frac{n}{2}}}{8} - \frac{1}{3} > 0, \\
\beta_{m,n} - a_{\frac{n}{2}-3} &= \frac{1}{3}\left(2^{\frac{n-4}{2}}(4m+3) + m + 1\right) - 2^{\frac{n}{2}-4}(4m+3) \\
&= \left(\frac{2^{\frac{n}{2}}}{12} + \frac{1}{3}\right) m + \frac{2^{\frac{n}{2}}}{16} + \frac{1}{3} > 0, \\
a_{\frac{n}{2}+2} - 2\beta_{m,n} &= \left(2^{\frac{n}{2}-2} - 2^{\frac{n}{2}-4}\right)(4m+3) + m + 1 \\
&\qquad - \frac{2}{3}\left(2^{\frac{n-4}{2}}(4m+3) + m + 1\right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{2^{\frac{n}{2}}}{12} + \frac{1}{3}\right) m + \frac{2^{\frac{n}{2}}}{16} + \frac{1}{3} > 0, \\
2\beta_{m,n} - a_{\frac{n}{2}+1} &= \frac{2}{3}\left(2^{\frac{n-4}{2}}(4m+3) + m + 1\right) \\
&\qquad - \left(\left(2^{\frac{n}{2}-2} - 2^{\frac{n}{2}-3}\right)(4m+3) + m + 1\right) \\
&= \left(\frac{2^{\frac{n}{2}}}{6} - \frac{1}{3}\right) m + \frac{2^{\frac{n}{2}}}{8} - \frac{1}{3} > 0.
\end{aligned}$$

By Ref. [1] (Theorem 1.1 (6)) and Lemma 1, we then see that:

$$T_{(b(m,n),n+2r)}(K(m,n,r)) = K(m,n,r),$$

since $b(m,n) - 1 - \beta_{m,n} = 2\beta_{m,n}$. Therefore, $K(m,n,r)$ is a $b(m,n)$-adic $(n+2r)$-digit non-regular Kaprekar constant for any $r \geq 2$.

By (a) and (b) above, Part (1) of Theorem 2 is proven.

(2) We assume that:

$$m = 1,\ n \equiv 3 \pmod 6 \text{ and } n \geq 9.$$

Let the notation be as in Theorem 1 (2). By the definition in *loc. cit.*, the $b(1,n)(= n+1)$-adic n-digit integer obtained by rearranging of the numbers of all digits $b_0, b_1, \ldots, b_{n-1}$ of $L(1,n)$ in descending order is:

$$(n\ (n-1) \cdots 3\ 2\ 1)_{b(1,n)}$$

given by all integers from 1–n. By Ref. [1] (Theorem 1.1 (8)) and Theorem 1 (2), we then see that:

$$\begin{aligned}
&T_{(b(1,n),n+2r)}(L(1,n,r)) \\
&= T_{(b(1,n),n)}\left(n \cdots \frac{2n+3}{3}\ \overbrace{\frac{2n}{3} \cdots \frac{2n}{3}}^{r}\ \frac{2n}{3} \cdots \frac{n+3}{3}\ \overbrace{\frac{n}{3} \cdots \frac{n}{3}}^{r}\ \frac{n}{3} \cdots 1\right) \\
&= L(1,n,r),
\end{aligned}$$

since $n \geq 9$ and $b(1,n) - 1 - \left(\frac{2n}{3} - \frac{n}{3}\right) = \frac{2n}{3}$. Therefore, $L(1,n,r)$ is a $b(1,n)$-adic $(n+2r)$-digit non-regular Kaprekar constant for any $r \geq 2$, and Part (2) of Theorem 2 is proven.

Remark 3. *Although we omit the proof here, we can see that for any integer $m \geq 2$ and odd integer $n \geq 3$, it is impossible to construct any $b(m,n)$-adic $(n+2r)$-digit non-regular Kaprekar constant by adding $\beta_{m,n}$'s and $(2\beta_{m,n})$'s to the $b(m,n)$-adic expression of the $b(m,n)$-adic n-digit regular Kaprekar constant $L(m,n)$, as well as in Part (1) of Theorem 2.*

2.4. A Proof of Corollary 2

We assume that:

$$m \equiv 1 \pmod 3 \text{ and } n \equiv 0 \pmod 4$$

(resp.

$$m = 1,\ n \equiv 3 \pmod 6 \text{ and } n \geq 9).$$

By Theorem 2, for any integer $r \geq 2$, we then have the $b(m,n)$-adic $(n+2r)$-digit non-regular Kaprekar constant $K(m,n,r)$ (resp. $L(1,n,r)$). Therefore, we see that:

$$\nu_{\text{non-reg}}(b, n+2r) \geq 1,$$

and Corollary 2 is proven.

3. On *n*-Digit Regular Kaprekar Constants with Specified *n*

3.1. Some Formulas for All n-Digit Regular Kaprekar Constants with Specified n

Let $K(n)$ and $L(n)$ be the progressions of n-digit regular Kaprekar constants defined in Definition 3(2) for even and odd positive integers n, respectively. On the other hand, it seems that it is very hard to obtain formulas for *all* n-digit regular Kaprekar constants. In this subsection, we shall obtain partial results on such formulas by case-by-case arguments.

Firstly, we shall see formulas for all n-digit regular Kaprekar constants in the cases where $n = 5, 7, 9, 11$ in Theorem 3. Note that, in the case where $n = 3$, Eldridge and Sagong [5] already proved that a three-digit integer x is a regular Kaprekar constant if and only if $x \in L(3)$, i.e., x is of the form:

$$(m(2m+1)(m+1))_{2m+2}$$

with $m \geq 1$.

Although one can obtain a similar result for each odd integer $n \geq 13$, the authors would not like to do tedious calculations for solving simultaneous equations obtained by the uniqueness of b-adic expressions of any positive integer for any integer $b \geq 2$.

Theorem 3. (1) *A five-digit integer x is a regular Kaprekar constant if and only if $x \in L(5)$, i.e., x is of the form:*

$$((2m+2)m(3m+2)(2m+1)(m+1))_{3m+3}$$

with $m \geq 1$.

(2) *A seven-digit integer x is a regular Kaprekar constant if and only if $x \in L(7)$, i.e., x is of the form:*

$$((3m+3)(2m+2)m(4m+3)(3m+2)(2m+1)(m+1))_{4m+4}$$

with $m \geq 1$.

(3) *For any integer $b \geq 2$, a b-adic nine-digit integer x is a regular Kaprekar constant if and only if x is of the form:*

$$((b-m-1)(b-2m-2)(b-3m-3)m(b-1)(b-m-2)(3m+2)(2m+1)(m+1))_b,$$

where the base b is in the range $5m+4 < b < 6m+5$ with $m \geq 1$.

In particular, when $b = 5m+5$, x is a member of $L(9)$.

(4) *An 11-digit integer x is a regular Kaprekar constant if and only if $x \in L(11)$, i.e., x is of the form:*

$$\begin{aligned}((5m+5)(4m+4)(3m+3)(2m+2)m(6m+5)\\ (5m+4)(4m+3)(3m+2)(2m+1)(m+1))_{6m+6}\end{aligned}$$

with $m \geq 1$.

Proof. By Theorem 1, it suffices to show that any regular Kaprekar constant in each case is of the form stated in the assertion. In the following, let $b \geq 2$ be any integer.

(1) For any b-adic five-digit regular Kaprekar constant x, we denote by $(c_4c_3c_2c_1c_0)_b$ with:

$$b-1 \geq c_4 > c_3 > c_2 > c_1 > c_0 \geq 0$$

the rearrangement in descending order of the numbers of all digits of x. By Ref. [1] (Theorem 1.1 (7)),

$$\begin{aligned}x &= T_{(b,5)}((c_4c_3c_2c_1c_0)_b)\\ &= ((c_4-c_0)(c_3-c_1-1)(b-1)(b-1-(c_3-c_1))(b-(c_4-c_0)))_b.\end{aligned}$$

We see the following magnitude relations among the numbers of all digits of x:

$$\begin{aligned}&b-1 \geq c_4-c_0 > c_3-c_1-1,\\ &b-1 > b-1-(c_3-c_1) > b-(c_4-c_0).\end{aligned}$$

Then, we obtain the following:

Lemma 3.

$$\begin{aligned}&b-1 = c_4, \quad c_4-c_0 = c_3, \quad b-1-(c_3-c_1) = c_2,\\ &b-(c_4-c_0) = c_1 \quad \textit{and} \quad c_3-c_1-1 = c_0.\end{aligned}$$

Proof. Since c_4 is the maximum number among all digits of x,

$$b-1 = c_4.$$

This implies that:

$$c_4-c_0 = b-1-c_0 \quad \text{and} \quad b-(c_4-c_0) = c_0+1.$$

Since c_1 is the second smallest number among all digits of x, we then see that:

$$b-(c_4-c_0) = c_1.$$

This implies that:

$$c_3-c_1-1 = c_0$$

by the two inequalities above. Moreover, we see that:

$$\begin{aligned} b-1-(c_3-c_1) &= b-2-c_0 \\ &< b-1-c_0 = c_4-c_0, \end{aligned}$$

which implies that:

$$c_4-c_0 = c_3 \quad \text{and} \quad b-1-(c_3-c_1) = c_2$$

as desired. □

We then see that the following equality holds:

$$\begin{aligned} &((c_4-c_0)(c_3-c_1-1)(b-1)(b-1-(c_3-c_1))(b-(c_4-c_0)))_b \\ =&(c_3c_0c_4c_2c_1)_b. \end{aligned}$$

This implies that $b = 3c_0+3$ and:

$$c_4 = 3c_0+2,\ c_3 = 2c_0+2,\ c_2 = 2c_0+1,\ c_1 = c_0+1.$$

Putting $m = c_0 \geq 0$, we then see that:

$$x = ((2m+2)m(3m+2)(2m+1)(m+1))_{3m+3}.$$

If $m = 0$, then we see a contradiction that $x = (20211)_3$ is not regular. Therefore, $m \geq 1$, and Part (1) is proven.

(2) For any b-adic seven-digit regular Kaprekar constant x, we denote by $(c_6c_5c_4c_3c_2c_1c_0)_b$ with:

$$b-1 \geq c_6 > c_5 > c_4 > c_3 > c_2 > c_1 > c_0 \geq 0$$

the rearrangement in descending order of the numbers of all digits of x. By the same argument as in the proof of Part (1), we then see that one of the following two equalities holds:

$$\begin{aligned} &((c_6-c_0)(c_5-c_1)(c_4-c_2-1)(b-1)(b-1-(c_4-c_2)) \\ &\qquad\qquad (b-1-(c_5-c_1))(b-(c_6-c_0)))_b \\ =&\begin{cases} (c_5c_2c_0c_6c_4c_3c_1)_b & \cdots\ \text{(i)} \\ (c_5c_3c_0c_6c_4c_2c_1)_b & \cdots\ \text{(ii)} \end{cases} \end{aligned}$$

The equality (i) implies a contradiction that $c_2 = -\dfrac{1}{2}$.

The equality (ii) implies that $b = 4c_0+4$ and:

$$\begin{aligned} c_6 &= 4c_0+3, & c_5 &= 3c_0+3, & c_4 &= 3c_0+2, \\ c_3 &= 2c_0+2, & c_2 &= 2c_0+1, & c_1 &= c_0+1. \end{aligned}$$

Putting $m = c_0 \geq 0$, we then see that:

$$x = ((3m+3)(2m+2)m(4m+3)(3m+2)(2m+1)(m+1))_{4m+4}.$$

If $m = 0$, then we see a contradiction that $x = (3203211)_4$ is not regular. Therefore, $m \geq 1$, and Part (2) is proven.

(3) For any b-adic nine-digit regular Kaprekar constant x, we denote by $(c_8c_7c_6c_5c_4c_3c_2c_1c_0)_b$ with:

$$b-1 \geq c_8 > c_7 > c_6 > c_5 > c_4 > c_3 > c_2 > c_1 > c_0 \geq 0$$

the rearrangement in descending order of the numbers of all digits of x. By the same argument as in the proof of Part (1), we then see that one of the following six equalities holds:

$$\begin{aligned}
&((c_8-c_0)(c_7-c_1)(c_6-c_2)(c_5-c_3-1)(b-1)(b-1-(c_5-c_3))\\
&\qquad(b-1-(c_6-c_2))(b-1-(c_7-c_1))(b-(c_8-c_0)))_b\\
&=\begin{cases}
(c_7c_5c_4c_0c_8c_6c_3c_2c_1)_b & \cdots \text{(i)}\\
(c_7c_5c_3c_0c_8c_6c_4c_2c_1)_b & \cdots \text{(ii)}\\
(c_7c_5c_2c_0c_8c_6c_4c_3c_1)_b & \cdots \text{(iii)}\\
(c_7c_4c_3c_0c_8c_6c_5c_2c_1)_b & \cdots \text{(iv)}\\
(c_7c_4c_2c_0c_8c_6c_5c_3c_1)_b & \cdots \text{(v)}\\
(c_7c_3c_2c_0c_8c_6c_5c_4c_1)_b & \cdots \text{(vi)}
\end{cases}
\end{aligned}$$

The equalities (i) and (v) imply a contradiction that $c_4 = c_3$.
The equalities (iii), (iv), and (vi) imply a contradiction that $c_5 = c_4$.
The equality (ii) implies that $b = c_3 + 3c_0 + 3$ and:

$$\begin{aligned}
&c_8 = c_3 + 3c_0 + 2, \quad c_7 = c_3 + 2c_0 + 2, \quad c_6 = c_3 + 2c_0 + 1,\\
&c_5 = c_3 + c_0 + 1, \quad c_4 = 3c_0 + 2, \quad c_2 = 2c_0 + 1, \quad c_1 = c_0 + 1.
\end{aligned}$$

Putting $m = c_0 \geq 0$, we then see that x is equal to:

$$((b-m-1)(b-2m-2)(b-3m-3)m(b-1)(b-m-2)(3m+2)(2m+1)(m+1))_b,$$

where the base b is in the range $5m + 4 < b < 6m + 5$, since:

$$c_4 = 3m + 2 > c_3 = b - 3m - 3 > c_2 = 2m + 1.$$

If $m = 0$, then we see a contradiction that b is in the range $4 < b < 5$. Therefore, $m \geq 1$, and Part (3) is proven.

(4) For any b-adic 11-digit regular Kaprekar constant x, we denote by $(c_{10}c_9c_8c_7c_6c_5c_4c_3c_2c_1c_0)_b$ with:

$$b - 1 \geq c_{10} > c_9 > c_8 > c_7 > c_6 > c_5 > c_4 > c_3 > c_2 > c_1 > c_0 \geq 0$$

the rearrangement in descending order of the numbers of all digits of x. By the same argument as in the proof of Part (1), we then see that one of the following twenty equalities holds:

$$\begin{aligned}
&((c_{10}-c_0)(c_9-c_1)(c_8-c_2)(c_7-c_3)(c_6-c_4-1)(b-1)(b-1-(c_6-c_4))\\
&(b-1-(c_7-c_3))(b-1-(c_8-c_2))(b-1-(c_9-c_1))(b-(c_{10}-c_0)))_b\\
&=\begin{cases}
(c_9c_7c_6c_5c_0c_{10}c_8c_4c_3c_2c_1)_b & \cdots\ \text{(i)}\\
(c_9c_7c_6c_4c_0c_{10}c_8c_5c_3c_2c_1)_b & \cdots\ \text{(ii)}\\
(c_9c_7c_6c_3c_0c_{10}c_8c_5c_4c_2c_1)_b & \cdots\ \text{(iii)}\\
(c_9c_7c_6c_2c_0c_{10}c_8c_5c_4c_3c_1)_b & \cdots\ \text{(iv)}\\
(c_9c_7c_5c_4c_0c_{10}c_8c_6c_3c_2c_1)_b & \cdots\ \text{(v)}\\
(c_9c_7c_5c_3c_0c_{10}c_8c_6c_4c_2c_1)_b & \cdots\ \text{(vi)}\\
(c_9c_7c_5c_2c_0c_{10}c_8c_6c_4c_3c_1)_b & \cdots\ \text{(vii)}\\
(c_9c_7c_4c_3c_0c_{10}c_8c_6c_5c_2c_1)_b & \cdots\ \text{(viii)}\\
(c_9c_7c_4c_2c_0c_{10}c_8c_6c_5c_3c_1)_b & \cdots\ \text{(ix)}\\
(c_9c_7c_3c_2c_0c_{10}c_8c_6c_5c_4c_1)_b & \cdots\ \text{(x)}\\
(c_9c_4c_3c_2c_0c_{10}c_8c_7c_6c_5c_1)_b & \cdots\ \text{(xi)}\\
(c_9c_5c_3c_2c_0c_{10}c_8c_7c_6c_4c_1)_b & \cdots\ \text{(xii)}\\
(c_9c_5c_4c_2c_0c_{10}c_8c_7c_6c_3c_1)_b & \cdots\ \text{(xiii)}\\
(c_9c_5c_4c_3c_0c_{10}c_8c_7c_6c_2c_1)_b & \cdots\ \text{(xiv)}\\
(c_9c_6c_3c_2c_0c_{10}c_8c_7c_5c_4c_1)_b & \cdots\ \text{(xv)}\\
(c_9c_6c_4c_2c_0c_{10}c_8c_7c_5c_3c_1)_b & \cdots\ \text{(xvi)}\\
(c_9c_6c_4c_3c_0c_{10}c_8c_7c_5c_2c_1)_b & \cdots\ \text{(xvii)}\\
(c_9c_6c_5c_2c_0c_{10}c_8c_7c_4c_3c_1)_b & \cdots\ \text{(xviii)}\\
(c_9c_6c_5c_3c_0c_{10}c_8c_7c_4c_2c_1)_b & \cdots\ \text{(xix)}\\
(c_9c_6c_5c_4c_0c_{10}c_8c_7c_3c_2c_1)_b & \cdots\ \text{(xx)}
\end{cases}
\end{aligned}$$

The equality (i) implies a contradiction that $c_5 \leq c_4$.
The equalities (ii), (x), (xii), and (xiii) imply a contradiction that $c_{10} = c_9$.
The equalities (iii), (iv), (vii), (xi), and (xviii) imply a contradiction that $c_7 < c_6$.
The equality (v) implies a contradiction that $c_6 < c_5$.
The equalities (viii) and (xvi) imply a contradiction that $c_7 = c_6$.
The equality (ix) implies that:

$$c_7 = c_2 + 2c_0 + 1, \quad c_6 = 4c_0 + 2, \quad c_3 = 2c_0 + 1,$$

which yields a contradiction that $c_2 > 2c_0 + 1 > c_2$.
The equality (xiv) implies a contradiction that $c_6 = c_5$.
The equality (xv) implies a contradiction that $c_8 = c_7$.
The equality (xvii) implies that:

$$c_7 = 2c_3, \quad c_6 = 3c_3 - 2c_0 - 1, \quad c_2 = 2c_0 + 1,$$

which implies a contradiction that $c_3 > 2c_0 + 1 > c_3$.
The equality (xix) implies a contradiction that $c_7 = 4c_0 + \dfrac{8}{3}$.
The equality (xx) implies a contradiction that $c_8 < c_7$.

The equality (vi) implies that $b = 6c_0 + 6$ and:

$$c_{10} = 6c_0 + 5,\ c_9 = 5c_0 + 5,\ c_8 = 5c_0 + 4,\ c_7 = 4c_0 + 4,\ c_6 = 4c_0 + 3,$$
$$c_5 = 3c_0 + 3,\ c_4 = 3c_0 + 2,\ c_3 = 2c_0 + 2,\ c_2 = 2c_0 + 1,\ c_1 = c + 1.$$

Putting $m = c_0 \geq 0$, we then see that:

$$x = ((5m+5)(4m+4)(3m+3)(3m+2)m(6m+5)$$
$$(5m+4)(4m+3)(3m+2)(2m+1)(m+1))_{6m+6}.$$

If $m = 0$, then we see a contradiction that $x = (54320543211)_6$ is not regular. Therefore, $m \geq 1$, and Part (4) is proven. $\square$

Secondly, we see formulas for all n-digit regular Kaprekar constants in the cases where $n = 2, 4, 6, 8$ in Theorem 4. Although one can obtain a similar result for each even integer $n \geq 10$, the authors would not like to do tedious calculations for solving simultaneous equations obtained by the uniqueness of b-adic expressions of any positive integer for any integer $b \geq 2$.

Note that we shall need more calculations of solving simultaneous equations in the proof for even cases in Theorem 4 than odd cases in Theorem 3, because, in the case where $n \geq 2$ is even, the Kaprekar transformation $T_{(b,n)}$ may not necessarily give us the maximum number $b - 1$ among the numbers of all digits.

Theorem 4. (1) *A two-digit integer x is a regular Kaprekar constant if and only if $x \in K(2) \cup \{(01)_2\}$, i.e, x is of the form:*

$$(m(2m+1))_{3m+2}$$

with $m \geq 0$.

(2) *A four-digit integer x is a regular Kaprekar constant if and only if $x = (3021)_4$ or $x \in K(4)$, i.e., x is of the form:*

$$((3m+3)m(4m+3)(2m+2))_{5m+5}$$

with $m \geq 1$.

(3) *A six-digit integer x is a regular Kaprekar constant if and only if x is equal to:*

$$(530421)_6,$$
$$((9m+6)(5m+3)(3m+1)(2m+7)(10m+6)(6m+4))_{15m+10},$$
$$((5m+4)(3m+2)m(6m+4)(4m+3)(2m+2))_{7m+6} \quad \text{or}$$
$$((7m+6)(5m+4)m(8m+6)(4m+1)(2m+2))_{9m+8}\ (\in K(6))$$

with $m \geq 1$.

(4) *An eight-digit integer x is a regular Kaprekar constant if and only if x is equal to:*

$$\begin{aligned}
&(97508421)_{10}, \quad (75306421)_8,\\
&((11m+7)(7m+4)(5m+3)(3m+1)(14m+8)\\
&\qquad\qquad\qquad (12m+7)(10m+6)(6m+4))_{17m+11},\\
&((15m+9)(9m+5)(7m+4)(3m+1)(18m+10)\\
&\qquad\qquad\qquad (14m+8)(12m+7)(6m+4))_{21m+13},\\
&((13m+10)(11m+8)(7m+5)m(14m+10)\\
&\qquad\qquad\qquad (8m+6)(4m+3)(2m+2))_{15m+12} \quad \textit{or}\\
&((15m+12)(13m+10)(9m+7)m(16m+12)\\
&\qquad\qquad\qquad (8m+6)(4m+3)(2m+2))_{17m+14}\ (\in K(8))
\end{aligned}$$

with $m \geq 1$.

Proof. (1) For any b-adic two-digit regular Kaprekar constant x, we denote by $x = (c_1c_0)_b$ with $b-1 \geq c_1 > c_0 \geq 0$ the rearrangement in descending order of numbers of all digits of x. By Ref. [1] (Theorem 1.1 (2)),

$$x = T_{(b,2)}((c_1c_0)_b) = ((c_1 - c_0 - 1)(b - (c_1 - c_0)))_b.$$

We then see that one of the following two equalities holds:

$$((c_1 - c_0 - 1)(b - (c_1 - c_0)))_b = \begin{cases} (c_1c_0)_b & \cdots \text{(i)} \\ (c_0c_1)_b & \cdots \text{(ii)} \end{cases}$$

The equality (i) implies a contradiction that $c_0 = -1$.

The equality (ii) implies that:

$$c_1 = \frac{2b-1}{3} \quad \text{and} \quad c_0 = \frac{b-2}{3}.$$

Putting $m = c_0 \geq 0$, we then see that:

$$b = 3m+2 \quad \text{and} \quad c_1 = 2m+1$$

as desired.

(2) For any b-adic four-digit regular Kaprekar constant x, we denote by $(c_3c_2c_1c_0)_b$ with $b-1 \geq c_3 > c_2 > c_1 > c_0 \geq 0$ the rearrangement in descending order of the numbers of all digits of x. By Ref. [1] (Theorem 1.1 (6)),

$$\begin{aligned}
x &= T_{(b,4)}((c_3c_2c_1c_0)_b)\\
&= ((c_3 - c_0)(c_2 - c_1 - 1)(b - 1 - (c_2 - c_1)))(b - (c_3 - c_0)))_b.
\end{aligned}$$

Since:

$$c_3 - c_0 > c_2 - c_1 - 1 \quad \text{and} \quad b - 1 - (c_2 - c_1) > b - (c_3 - c_0),$$

we see that one of the following six equalities holds:

$$((c_3 - c_0)(c_2 - c_1 - 1)(b - 1 - (c_2 - c_1))(b - (c_3 - c_0)))_b$$
$$= \begin{cases} (c_3c_2c_1c_0)_b & \cdots \text{ (i)} \\ (c_3c_1c_2c_0)_b & \cdots \text{ (ii)} \\ (c_3c_0c_2c_1)_b & \cdots \text{ (iii)} \\ (c_1c_0c_3c_2)_b & \cdots \text{ (iv)} \\ (c_2c_0c_3c_1)_b & \cdots \text{ (v)} \\ (c_2c_1c_3c_0)_b & \cdots \text{ (vi)} \end{cases}$$

The equalities (i), (ii), and (vi) imply a contradiction that $c_3 = b$.
The equality (iii) implies that $x = (3021)_4$.
The equality (iv) implies a contradiction that $c_3 < c_2$.
The equality (v) implies that $b = 5c_0 + 5$ and:

$$c_3 = 4c_0 + 3, \quad c_2 = 3c_0 + 3, \quad c_1 = 2c_0 + 2.$$

Putting $m = c_0 \geq 0$, we then see that:

$$x = ((3m+3)m(4m+3)(2m+2))_{5m+5}.$$

If $m = 0$, then we see a contradiction that $x = (3032)_5$ is not regular. Therefore, $m \geq 1$, and Part (2) is proven.

(3) For any b-adic six-digit regular Kaprekar constant x, we denote by $(c_5c_4c_3c_2c_1c_0)_b$ with:

$$b - 1 \geq c_5 > c_4 > c_3 > c_2 > c_1 > c_0 \geq 0$$

the rearrangement in descending order of the numbers of all digits of x. By Ref. [1] (Theorem 1.1 (6)),

$$\begin{aligned} x &= T_{(b,6)}((c_5c_4c_3c_2c_1c_0)_b) \\ &= ((c_5 - c_0)(c_4 - c_1)(c_3 - c_2 - 1)(b - 1 - (c_3 - c_2)) \\ &\qquad (b - 1 - (c_4 - c_1))(b - (c_5 - c_0)))_b. \end{aligned}$$

Since $c_5 - c_0 > c_4 - c_1 > c_3 - c_2 - 1$ and:

$$b - 1 - (c_3 - c_2) > b - 1 - (c_4 - c_1) > b - (c_5 - c_0),$$

we see that $c_3 - c_2 - 1 = c_0$ or $b - (c_5 - c_0) = c_0$. The equality $b - (c_5 - c_0) = c_0$ implies a contradiction that $b = c_5$, and the equality $c_4 - c_1 = c_4$ implies a contradiction that $c_1 = 0 > c_0$. Therefore, we see that one of the following nine equalities holds:

$$
\begin{aligned}
&((c_5 - c_0)(c_4 - c_1)(c_3 - c_2 - 1)(b - 1 - (c_3 - c_2)) \\
&\qquad\qquad\qquad\qquad (b - 1 - (c_4 - c_1))(b - (c_5 - c_0)))_b \\
&= \begin{cases}
(c_5c_3c_0c_4c_2c_1)_b & \cdots \text{(i)} \\
(c_5c_2c_0c_4c_3c_1)_b & \cdots \text{(ii)} \\
(c_5c_1c_0c_4c_3c_2)_b & \cdots \text{(iii)} \\
(c_2c_1c_0c_5c_4c_3)_b & \cdots \text{(iv)} \\
(c_3c_1c_0c_5c_4c_2)_b & \cdots \text{(v)} \\
(c_3c_2c_0c_5c_4c_1)_b & \cdots \text{(vi)} \\
(c_4c_1c_0c_5c_3c_2)_b & \cdots \text{(vii)} \\
(c_4c_2c_0c_5c_3c_1)_b & \cdots \text{(viii)} \\
(c_4c_3c_0c_5c_2c_1)_b & \cdots \text{(ix)}
\end{cases}
\end{aligned}
$$

The equality (i) implies that $x = (530421)_6$.
The equality (ii) and (iii) imply a contradiction that $c_2 = c_1$.
The equality (iv) implies that $c_2 = c_0 + 1$, which contradicts the condition that $c_2 > c_1 > c_0$.
The equality (vi) implies a contradiction that $c_2 = c_0$.
The equality (vii) implies a contradiction that $x = (420432)_6$ is not regular.
The equality (v) implies that $b = 5c_0 + 5$ and:

$$
\begin{aligned}
&c_5 = 4c_0 + 3, \quad c_4 = \frac{10c_0 + 8}{3}, \quad c_3 = 3c_0 + 3, \\
&c_2 = 2c_0 + 2, \quad c_1 = \frac{5c_0 + 4}{3}.
\end{aligned}
$$

Putting $c_0 = 3m + 1$ with $m \geq 0$, we then see that:

$$x = ((9m + 6)(5m + 3)(3m + 1)(12m + 7)(10m + 6)(6m + 4))_{15m+10}.$$

If $m = 0$, then we see a contradiction that $x = (631764)_{10}$ is not regular. Therefore, $m \geq 1$.
The equality (viii) implies that $b = 7c_0 + 6$ and:

$$
\begin{aligned}
&c_5 = 6c_0 + 4, \quad c_4 = 5c_0 + 4, \quad c_3 = 4c_0 + 3, \\
&c_2 = 3c_0 + 2, \quad c_1 = 2c_0 + 2.
\end{aligned}
$$

Putting $m = c_0 \geq 0$, we then see that:

$$x = ((5m + 4)(3m + 2)m(6m + 4)(4m + 3)(2m + 2))_{7m+6}.$$

If $m = 0$, then we see a contradiction that $x = (420432)_6$ is not regular. Therefore, $m \geq 1$.
The equality (ix) implies that $b = 9c_0 + 8$ and:

$$
\begin{aligned}
&c_5 = 8c_0 + 6, \quad c_4 = 7c_0 + 6, \quad c_3 = 5c_0 + 4, \\
&c_2 = 4c_0 + 3, \quad c_1 = 2c_0 + 2.
\end{aligned}
$$

Putting $m = c_0 \geq 0$, we then see that:

$$x = ((7m + 6)(5m + 4)m(8m + 6)(4m + 3)(2m + 2))_{9m+8}.$$

If $m = 0$, then we see a contradiction that $x = (640632)_8$ is not regular. Therefore, $m \geq 1$, and Part (3) is proven.

(4) For any b-adic eight-digit regular Kaprekar constant x, we denote by $(c_7c_6c_5c_4c_3c_2c_1c_0)_b$ with:

$$b-1 \geq c_7 > c_6 > c_5 > c_4 > c_3 > c_2 > c_1 > c_0 \geq 0$$

the rearrangement in descending order of the numbers of all digits of x. By Ref. [1] (Theorem 1.1 (6)),

$$\begin{aligned} x &= T_{(b,8)}((c_7c_6c_5c_4c_3c_2c_1c_0)_b) \\ &= ((c_7-c_0)(c_6-c_1)(c_5-c_2)(c_4-c_3-1)(b-1-(c_4-c_3)) \\ &\qquad (b-1-(c_5-c_2))(b-1-(c_6-c_1))(b-(c_7-c_0)))_b. \end{aligned}$$

Since $c_7 - c_0 > c_6 - c_1 > c_5 - c_2 > c_4 - c_3 - 1$ and:

$$b-1-(c_4-c_3) > b-1-(c_5-c_2) > b-1-(c_6-c_1) > b-(c_7-c_0),$$

we see that $c_4 - c_3 - 1 = c_0$ or $b - (c_7 - c_0) = c_0$. The equality $b - (c_7 - c_0) = c_0$ implies a contradiction that $b = c_7$, and the equality $c_6 - c_1 = c_6$ implies a contradiction that $c_1 = 0 > c_0$. Therefore, we see that one of the following thirty equalities holds:

$$
\begin{aligned}
&((c_7 - c_0)(c_6 - c_1)(c_5 - c_2)(c_4 - c_3 - 1)(b - 1 - (c_4 - c_3)) \\
&\qquad (b - 1 - (c_5 - c_2))(b - 1 - (c_6 - c_1))(b - (c_7 - c_0)))_b
\end{aligned}
$$

$$
= \begin{cases}
(c_7c_5c_4c_0c_6c_3c_2c_1)_b & \cdots \text{(i)} \\
(c_7c_5c_3c_0c_6c_4c_2c_1)_b & \cdots \text{(ii)} \\
(c_7c_5c_2c_0c_6c_4c_3c_1)_b & \cdots \text{(iii)} \\
(c_7c_5c_1c_0c_6c_4c_3c_2)_b & \cdots \text{(iv)} \\
(c_7c_4c_3c_0c_6c_5c_2c_1)_b & \cdots \text{(v)} \\
(c_7c_4c_2c_0c_6c_5c_3c_1)_b & \cdots \text{(vi)} \\
(c_7c_4c_1c_0c_6c_5c_3c_2)_b & \cdots \text{(vii)} \\
(c_7c_3c_2c_0c_6c_5c_4c_1)_b & \cdots \text{(viii)} \\
(c_7c_3c_1c_0c_6c_5c_4c_2)_b & \cdots \text{(ix)} \\
(c_7c_2c_1c_0c_6c_5c_4c_3)_b & \cdots \text{(x)} \\
(c_3c_2c_1c_0c_7c_6c_5c_4)_b & \cdots \text{(xi)} \\
(c_4c_2c_1c_0c_7c_6c_5c_3)_b & \cdots \text{(xii)} \\
(c_4c_3c_1c_0c_7c_6c_5c_2)_b & \cdots \text{(xiii)} \\
(c_4c_3c_2c_0c_7c_6c_5c_1)_b & \cdots \text{(xiv)} \\
(c_5c_2c_1c_0c_7c_6c_4c_3)_b & \cdots \text{(xv)} \\
(c_5c_3c_1c_0c_7c_6c_4c_2)_b & \cdots \text{(xvi)} \\
(c_5c_3c_2c_0c_7c_6c_4c_1)_b & \cdots \text{(xvii)} \\
(c_5c_4c_1c_0c_7c_6c_3c_2)_b & \cdots \text{(xviii)} \\
(c_5c_4c_2c_0c_7c_6c_3c_1)_b & \cdots \text{(xix)} \\
(c_5c_4c_3c_0c_7c_6c_2c_1)_b & \cdots \text{(xx)} \\
(c_6c_2c_1c_0c_7c_5c_4c_3)_b & \cdots \text{(xxi)} \\
(c_6c_3c_1c_0c_7c_5c_4c_2)_b & \cdots \text{(xxii)} \\
(c_6c_3c_2c_0c_7c_5c_4c_1)_b & \cdots \text{(xxiii)} \\
(c_6c_4c_1c_0c_7c_5c_3c_2)_b & \cdots \text{(xxiv)} \\
(c_6c_4c_2c_0c_7c_5c_3c_1)_b & \cdots \text{(xxv)} \\
(c_6c_4c_3c_0c_7c_5c_2c_1)_b & \cdots \text{(xxvi)} \\
(c_6c_5c_1c_0c_7c_4c_3c_2)_b & \cdots \text{(xxvii)} \\
(c_6c_5c_2c_0c_7c_4c_3c_1)_b & \cdots \text{(xxviii)} \\
(c_6c_5c_3c_0c_7c_4c_2c_1)_b & \cdots \text{(xxix)} \\
(c_6c_5c_4c_0c_7c_3c_2c_1)_b & \cdots \text{(xxx)}
\end{cases}
$$

The equality (i) implies that $x = (97508421)_{10}$.
The equality (ii) implies that $x = (75306421)_8$.
The equality (iii) implies a contradiction that $c_6 = c_4$.
The equality (iv) implies a contradiction that $c_5 = c_3$.
The equalities (v), (x), (xv), and (xxi) imply a contradiction that $c_6 = c_5$.
The equality (vi) implies a contradiction that $c_2 = \frac{5}{3}$.
The equality (vii) implies a contradiction that $c_7 < c_6$.

The equalities (viii) and (ix) imply a contradiction that $c_3 = c_1$.
The equalities (xi), (xii), (xiii), and (xiv) imply a contradiction that $c_2 = c_1$.
The equality (xvii) implies a contradiction that $c_1 = c_0 = -2$.
The equality (xviii) implies a contradiction that $b = 5c_0 + \dfrac{14}{3}$.
The equality (xix) implies a contradiction that $b = 2c_2 - \dfrac{2}{3}$.
The equality (xx) implies a contradiction that $c_7 = c_5$.
The equality (xxii) implies a contradiction that $4 > c_1 > 3$.
The equality (xxiv) implies a contradiction that $b = 2c_1 + \dfrac{7}{3}$.
The equality (xxv) implies a contradiction that $c_5 = 6c_1 + \dfrac{14}{3}$.
The equality (xxvi) implies a contradiction that $c_4 = c_1$.
The equality (xxvii) implies a contradiction that $c_0 = -1$.
The equality (xxviii) implies a contradiction that $c_7 = c_4$.
The equality (xvi) implies that $b = \dfrac{17c_0 + 16}{3}$ and:

$$\begin{aligned}
&c_7 = \frac{14c_0 + 10}{3}, \quad c_6 = 4c_0 + 3, \quad c_5 = \frac{11c_0 + 10}{3}, \quad c_4 = \frac{10c_0 + 8}{3}, \\
&c_3 = \frac{7c_0 + 5}{3}, \quad c_2 = 2c_0 + 2, \quad c_1 = \frac{5c_0 + 4}{3}.
\end{aligned}$$

Putting $c_0 = 3m + 1$ with $m \geq 0$, we then see that:

$$\begin{aligned}
x = ((11m + 7)(7m + 4)(5m + 3)(3m + 1) \\
(4m + 8)(12m + 7)(10m + 6)(6m + 4))_{17m+11}.
\end{aligned}$$

If $m = 0$, then we see a contradiction that $x = (74318764)_{11}$ is not regular. Therefore, $m \geq 1$.
The equality (xxiii) implies that $b = 7c_0 + 6$ and:

$$\begin{aligned}
&c_7 = 6c_0 + 4, \quad c_6 = 5c_0 + 4, \quad c_5 = \frac{14c_0 + 10}{3}, \quad c_4 = 4c_0 + 3, \\
&c_3 = 3c_0 + 2, \quad c_2 = \frac{7c_0 + 5}{3}, \quad c_1 = 2c_0 + 2.
\end{aligned}$$

Putting $c_0 = 3m + 1$ with $m \geq 0$, we then see that:

$$\begin{aligned}
x = ((15m + 9)(9m + 5)(7m + 4)(3m + 1) \\
(18m + 10)(14m + 8)(12m + 7)(6m + 4))_{21m+13}.
\end{aligned}$$

If $m = 0$, then we see a contradiction that $x = (9541(10)874)_{13}$ is not regular. Therefore, $m \geq 1$.
The equality (xxix) implies that $b = 15c_0 + 12$ and:

$$\begin{aligned}
&c_7 = 14c_0 + 10, \quad c_6 = 13c_0 + 10, \quad c_5 = 11c_0 + 8, \quad c_4 = 8c_0 + 6, \\
&c_3 = 7c_0 + 5, \quad c_2 = 4c_0 + 3, \quad c_1 = 2c_0 + 2.
\end{aligned}$$

Putting $m = c_0 \geq 0$, we then see that:

$$\begin{aligned}
x = ((13m + 10)(11m + 8)(7m + 5)m \\
(14m + 10)(8m + 6)(4m + 3)(2m + 2))_{15m+12}.
\end{aligned}$$

If $m = 0$, then we see a contradiction that $x = ((10)850(10)632)_{12}$ is not regular. Therefore, $m \geq 1$.

The equality (xxx) implies that $b = 17c_0 + 14$ and:

$$c_7 = 16c_0 + 12, \quad c_6 = 15c_0 + 12, \quad c_5 = 13c_0 + 10, \quad c_4 = 9c_0 + 7,$$
$$c_3 = 8c_0 + 6, \quad c_2 = 4c_0 + 3, \quad c_1 = 2c_0 + 2.$$

Putting $m = c_0 \geq 0$, we then see that:

$$x = ((15m+12)(13m+10)(9m+7)m \\ (16m+12)(8m+6)(4m+3)(2m+2))_{17m+14}.$$

If $m = 0$, then we see a contradiction that $x = ((12)(10)70(12)632)_{14}$ is not regular. Therefore, $m \geq 1$, and Part (4) is proven. $\square$

We shall also obtain some conditional results on formulas for n-digit regular Kaprekar constants in the following proposition for which we omit the proof because one can prove them by the same arguments as in the proof of Theorem 3:

Proposition 1. *Let the notation be as in Theorem 3. For any integer $b \geq 2$, we see the following:*

(1) *A b-adic 13-digit integer $x = (a_{12} \cdots a_0)_b$ with $0 \leq a_0, \ldots, a_{12} \leq b-1$ satisfying the condition:*

$$a_{11} > a_4 > a_{10} > a_3 > a_9 > a_2 > a_8 > a_1$$

is a regular Kaprekar constant if and only if $x \in L(13)$ with $b \in b(13)$, i.e., x is of the form:

$$((6m+6)(5m+5)(4m+4)(3m+3)(2m+2)m \\ (7m+6)(6m+5)(5m+4)(4m+3)(3m+2)(2m+1)(m+1))_{7m+7}$$

with $m \geq 1$.

(2) *A b-adic 15-digit integer $x = (a_{14} \cdots a_0)_b$ with $0 \leq a_0, \ldots, a_{14} \leq b-1$ satisfying the condition:*

$$a_{13} > a_5 > a_{12} > a_4 > a_{11} > a_3 > a_{10} > a_2 > a_9 > a_1$$

is a regular Kaprekar constant if and only if x is of the form:

$$((b-m_1-1)(b-2m_1-2)(b-3m_1-3)(b-2m_1-m_2-2) \\ (b-3m_1-m_2-3)m_2m_1(b-1)(b-m_1-2)(b-m_2-1) \\ (3m_1+m_2+2)(2m_1+m_2+1)(3m_1+2)(2m_1+1)(m_1+1))_b,$$

where $m_1 \geq 1$, m_2 is in the range:

$$2m_1 + 1 < m_2 < 3m_1 + 2$$

and b is in the range:

$$6m_1 + m_2 + 5 < b < 5m_1 + 2m_2 + 4.$$

(3) *A b-adic 17-digit integer $x = (a_{16} \cdots a_0)_b$ with $0 \leq a_0, \ldots, a_{16} \leq b-1$ satisfying the condition:*

$$a_{15} > a_6 > a_{14} > a_5 > a_{13} > a_4 > a_{12} > a_3 > a_{11} > a_2 > a_{10} > a_1$$

is a regular Kaprekar constant if and only if x is of the form:

$$\begin{aligned}&((b-m-1)(b-2m-2)(b-3m-3)\\&\quad\left(\frac{3b-7m-7}{4}\right)\left(\frac{3b-11m-11}{4}\right)\left(\frac{b-3m-2}{2}\right)\left(\frac{b-m-1}{4}\right)\\&\quad m(b-1)(b-m-2)\\&\quad\left(\frac{3b+m-3}{4}\right)\left(\frac{b+3m+1}{2}\right)\left(\frac{b+11m+7}{4}\right)\left(\frac{b+7m+3}{4}\right)\\&\quad(3m+2)(2m+1)(m+1))_b,\end{aligned}$$

where b satisfies the conditions:

$$9m+7<b<11m+9 \quad \textit{and} \quad b\equiv m+1 \pmod 4$$

with $m \geq 1$.

3.2. Some Observations on $\nu_{\mathrm{reg}}(b,n)$ with Specified n

As a corollary to Theorems 3 and 4, we can make some observations on the numbers $\nu_{\mathrm{reg}}(b,n)$ of all b-adic n-digit regular Kaprekar constants for $n = 2,4,5,6,7,8,9,11$ as in the following:

Corollary 3. *Let* $b \geq 2$ *be any integer. Then, we see the following:*

(1) $$\nu_{\mathrm{reg}}(b,2) = \begin{cases} 1 & \text{if } 3 \mid (b+1), \\ 0 & \text{otherwise.} \end{cases}$$

(2) $$\nu_{\mathrm{reg}}(b,4) = \begin{cases} 1 & \text{if } b = 4 \text{ or, } b \geq 10 \text{ and } 5 \mid b, \\ 0 & \text{otherwise.} \end{cases}$$

(3) $$\nu_{\mathrm{reg}}(b,5) = \begin{cases} 1 & \text{if } b \geq 6 \text{ and } 3 \mid b, \\ 0 & \text{otherwise.} \end{cases}$$

(4) $$\nu_{\mathrm{reg}}(b,6) = \begin{cases} 2 & \text{if } b \in (A_1 \cap A_2) \cup (A_2 \cap A_3), \\ 1 & \text{otherwise,} \\ 0 & \text{if } b \neq 6 \text{ and } b \notin A_1 \cup A_2 \cup A_3, \end{cases}$$

where the sets A_1, A_2, *and* A_3 *are defined as:*

$$\begin{aligned}A_1 &= \{b \in \mathbb{Z} \mid b \geq 25 \text{ and } b \equiv 10 \pmod{15}\},\\ A_2 &= \{b \in \mathbb{Z} \mid b \geq 13 \text{ and } b \equiv 6 \pmod{7}\},\\ A_3 &= \{b \in \mathbb{Z} \mid b \geq 17 \text{ and } b \equiv 8 \pmod{9}\}.\end{aligned}$$

(5) $$\nu_{\mathrm{reg}}(b,7) = \begin{cases} 1 & \text{if } b \geq 8 \text{ and } 4 \mid b, \\ 0 & \text{otherwise.} \end{cases}$$

(6) $$\nu_{\mathrm{reg}}(b,8) = \begin{cases} 2 & \text{if } b \in (B_1 \cap B_2) \cup (B_1 \cap B_3) \cup (B_2 \cap B_3) \cup (B_3 \cap B_4), \\ 1 & \text{otherwise,} \\ 0 & \text{if } b \neq 8, 10 \text{ and } b \notin B_1 \cup B_2 \cup B_3 \cup B_4. \end{cases}$$

where the sets B_1, B_2, B_3, *and* B_4 *are defined as:*

$$\begin{aligned}
B_1 &= \{b \in \mathbb{Z} \mid b \geq 28 \text{ and } b \equiv 11 \pmod{17}\},\\
B_2 &= \{b \in \mathbb{Z} \mid b \geq 34 \text{ and } b \equiv 13 \pmod{21}\},\\
B_3 &= \{b \in \mathbb{Z} \mid b \geq 27 \text{ and } b \equiv 12 \pmod{15}\},\\
B_4 &= \{b \in \mathbb{Z} \mid b \geq 31 \text{ and } b \equiv 14 \pmod{17}\}.
\end{aligned}$$

(7) $$\nu_{\mathrm{reg}}(b,9) = \begin{cases} \left[\dfrac{b}{30}\right] + 1 & \text{if } b \equiv 10, 15, 16, 20, 21, 22, 25, 26, 27, 28 \pmod{30},\\[2ex] \left[\dfrac{b}{30}\right] & \text{otherwise.} \end{cases}$$

(8) $$\nu_{\mathrm{reg}}(b,11) = \begin{cases} 1 & \text{if } b \geq 12 \text{ and } 6 \mid b,\\ 0 & \text{otherwise.} \end{cases}$$

Remark 4. (1) *The intersections of the sets* A_1, A_2, *and* A_3 *in Corollary 3 (4) are the following:*

$$\begin{aligned}
A_1 \cap A_2 &= \{b \in \mathbb{Z} \mid b \geq 55 \text{ and } b \equiv 55 \pmod{105}\},\\
A_2 \cap A_3 &= \{b \in \mathbb{Z} \mid b \geq 62 \text{ and } b \equiv 62 \pmod{63}\},\\
A_1 \cap A_3 &= \emptyset.
\end{aligned}$$

(2) *The intersections of the sets* B_1, B_2, B_3, *and* B_4 *in Corollary 3 (6) are the following:*

$$\begin{aligned}
B_1 \cap B_2 &= \{b \in \mathbb{Z} \mid b \geq 181 \text{ and } b \equiv 181 \pmod{357}\},\\
B_1 \cap B_3 &= \{b \in \mathbb{Z} \mid b \geq 147 \text{ and } b \equiv 147 \pmod{255}\},\\
B_2 \cap B_4 &= \{b \in \mathbb{Z} \mid b \geq 286 \text{ and } b \equiv 286 \pmod{357}\},\\
B_3 \cap B_4 &= \{b \in \mathbb{Z} \mid b \geq 255 \text{ and } b \equiv 255 \pmod{255}\},\\
B_1 \cap B_4 &= B_2 \cap B_3 = \emptyset.
\end{aligned}$$

Remark 5. *We can see that Corollary 3(1)–(5) matches the values of* ν_{r} *in the list in Example 2.*

Proof. We see immediately that Parts (1)–(6) and (8) are implied by the respective formulas obtained in Theorem 3(1), (2), (4) and Theorem 4 for the respective digits n, since these formulas give distinct n-digit regular Kaprekar constants for distinct positive integers m, and we see that:

$$A_1 \cap A_3 = B_1 \cap B_4 = B_2 \cap B_3 = \emptyset$$

as mentioned in Remark 4.

Now, we prove Part (7) for the case where $n = 9$. Since the formula obtained in Theorem 3(3) gives distinct b-adic nine-digit regular Kaprekar constants for distinct pairs (b, m) of suitable integers b and m, we see that:

$$\nu_{\mathrm{reg}}(b,9) = \sharp \left\{ m \in \mathbb{Z} \;\middle|\; m \geq 1,\ \frac{b-5}{6} < m < \frac{b-4}{5} \right\},$$

where the symbol $\sharp$ stands for the number of all elements in the set.

For any integer $b' \geq 0$, we then see that:

$$\nu_{\mathrm{reg}}(b,9) = \begin{cases} b' & \text{if } 30b'+2 \leq b \leq 30b'+9, \\ b'+1 & \text{if } b = 30b'+10, \\ b' & \text{if } 30b'+11 \leq b \leq 30b'+14, \\ b'+1 & \text{if } 30b'+15 \leq b \leq 30b'+16, \\ b' & \text{if } 30b'+17 \leq b \leq 30b'+19, \\ b'+1 & \text{if } 30b'+20 \leq b \leq 30b'+22, \\ b' & \text{if } 30b'+23 \leq b \leq 30b'+24, \\ b'+1 & \text{if } 30b'+25 \leq b \leq 30b'+28, \\ b' & \text{if } b = 30b'+29, \\ b'+1 & \text{if } 30b'+30 \leq b \leq 30b'+31. \end{cases}$$

Therefore, Part (7) is proven. □

Moreover, as a corollary to Proposition 1, we can obtain lower bounds for $\nu_{\mathrm{reg}}(b,n)$ with $n = 13, 15, 17$ as in the following:

Corollary 4. *Let $b \geq 2$ be any integer. Then, we have the following estimations:*

(1) $\nu_{\mathrm{reg}}(b,13) \geq 1$ if $b \geq 14$ and $7 \mid b$.

(2) $\nu_{\mathrm{reg}}(b,15) \geq \sum_{\frac{b-7}{9} \leq m \leq \frac{b-8}{8}} (b-8m-7) + \sum_{\frac{b-5}{11} \leq m \leq \frac{b-8}{9}} \left(m - \left[\frac{b-9m}{2}\right] + 3\right)$*, where the symbol m in the sums stands for positive integers.*

(3) $\nu_{\mathrm{reg}}(b,17) \geq \sharp \left\{ k \in \mathbb{Z} \,\middle|\, k \geq 2, b \equiv k \pmod 4, 0 \leq \frac{b-9k}{4} \leq \left[\frac{k}{2}\right] - 1 \right\}.$

Proof. (1) We see immediately that Part (1) is implied by the conditional formula obtained in Proposition 1(1), since the formula gives distinct $(7m+7)$-adic 13-digit regular Kaprekar constants for distinct positive integers m.

(2) Since the conditional formula obtained in Proposition 1(2) gives distinct b-adic 15-digit regular Kaprekar constants for distinct triples (b, m_1, m_2) of suitable integers b, m_1, and m_2, we see that:

$$\begin{aligned} \nu_{\mathrm{reg}}(b,15) \geq \sharp\{(m_1,m_2) \in \mathbb{Z} \times \mathbb{Z} \mid m_1 \geq 1,\ & 2m_1+1 < m_2 < 3m_1+2, \\ & 6m_1+m_2+5 < b < 5m_1+2m_2+4\}. \end{aligned}$$

For any integer $m_1 \geq 1$, the list of m_2 and b satisfying the conditions:

$$2m_1+1 < m_2 < 3m_1+2, \quad 6m_1+m_2+5 < b < 5m_1+2m_2+4$$

is the following:

m_2	b
$2m_1+2$	$8m_1+8, \ldots, 9m_1+7$
$2m_1+3$	$8m_1+9, \ldots, 9m_1+8,\ \ 9m_1+9$
$\vdots$	$\vdots \quad \ddots \quad \vdots \qquad \vdots \quad \ddots$
$3m_1+1$	$9m_1+7, \ldots, 10m_1+6, 10m_1+7, \ldots, 11m_1+5$

Since the number of b's appearing in the list above is equal to:

$$\begin{cases} (b+1)-(8m_1+8) & \text{if } 8m_1+8 \le b \le 9m_1+7, \\ (m_1-1)-\left[\dfrac{b-(9m_1+8)}{2}\right] & \text{if } 9m_1+8 \le b \le 11m_1+5, \end{cases}$$

the right-hand side in the inequality above is equal to:

$$\sum_{\frac{b-7}{9} \le m \le \frac{b-8}{8}} (b-8m-7) + \sum_{\frac{b-5}{11} \le m \le \frac{b-8}{9}} \left(m - \left[\frac{b-9m}{2}\right] + 3 \right),$$

where the symbol m in the sums stands for positive integers. Therefore, Part (2) is proven.

(3) Since the conditional formula obtained in Proposition 1(3) gives distinct b-adic 17-digit regular Kaprekar constants for distinct pairs (b,m) of suitable integers b and m, we see that:

$$\nu_{\mathrm{reg}}(b,17) \ge \sharp\{m \in \mathbb{Z} \mid m \ge 1, 9m+7 < b < 11m+9, b \equiv m+1 \pmod 4\}.$$

For any integer $m \ge 1$, the first term and the final term in the range $9m+7 < b < 11m+9$ of the arithmetic progression with the common difference of four, which are congruent to $m+1$ modulo four, are $9m+9$ and $(9m+9)+4\left(\left[\dfrac{m+1}{2}\right]-1\right)$, respectively. Putting $k = m+1$, we then see that:

$$\begin{aligned} &\sharp\{m \in \mathbb{Z} \mid m \ge 1,\ 9m+7 < b < 11m+9,\ b \equiv m+1 \pmod 4\} \\ =\ &\sharp\left\{k \in \mathbb{Z} \;\middle|\; k \ge 2,\ b \equiv k \pmod 4,\ 0 \le \frac{b-9k}{4} \le \left[\frac{k}{2}\right]-1\right\}, \end{aligned}$$

and Part (3) is proven. □

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Errata of [1]: Since the reference [1] is very important to readers of this article, we would like to describe the errata of [1] here:

p. 263, ℓ. 32, $N(b,2)$ and $\ell(b,2) \to N(b,5)$ and $\ell(b,5)$
p. 266, ℓ.7, 14, 16, 18, 19, 20, 21, 23, 24: $(c0)_2 \to (c0)_b$
p. 266, ℓ.16: $((c-1)(b-c))_2 \to ((c-1)(b-c))_b$
p. 266, ℓ.14, 19: $((\delta_1(c)-1)(b-\delta_1(c)))_2 \to ((\delta_1(c)-1)(b-\delta_1(c)))_b$
p. 266, ℓ.21: $(c-1)(b-c))_2 \to ((c-1)(b-c))_b$
p. 266, ℓ.24: $((\delta_{v_2(b+1)-v_2+1}(c)-1)(b-\delta_{v_2(b+1)-v_2+1}(c)))_2$
$\to ((\delta_{v_2(b+1)-v_2(c)+1}(c)-1)(b-\delta_{v_2(b+1)-v_2(c)+1}(c)))_b$
p. 267, ℓ.2, 3: $(c0)_2 \to (c0)_b$
p. 269, ℓ.11: $n \ge 7$ and $\to n \ge 7$; n is odd and
p. 269, ℓ.12: $c_{\frac{n}{2}-2} \to c_{\frac{n-1}{2}-2}$
p.280, ℓ.16: Delete the sentence "A.L. Ludington, A bound on Kaprekar constants, J. Reine Angew. Math. 310 (1979) 196–203."

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