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Generalized Weyl-Heisenberg Algebra, Qudit Systems and Entanglement Measure of Symmetric States via Spin Coherent States. Part II: The Perma-Concurrence Parameter

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Abstract: This paper deals with separable and entangled qudits $|\psi_d\rangle$ (quantum states in dimension d) constructed from Dicke states made of $N = d - 1$ qubits. Such qudits present the property to be totally symmetric under the interchange of the N qubits. We discuss the notion of perma-concurrence P_d for the qudit $|\psi_d\rangle$, introduced by the authors (*Entropy* 2018, 20, 292), as a parameter for characterizing the entanglement degree of $|\psi_d\rangle$. For $d = 3$, the perma-concurrence P_3 constitutes an alternative to the concurrence C for symmetric two-qubit states. We give several expressions of P_d (in terms of matrix permanent and in terms of unit vectors of $\mathbb{R}^3$ pointing on the Bloch sphere) and precise the range of variation of P_d (going from separable to maximally entangled states). Numerous examples are presented for P_d . Special attention is devoted to states of W type and to maximally entangled states of Bell and Greenberger–Horne–Zeiling type.

Keywords: generalized Weyl-Heisenberg algebra; qubits and qudits; Dicke states; SU(2) coherent states; entanglement; concurrence; perma-concurrence; Majorana stars

1. Introduction

Entanglement in multi-qubit systems is a valuable resource in implementing several quantum protocols offering significant advantages in the communication and processing of information [1]. Quantum teleportation [2,3] and quantum cryptography or quantum key distribution [4] constitute promising applications in quantum information science.

The characterisation, classification and identification of genuine entanglement in multipartite quantum systems remain a challenging issue and continue presently to attract attention in the field of quantum information theory. Several measures of quantum correlations in bipartite and multipartite quantum systems have been introduced in the literature [5–8]. The Wootters concurrence and the entanglement of formation are two examples of quantitative measures of entanglement [9]; the concurrence is a simple tool to characterize the degree of entanglement of two-qubit systems in pure states. Another indicator of quantum correlations in bipartite quantum systems is the quantum discord based on the von Neumann entropy. This parameter was introduced by Ollivier and Zurek [10] and by Henderson and Vedral [11]; it goes beyond entanglement and provide a proper tool to investigate the quantum correlations in an arbitrary bipartite state even for a separable state. In general, the analytical evaluation of the quantum discord is difficult and only partial results were obtained for some special two-qubit states.

Unfortunately, there is no simple way of measuring entanglement for multi-qubit systems. It is thus of paramount importance to find an adequate formalism to understand the genuine entanglement in multi-qubit systems. The first attempt to find a measure of three-qubit entanglement was made by Coffmann et al. by introducing the notion of 3-tangle [12]. This idea was extended to N -tangle by Wong and Christensen [13] for an even number N of qubits. Clearly, developing the tools to distinguish between different classes of multipartite entangled states is essential in quantum information theory. In this respect, the classification of mutli-entangled states was approached via local unitary operations, local operations and classical communication, and stochastic local operations assisted by classical communication (see the list of references quoted in [14]). Another way to deal with entanglement in multi-qubit systems consists of using the entanglement witness method [7,15]. Despite the intensive efforts dedicated to this problem, the determination of quantum correlations in multi-qubit states remains a highly non-trivial issue in entanglement theory. Even though several tools and parameters were introduced, it remains difficult in several cases to decide between entanglement and separability of a given multipartite quantum state. This problem was shown to belong to the complexity class NP in the dimension of the quantum system.

In this paper, we shall use the notion of the perma-concurrence P_d introduced in [16] to deal with the entanglement of a state $|\psi_d\rangle$ involving $N = d - 1$ ($d \geq 3$) qubits, state invariant under the group S_N . The paper is organized as follows. Section 2 is concerned with some generalities (generalized Weyl-Heisenberg algebra, Dicke states and Majorana states). The perma-concurrence P_d is studied with numerous examples in Section 3. Sections 4 and 5 are devoted to the perma-concurrence for states of W type and states of Bell and Greenberger–Horne–Zeilinger (GHZ) type, respectively. Some conclusions are given in a last section.

In the present article, we use the following notations: $A^\dagger$, $[A, B]$, $\mathbb{I}$, $\delta_{i,j}$, and C_N^k stand for the adjoint of the operator A , the commutator of the operators A and B , the identity operator, the Kronecker δ , and the binomial coefficient, respectively.

2. Generalities

In this section we briefly review, in a quantum information context, the main ingredients of the generalized Weyl-Heisenberg algebra [17,18] for which the Dicke states [16,19–21] and the Majorana states [22] play a central role. The interest of Dicke states and Majorana states is important for quantum information. For instance, some particular classes of Dicke states are known to be decoherence free or noiseless under the environment effects [23]. This important property makes this kind of states very useful in encoding quantum information in qudit systems [24]. Furthermore, it has been mentioned that the Majorana representation of Dicke states can be of special interest in investigating and understanding decoherence free or noiseless multi-qubit systems [25].

2.1. Generalized Weyl-Heisenberg Algebra

Let $\mathcal{H}_2$ be the two-dimensional Hilbert space with an orthonormal basis $\{|0\rangle, |1\rangle\}$. In quantum mechanics, the vectors $|0\rangle$ and $|1\rangle$ can represent the states for a two-level system (e.g., $|0\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ and $|1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ for a $\frac{1}{2}$ -spin particle); the states $|0\rangle$ and $|1\rangle$ are then refereed to as qubits. The operators

$$q^- = |0\rangle\langle 1|, \quad q^+ = |1\rangle\langle 0|, \quad K = |1\rangle\langle 1|$$

satisfy the relations

$$(q^-)^\dagger = q^+, \quad K^\dagger = K, \quad [q^-, q^+] = \mathbb{I} - 2K, \quad [K, q^+] = q^+, \quad [K, q^-] = -q^- \quad (1)$$

The 2^N -dimensional Hilbert space $\mathcal{H}_{2^N}$ for a system of N qubits is made of linear combinations of vectors belonging to

$$\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2$$

with N factors. Let us consider the set $\{q_i^-, q_i^+, K_i : i = 1, 2, \dots, N\}$ of operators acting on $\mathcal{H}_{2^N}$. As an extension of Equation (1), we take

$$(q_i^-)^\dagger = q_i^+, \quad K_i^\dagger = K_i, \quad [q_i^-, q_j^-] = [q_i^+, q_j^+] = 0$$

$$[q_i^-, q_j^+] = \delta_{i,j}(\mathbb{I} - 2K_i), \quad [K_i, q_j^+] = \delta_{i,j}q_i^+, \quad [K_i, q_j^-] = -\delta_{i,j}q_i^-$$

By introducing

$$q^- = \sum_{i=1}^N q_i^-, \quad q^+ = \sum_{i=1}^N q_i^+, \quad K = \sum_{i=1}^N K_i$$

we easily obtain the commutation relations

$$[q^-, q^+] = N\mathbb{I} - 2K, \quad [K, q^+] = q^+, \quad [K, q^-] = -q^-$$

which generalize Equation (1). By putting

$$a^\pm = \frac{1}{\sqrt{N}}q^\pm$$

then Equation (1) can be rewritten as

$$[a^-, a^+] = \mathbb{I} + 2\kappa K, \quad [K, a^+] = a^+, \quad [K, a^-] = -a^-$$

where

$$\kappa = -\frac{1}{N}$$

so that the set $\{a^-, a^+, K, \mathbb{I}\}$ generates the generalized Weyl-Heisenberg algebra $\mathcal{A}_\kappa$ defined in [17,18].

As a conclusion, the algebra $\mathcal{A}_\kappa$ can be described by a set of N qubits. Here, $\kappa < 0$. Therefore, according to the representation theory of $\mathcal{A}_\kappa$, the algebra $\mathcal{A}_\kappa$ admits finite-dimensional representations (for more details, see Ref. [17,18] and references therein). The Dicke states (to be defined below) can serve to construct a representation of dimension $d = N + 1$ of $\mathcal{A}_\kappa$.

2.2. Dicke States

The Hilbert space $\mathcal{H}_{2^N}$ can be partitioned as

$$\mathcal{H}_{2^N} = \bigoplus_{k=0}^N \mathcal{F}_{N,k}$$

where the subspace $\mathcal{F}_{N,k}$, of dimension C_N^k , is spanned by the orthonormal set

$$S_{N,k} = \{|\eta_1\eta_2\cdots\eta_N\rangle : \eta_1 + \eta_2 + \cdots + \eta_N = k\}$$

where in each vector

$$|\eta_1\eta_2\cdots\eta_N\rangle = |\eta_1\rangle \otimes |\eta_2\rangle \otimes \cdots \otimes |\eta_N\rangle, \quad \eta_i = 0, 1, \quad i = 1, 2, \dots, N$$

of $S_{N,k}$ there are $N - k$ qubits $|0\rangle$ and k qubits $|1\rangle$ (i.e., $N - k + k = N$ qubits in all).

The Dicke state $|N; k\rangle$ is the normalized symmetrical superposition [16,19–21]

$$|N; k\rangle = (C_N^k)^{-\frac{1}{2}} \sum_{|x\rangle \in S_{N,k}} |x\rangle$$

where the summation over $|x\rangle$ is extended over all the states of $S_{N,k}$. Note that

$$\langle N; k | N; \ell \rangle = \delta_{k,\ell}, \quad k, \ell = 0, 1, \dots, N$$

so that the set

$$\{|N; k\rangle : k = 0, 1, \dots, N\}$$

of the $N + 1$ symmetrical vectors $|N; k\rangle$ constitutes an orthonormal basis for a subspace of $\mathcal{H}_{2N}$; let us denote as $\mathcal{G}_d$ this subspace of dimension $d = N + 1$.

Any normalized vector

$$|\psi_d\rangle = c_0|N; 0\rangle + c_1|N; 1\rangle + \dots + c_N|N; N\rangle, \quad N = d - 1, \quad \sum_{k=0}^N |c_k|^2 = 1, \quad c_k \in \mathbb{C} \quad (2)$$

of $\mathcal{G}_d$ is a qudit expressed in terms of N qubits. Each state $|\psi_d\rangle$ is a basis vector for the symmetric irreducible representation $[N]$ of the permutation group S_N . When $N \rightarrow \infty$, a convenient limiting procedure shows that the space $\mathcal{G}_\infty$ corresponds to the space of the states of the harmonic oscillator (for N going to infinity, then the generalized Weyl-Heisenberg algebra $\mathcal{A}_{-\frac{1}{N}}$ goes to the harmonic oscillator algebra).

As a non-trivial example, it can be verified that the 6 vectors

$$\begin{aligned} |5; 0\rangle &= |00000\rangle \\ |5; 1\rangle &= \frac{1}{\sqrt{5}}(|00001\rangle + |00010\rangle + |00100\rangle + |01000\rangle + |10000\rangle) \\ |5; 2\rangle &= \frac{1}{\sqrt{10}}(|00011\rangle + |00101\rangle + |00110\rangle + |01001\rangle + |01010\rangle \\ &\quad + |01100\rangle + |10001\rangle + |10010\rangle + |10100\rangle + |11000\rangle) \\ |5; 3\rangle &= \frac{1}{\sqrt{10}}(|00111\rangle + |01011\rangle + |01101\rangle + |01110\rangle + |10011\rangle \\ &\quad + |10101\rangle + |10110\rangle + |11001\rangle + |11010\rangle + |11100\rangle) \\ |5; 4\rangle &= \frac{1}{\sqrt{5}}(|01111\rangle + |10111\rangle + |11011\rangle + |11101\rangle + |11110\rangle) \\ |5; 5\rangle &= |11111\rangle \end{aligned}$$

form an orthonormal basis for the space $\mathcal{G}_6$.

2.3. Majorana States

By using the spin $\frac{1}{2}$ -spin coherent states (the so-called SU(2) coherent states)

$$|z_i\rangle = N_i(|0\rangle + z_i|1\rangle), \quad N_i = \frac{1}{\sqrt{1 + \bar{z}_i z_i}}, \quad i = 1, 2, \dots, N$$

the state $|\psi_d\rangle$ in (2) can be rewritten in the Majorana representation [22]. Note that, when necessary, the state $|z_i\rangle$ shall be denoted as $|z_i\rangle_M$ in order to avoid a confusion with the qubit $|1\rangle$; the state $|z_i = 0\rangle_M$ is identical to the qubit $|0\rangle$ but the limit state $|z_i\rangle_M$ with $|z_i| = \infty$ corresponds to the qubit $|1\rangle$ while the state $|z_i = 1\rangle_M$ is nothing but the qubit $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. In the Majorana representation and in view of the invariance of $|\psi_d\rangle$ over S_N , the state $|\psi_d\rangle$ takes the form

$$|\psi_d\rangle = \mathcal{N}_d \sum_{\sigma \in S_N} \sigma(|z_1\rangle \otimes |z_2\rangle \otimes \dots \otimes |z_N\rangle) \quad (3)$$

where the summation over σ runs on all the permutations of S_N and $\mathcal{N}_d$ is a normalization factor given by

$$|\mathcal{N}_d|^{-2} = N! \sum_{\sigma \in S_N} \prod_{i=1}^N \langle z_i | z_{\sigma(i)} \rangle = N! \frac{1}{\prod_{j=1}^N (1 + \bar{z}_j z_j)} \sum_{\sigma \in S_N} \prod_{i=1}^N (1 + \bar{z}_i z_{\sigma(i)}).$$

The complex numbers z_i for $i = 1, 2, \dots, N$ are the roots (called Majorana roots or Majorana stars) of the d'Alembert equation of degree N (for $c_0 \neq 0$)

$$\sum_{k=0}^N (-1)^k \sqrt{\frac{N!}{k!(N-k)!}} c_k z^{N-k} = 0 \quad (4)$$

in terms of the coefficients c_k for $k = 1, 2, \dots, N$. Conversely, the coefficients c_k follow from

$$c_k = N! N_1 N_2 \cdots N_N \mathcal{N}_d \sqrt{\frac{k!(N-k)!}{N!}} s_k(z_1 z_2 \cdots z_N) \quad (5)$$

where

$$s_0(z_1 z_2 \cdots z_N) = 1, \quad s_k(z_1 z_2 \cdots z_N) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} z_{i_1} z_{i_2} \cdots z_{i_k}, \quad k = 1, 2, \dots, N. \quad (6)$$

In the case where the solutions of (4) are degenerate, i.e.,

$$z = z_1 = z_2 = \cdots = z_N \quad (7)$$

the state (3) becomes the completely separable state

$$|\psi_d\rangle = |z\rangle \otimes |z\rangle \otimes \cdots \otimes |z\rangle = \frac{1}{(1 + \bar{z}z)^{\frac{N}{2}}} \sum_{k=0}^N \sqrt{\frac{N!}{k!(N-k)!}} z^k |N; k\rangle$$

which is a Perelomov SU(2) coherent state.

2.4. Procedures for Calculating Majorana States

Given the state $|\psi_d\rangle$ in the form (2) with $c_0 \neq 0$, the Majorana stars z_i ($i = 1, 2, \dots, N$) can be calculated from Equation (4). In the case where $c_k = 0$ for $k = 0, 1, \dots, k_{\max}$ (with $k_{\max} \leq N - 1$), part of the Majorana stars z_i can be calculated from (4) and part from the zeros of the Bargmann function associated with the state $|\psi_d\rangle$ (see Ref. [16]); in this case, Equations (5) and (6) can also be very useful for determining the Majorana stars z_i (or checking the obtained values for z_i).

3. The Perma-Concurrence

3.1. Expressions and Properties of the Perma-Concurrence

We are now in a position to investigate the parameter P_d associated with the qudit $|\psi_d\rangle$, or more precisely $P_d(\psi_d)$, introduced in Ref. [16] via

$$P_d = \frac{1}{N!} \text{perm}(A_N) \quad (8)$$

where $\text{perm}(A_N)$ is the permanent of the matrix A_N , of dimension $N \times N$, with elements

$$(A_N)_{ij} = \langle z_i | z_j \rangle = \frac{1 + \bar{z}_i z_j}{\sqrt{(1 + |z_i|^2)(1 + |z_j|^2)}} \quad (9)$$

for $i, j = 1, 2, \dots, N$. A simple development of $\text{perm}(A_N)$

$$\text{perm}(A_N) = \sum_{\sigma \in S_N} \prod_{i=1}^N \langle z_i | z_{\sigma(i)} \rangle = \frac{1}{\prod_{j=1}^N (1 + \bar{z}_j z_j)} \sum_{\sigma \in S_N} \prod_{i=1}^N (1 + \bar{z}_i z_{\sigma(i)}) \quad (10)$$

that gives

$$\text{perm}(A_N) = \frac{1}{N!} |\mathcal{N}_d|^{-2}$$

and finally we have

$$P_d = \frac{1}{(N! |\mathcal{N}_d|)^2} \quad (11)$$

a very useful relation for calculating P_d .

In the case where $d = 3$, the parameter P_d is an alternative to the parameter C , called concurrence, for symmetric two-qubit states (see Section 3.3 too). For d arbitrary, the parameter $P_d(\psi_d)$, called perma-concurrence, can serve as a measure of the entanglement of the totally symmetric N -qubit state $|\psi_d\rangle$. The range of variations of P_d is

$$\frac{1}{2^{N-1}} \leq P_d \leq 1, \quad N = d - 1$$

(the range of variations $\frac{1}{N} \leq P_d \leq 1$ mentioned in [16] does not include maximally entangled states, except for $N = 2$). The upper limit $P_d = 1$ corresponds to separable states for which all the Majorana roots (7) are identical while the lower limit $P_d = \frac{1}{2^{N-1}}$ corresponds to maximally entangled states.

As an important property, it can be proved that

$$P_d(|\psi_d\rangle) = P_d(|\varphi_d\rangle)$$

where the qudit $|\varphi_d\rangle$ is deduced from the qudit $|\psi_d\rangle$ by interchanging the qubits $|0\rangle$ and $|1\rangle$ in $|\psi_d\rangle$.

Another useful expression of P_d can be obtained by introducing the inner products $n_i \cdot n_j$ ($i, j = 1, 2, \dots, N$), where the unit vectors n_i and n_j of $\mathbb{R}^3$ are defined through

$$n_k = \left(\frac{z_k + \bar{z}_k}{1 + z_k \bar{z}_k}, -i \frac{z_k - \bar{z}_k}{1 + z_k \bar{z}_k}, \frac{1 - z_k \bar{z}_k}{1 + z_k \bar{z}_k} \right)$$

with $i = \sqrt{-1}$. Indeed, we have

$$n_i \cdot n_j = n_j \cdot n_i = 2|\langle z_i | z_j \rangle|^2 - 1 \Leftrightarrow |\langle z_i | z_j \rangle|^2 = \frac{1}{2}(1 + n_i \cdot n_j) \quad (12)$$

that makes it possible to rewrite (10) in terms of $n_i \cdot n_j$ (note that $-1 \leq n_i \cdot n_j \leq 1 \Leftrightarrow 0 \leq |\langle z_i | z_j \rangle|^2 \leq 1$).

3.2. Procedures for Calculating the Perma-Concurrence

For a given expression of $|\psi_d\rangle$ in terms of Dicke states, once all the necessary Majorana stars are calculated, the corresponding value of P_d can be obtained from Equations (8)–(10). Another way to obtain the value of P_d is to use (11) where $\mathcal{N}_d$ follows with the help of (5) and (6).

We continue with two illustrative calculations of P_d : one for which Equation (4) is sufficient for calculating all the necessary Majorana stars and another one for which (4) allows obtaining only a part of the necessary Majorana stars.

3.2.1. Example 1

The Majorana stars for the state

$$|\psi_5\rangle = \sqrt{\frac{6}{3931}} \left(|4;0\rangle - |4;1\rangle - \frac{13}{\sqrt{6}} |4;2\rangle + 7|4;3\rangle + 24|4;4\rangle \right)$$

or

$$\begin{aligned} |\psi_5\rangle = & \sqrt{\frac{6}{3931}} \left[|0000\rangle - \frac{1}{2} \left(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle \right) \right. \\ & - \frac{13}{6} \left(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle \right) \\ & \left. + \frac{7}{2} \left(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle \right) + 24|1111\rangle \right] \end{aligned}$$

can be derived by introducing the coefficients

$$c_0 = \sqrt{\frac{6}{3931}}, \quad c_1 = -c_0, \quad c_2 = -\frac{13}{\sqrt{6}}c_0, \quad c_3 = 7c_0, \quad c_4 = 24c_0$$

into Equation (4) for $N = 4$. This leads to

$$z^4 + 2z^3 - 13z^2 - 14z + 24 = 0$$

that admits the solutions

$$z_1 = -4, \quad z_2 = -2, \quad z_3 = 1, \quad z_4 = 3$$

from which the inner products $\langle z_i | z_j \rangle$ with $i, j = 1, 2, 3, 4$ can be straightforwardly calculated. The value of P_5 follows from the application of (8)–(10). This leads to

$$P_5 = \frac{3931}{2^3 \times 3 \times 5^2 \times 17}$$

in accordance with $\frac{1}{8} \leq P_5 \leq 1$. Remark that P_5 also follows from (11) where $\mathcal{N}_d$ can be deduced from the coefficient c_0 given by (5). The result for P_5 shows that $|\psi_5\rangle$ is an entangled state but not a maximally entangled state.

3.2.2. Example 2

For the state

$$|\psi_4\rangle = \frac{1}{2} \left(\sqrt{3}|3;1\rangle + |3;3\rangle \right) = \frac{1}{2} \left(|001\rangle + |010\rangle + |100\rangle + |111\rangle \right)$$

the *a priori* fourth-degree d'Alembert equation, see (4), is reduced to the equation of degree 2

$$3z^2 + 1 = 0$$

that yields the two Majorana stars

$$z_1 = \frac{i}{\sqrt{3}}, \quad z_2 = -\frac{i}{\sqrt{3}}$$

while the third Majorana star z_3 is such that

$$|z_3| = \infty$$

corresponding to the zero of the Bargmann function related to the state $|\psi_4\rangle$ (see Ref. [16]). From Equation (3), it follows that $|\psi_4\rangle$ can be rewritten as

$$|\psi_4\rangle = \mathcal{N}_4 \sum_{\sigma \in \mathcal{S}_3} \sigma(|z_1\rangle_M \otimes |z_2\rangle_M \otimes |z_3\rangle_M)$$

where

$$|z_1\rangle_M = \frac{1}{2}(\sqrt{3}|0\rangle + i|1\rangle), \quad |z_2\rangle_M = \frac{1}{2}(\sqrt{3}|0\rangle - i|1\rangle), \quad |z_3\rangle_M = |1\rangle.$$

The overlaps between the three states $|z_i\rangle_M$ are given by

$${}_M\langle z_1|z_2\rangle_M = \frac{1}{2}, \quad {}_M\langle z_1|z_3\rangle_M = -\frac{i}{2}, \quad {}_M\langle z_2|z_3\rangle_M = \frac{i}{2}$$

so that the 3×3 matrix A_3 is

$$A_3 = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & 1 & \frac{i}{2} \\ \frac{i}{2} & -\frac{i}{2} & 1 \end{pmatrix}$$

cf. Equation (9). The calculation of the permanent of A_3 yields $\text{perm}(A_3) = \frac{3}{2}$. Finally, Equation (8) leads to $P_4 = \frac{1}{4}$, a result showing that the 3-qubit state $|\psi_4\rangle$ is maximally entangled.

3.3. Perma-Concurrence for $N = 2$ ($d = 3$)

Given the symmetric two-qubit state

$$|\psi_3\rangle = c_0|2;0\rangle + c_1|2;1\rangle + c_2|2;2\rangle, \quad |c_0|^2 + |c_1|^2 + |c_2|^2 = 1$$

it is easy to show that $|\psi_3\rangle$ can be rewritten as

$$|\psi_3\rangle = \frac{1}{2\sqrt{P_3}}(|z_1\rangle \otimes |z_2\rangle + |z_2\rangle \otimes |z_1\rangle)$$

with

$$P_3 = \frac{1}{2}\text{perm}(A_2) = \frac{1}{2}(1 + |\langle z_1|z_2\rangle|^2)$$

where the Majorana stars z_1 and z_2 can be determined from the procedures in Section 2.4 with

$$c_0z^2 - \sqrt{2}c_1z + c_2 = 0$$

cf. Equation (4) for $N = 2$. The perma-concurrence P_3 reads

$$P_3 = \frac{1}{4}(3 + n_1 \cdot n_2) \quad (13)$$

in terms of the units vectors n_1 and n_2 .

It can be shown that

$$|\langle z_1|z_2\rangle|^2 = \frac{1-C}{1+C}$$

where C is the concurrence

$$C = |c_1^2 - 2c_0c_2|$$

so that

$$P_3 = \frac{1}{1+C} \Leftrightarrow C = \frac{1}{P_3} - 1$$

and the range of variations of P_3 and C are clearly

$$\frac{1}{2} \leq P_3 \leq 1 \Leftrightarrow 1 \geq C \geq 0. \quad (14)$$

Equation (14) shows that the situation where $C = 0 \Leftrightarrow P_3 = 1$ characterizes a non entangled or separable state $|\psi_3\rangle$; in this case $\langle z_1|z_2\rangle = 1 \Leftrightarrow z_1 = z_2 \equiv z$ and $|\psi_3\rangle = |z\rangle \otimes |z\rangle$. Furthermore, the situation $C = 1 \Leftrightarrow P_3 = \frac{1}{2}$ describes a maximally entangled state $|\psi_3\rangle$; in this case $\langle z_1|z_2\rangle = 0 \Leftrightarrow z_1 = -\frac{1}{z_2} \equiv z$ and $|\psi_3\rangle = |z\rangle \otimes |-\frac{1}{z}\rangle$.

As an illustration, Table 1 lists the values of the perma-concurrence P_3 and the concurrence C for some states $|\psi_3\rangle$.

Table 1. The perma-concurrence P_3 and the concurrence C for some states $c_0|00\rangle + c_1\frac{|01\rangle+|10\rangle}{\sqrt{2}} + c_2|11\rangle$. The Majorana state vectors $|z_1\rangle_M$ and $|z_2\rangle_M$ are expressed in terms of the qubits $|0\rangle$ and $|1\rangle$. The first three states are the maximally entangled symmetric Bell states (with $P_3 = \frac{1}{2}$) and the last two states are separable symmetric states (with $P_3 = 1$); the value of P_3 for the other states interpolates between $\frac{1}{2}$ and 1.

c_0	c_1	c_2	z_1	$ z_1\rangle_M$	z_2	$ z_2\rangle_M$	P_3	C
$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	i	$\frac{ 0\rangle+i 1\rangle}{\sqrt{2}}$	-i	$\frac{ 0\rangle-i 1\rangle}{\sqrt{2}}$	$\frac{1}{2}$	1
$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	1	$\frac{ 0\rangle+ 1\rangle}{\sqrt{2}}$	-1	$\frac{ 0\rangle- 1\rangle}{\sqrt{2}}$	$\frac{1}{2}$	1
0	1	0	0	$ 0\rangle$	$ z_2 = \infty$	$ 1\rangle$	$\frac{1}{2}$	1
0	$c_1 \neq 0$	c_2	$\frac{c_2}{\sqrt{2}c_1}$	$\frac{ 0\rangle+z_1 1\rangle}{\sqrt{1+ z_1 ^2}}$	$ z_2 = \infty$	$ 1\rangle$	$\frac{1}{1+ c_1 ^2}$	$ c_1 ^2$
$\frac{1}{2\sqrt{2}}$	$\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{3}}{2}$	$\frac{1+\sqrt{1+2\sqrt{6}}}{\sqrt{2}}$	$\frac{ 0\rangle+z_1 1\rangle}{\sqrt{1+ z_1 ^2}}$	$\frac{1-\sqrt{1+2\sqrt{6}}}{\sqrt{2}}$	$\frac{ 0\rangle+z_2 1\rangle}{\sqrt{1+ z_2 ^2}}$	$\frac{8}{57}(9-2\sqrt{6})$	$\frac{1+2\sqrt{6}}{8}$
0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{2} 0\rangle+ 1\rangle}{\sqrt{3}}$	$ z_2 = \infty$	$ 1\rangle$	$\frac{2}{3}$	$\frac{1}{2}$
0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{\sqrt{2} 0\rangle- 1\rangle}{\sqrt{3}}$	$ z_2 = \infty$	$ 1\rangle$	$\frac{2}{3}$	$\frac{1}{2}$
1	0	0	0	$ 0\rangle$	0	$ 0\rangle$	1	0
0	0	1	$ z_1 = \infty$	$ 1\rangle$	$ z_2 = \infty$	$ 1\rangle$	1	0

3.4. Perma-Concurrence for $N = 3$ ($d = 4$)

For $N = 3$, the Majorana stars z_i ($i = 1, 2, 3$) follow from the procedures in Section 2.4 used in conjunction with the cubic equation

$$c_0z^3 - \sqrt{3}c_1z^2 + \sqrt{3}c_2z - c_3 = 0$$

cf. (4). Then, Equations (8)–(10) lead to

$$P_4 = \frac{1}{6}(1 + |\langle z_1|z_2\rangle|^2 + |\langle z_2|z_3\rangle|^2 + |\langle z_3|z_1\rangle|^2 + \langle z_1|z_2\rangle\langle z_2|z_3\rangle\langle z_3|z_1\rangle + \langle z_1|z_3\rangle\langle z_3|z_2\rangle\langle z_2|z_1\rangle)$$

which can be simplified as

$$P_4 = \frac{1}{3}(|\langle z_1|z_2\rangle|^2 + |\langle z_2|z_3\rangle|^2 + |\langle z_3|z_1\rangle|^2)$$

which in turn yields

$$P_4 = \frac{1}{6}(3 + n_1 \cdot n_2 + n_2 \cdot n_3 + n_3 \cdot n_1) \quad (15)$$

owing to Equation (12).

We give in Table 2 some examples of states $|\psi_4\rangle$ for which P_4 goes from $\frac{1}{4}$ (for maximally entangled states) to 1 (for separable states).

Table 2. The perma-concurrence P_4 for some states $c_0|000\rangle + c_1\frac{|001\rangle+|010\rangle+|100\rangle}{\sqrt{3}} + c_2\frac{|011\rangle+|101\rangle+|110\rangle}{\sqrt{3}} + c_3|111\rangle$. The parameters ω , a and b are $\omega = \exp(i\frac{2\pi}{3})$, $a = \sqrt[3]{4}$ and $b = \sqrt[3]{2}$. The first three states are maximally entangled symmetric states of GHZ type (with $P_4 = \frac{1}{4}$) and the last two states are separable symmetric states (with $P_4 = 1$); the value of P_4 for the other states interpolates between $\frac{1}{4}$ and 1.

c_0	c_1	c_2	c_3	z_1	z_2	z_3	P_4
$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$	1	ω	ω^2	$\frac{1}{4}$
$\frac{1}{2}$	0	$\frac{\sqrt{3}}{2}$	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	$\frac{1}{4}$
0	$\frac{\sqrt{3}}{2}$	0	$\frac{1}{2}$	$\frac{i}{\sqrt{3}}$	$-\frac{i}{\sqrt{3}}$	$ z_3 = \infty$	$\frac{1}{4}$
$\frac{1}{\sqrt{8}}$	$\frac{\sqrt{3}}{\sqrt{8}}$	$-\frac{\sqrt{3}}{\sqrt{8}}$	$\frac{1}{\sqrt{8}}$	$a + b + 1$	$\frac{2-a-b}{2} + i\sqrt{3}\frac{a-b}{2}$	$\frac{2-a-b}{2} - i\sqrt{3}\frac{a-b}{2}$	$\frac{4b}{6+3a+4b}$
0	1	0	0	0	0	$ z_3 = \infty$	$\frac{1}{3}$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	0	0	0	0	3	$\frac{2}{5}$
0	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$ z_2 = \infty$	$ z_3 = \infty$	$\frac{2}{5}$
1	0	0	0	0	0	0	1
0	0	0	1	$ z_1 = \infty$	$ z_2 = \infty$	$ z_3 = \infty$	1

3.5. Perma-Concurrence for $N = 4$ ($d = 5$)

From Equations (4), (8)–(10) and (12), the basic ingredients for dealing with the case $N = 4$ are the following:

$$c_0z^4 - 2c_1z^3 + \sqrt{6}c_2z^2 - 2c_3z + c_4 = 0$$

and

$$P_5 = \frac{1}{12} \left[|\langle z_1|z_2\rangle|^2 |\langle z_3|z_4\rangle|^2 + |\langle z_1|z_3\rangle|^2 |\langle z_2|z_4\rangle|^2 + |\langle z_1|z_4\rangle|^2 |\langle z_2|z_3\rangle|^2 \right. \\ \left. + 2(|\langle z_1|z_2\rangle|^2 + |\langle z_1|z_3\rangle|^2 + |\langle z_1|z_4\rangle|^2 + |\langle z_2|z_3\rangle|^2 + |\langle z_2|z_4\rangle|^2 + |\langle z_3|z_4\rangle|^2) - 3 \right]$$

which can be equally written as

$$P_5 = \frac{1}{48} \left[(n_1 \cdot n_2)(n_3 \cdot n_4) + (n_1 \cdot n_3)(n_2 \cdot n_4) + (n_1 \cdot n_4)(n_2 \cdot n_3) \right. \\ \left. + 5(n_1 \cdot n_2 + n_1 \cdot n_3 + n_1 \cdot n_4 + n_2 \cdot n_3 + n_2 \cdot n_4 + n_3 \cdot n_4) + 15 \right] \quad (16)$$

in terms of the unit vectors n_i with $i = 1, 2, 3, 4$.

3.6. Perma-Concurrence for $N = 5$ ($d = 6$)

The calculation of P_6 gives

$$P_6 = \frac{1}{80} \left(15 + 5P_{6,1} + P_{6,2} \right)$$

where

$$P_{6,1} = n_1 \cdot n_2 + n_1 \cdot n_3 + n_1 \cdot n_4 + n_1 \cdot n_5 + n_2 \cdot n_3 + n_2 \cdot n_4 + n_2 \cdot n_5 + n_3 \cdot n_4 + n_3 \cdot n_5 + n_4 \cdot n_5$$

and

$$\begin{aligned} P_{6,2} = & (n_1 \cdot n_2)(n_3 \cdot n_4) + (n_1 \cdot n_2)(n_3 \cdot n_5) + (n_1 \cdot n_2)(n_4 \cdot n_5) \\ & + (n_1 \cdot n_3)(n_2 \cdot n_4) + (n_1 \cdot n_3)(n_2 \cdot n_5) + (n_1 \cdot n_3)(n_4 \cdot n_5) \\ & + (n_1 \cdot n_4)(n_2 \cdot n_3) + (n_1 \cdot n_4)(n_2 \cdot n_5) + (n_1 \cdot n_4)(n_3 \cdot n_5) \\ & + (n_1 \cdot n_5)(n_2 \cdot n_3) + (n_1 \cdot n_5)(n_2 \cdot n_4) + (n_1 \cdot n_5)(n_3 \cdot n_4) \\ & + (n_2 \cdot n_3)(n_4 \cdot n_5) + (n_2 \cdot n_4)(n_3 \cdot n_5) + (n_2 \cdot n_5)(n_3 \cdot n_4). \end{aligned}$$

4. Perma-Concurrence for Symmetric N -Qubit States of W Type

In this section, we consider qudits $|\psi_d\rangle$, $d = N - 1$, for which

$$\langle z_1 | z_2 \rangle = 0, \quad \langle z_1 | z_3 \rangle = 0, \quad \dots, \quad \langle z_1 | z_N \rangle = 0 \quad (17)$$

up to an arbitrary permutation of the indices $1, 2, \dots, N$. Then, the corresponding perma-concurrence P_d assumes the following particular expression

$$P_d = \frac{1}{N!} \sum_{\sigma \in S_{N-1}} \prod_{i=2}^N \langle z_i | z_{\sigma(i)} \rangle$$

where the sum on σ and the product on i are to be performed on S_{N-1} (instead of S_N) and from 2 to N (instead of 1 to N), respectively. The conditions (17) imply that

$$z_2 = z_3 = \dots = z_N = -\frac{1}{\bar{z}}, \quad z = z_1$$

so that P_d takes the simple form

$$P_d = \frac{1}{N} = \frac{1}{d-1}$$

a value compatible with the domain $\frac{1}{2^{N-1}} \leq P_d \leq 1$.

Consequently, the state $|\psi_d\rangle$ with $P_d = \frac{1}{d-1}$ reads in the Majorana representation as

$$|\psi_d\rangle = \frac{1}{\sqrt{N}} \left(|z, -\frac{1}{\bar{z}}, \dots, -\frac{1}{\bar{z}}\rangle_M + |-\frac{1}{\bar{z}}, z, \dots, -\frac{1}{\bar{z}}\rangle_M + \dots + |-\frac{1}{\bar{z}}, -\frac{1}{\bar{z}}, \dots, z\rangle_M \right) \quad (18)$$

which is a state of W type [26]. Such a general state includes the special states

$$|W\rangle_N = \frac{1}{\sqrt{N}} \left(|0, 1, \dots, 1\rangle + |1, 0, \dots, 1\rangle + \dots + |1, \dots, 1, 0\rangle \right)$$

and

$$|\bar{W}\rangle_N = \frac{1}{\sqrt{N}} \left(|1, 0, \dots, 0\rangle + |0, 1, \dots, 0\rangle + \dots + |0, \dots, 0, 1\rangle \right)$$

corresponding to $z \rightarrow 0$ and $|z| \rightarrow \infty$, respectively.

The entanglement (not maximum entanglement) of the general state (18) can be described geometrically in terms of unit vectors. For instance, let us consider the cases $N = 2, 3$ and 4. For $N = 2$, the perma-concurrence P_3 given by (13) takes its minimal value $P_3 = \frac{1}{N} = \frac{1}{2}$ (corresponding to maximally entangled states) for $n_1 \cdot n_2 = -1$ showing that the two vectors n_1 and n_2 have opposite directions. For $N = 3$, Equation (15) gives $P_4 = \frac{1}{N} = \frac{1}{3}$ for entangled 3-qubit states such that

$n_1 \cdot n_2 = -1$ and $n_1 \cdot n_3 = -1$ which imply $n_2 \cdot n_3 = 1$; therefore, two unit vectors coincide and point in the opposite direction of the third one. For $N = 4$, the value $P_5 = \frac{1}{N} = \frac{1}{4}$ of the perma-concurrence can be obtained from Equation (16) with

$$n_1 \cdot n_2 = -1, \quad n_1 \cdot n_3 = -1, \quad n_1 \cdot n_4 = -1$$

which imply

$$n_2 \cdot n_3 = 1, \quad n_2 \cdot n_4 = 1, \quad n_3 \cdot n_4 = 1$$

so that for entangled states with $P_5 = \frac{1}{4}$ the unit vectors n_2, n_3 and n_4 are identical with an opposite direction to n_1 .

5. Perma-Concurrence for Symmetric N -Qubit States of Bell and GHZ Type

We now consider the N -qubit state

$$|\psi_d\rangle = \frac{1}{\sqrt{2}}(|N;0\rangle + |N;N\rangle) = \frac{1}{\sqrt{2}}(|00 \dots 0\rangle + |11 \dots 1\rangle)$$

which corresponds to

$$c_1 = c_2 = \dots = c_{N-1} = 0, \quad c_0 = c_N = \frac{1}{\sqrt{2}}$$

in (2). Hence, $|\psi_d\rangle$ is a Bell state for $N = 2$ and a GHZ state for $N > 2$. The corresponding N Majorana stars z_k are here entirely determined by

$$z^N + (-1)^N = 0$$

that follows from (4). Thus, the Majorana roots z_k are

$$z_k = \exp\left(i(2k+1)\pi\frac{N+1}{N}\right), \quad k = 1, 2, \dots, N \quad (19)$$

from which we can calculate the perma-concurrence P_d . It is sufficient to use (11). The normalisation factor $\mathcal{N}_d$ in (11) follows from (5) with $k = 0$, viz,

$$\frac{1}{\sqrt{2}} = N!N_1N_2 \dots N_N\mathcal{N}_d \quad (20)$$

with $N_1 = N_2 = \dots = N_N = \frac{1}{\sqrt{2}}$. This yields

$$P_d = \frac{1}{2^{N-1}} = \frac{1}{2^{d-2}} \quad (21)$$

which corresponds to the minimum value of P_d .

It is easy to check that the result given by Equation (21) holds for the states

$$|\psi_d\rangle = \frac{1}{\sqrt{2}}(|00 \dots 0\rangle - |11 \dots 1\rangle)$$

for $N \geq 2$ and

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

for $N = 2$. We thus recover that the symmetric Bell states and the GHZ states are maximally entangled.

As an example, let us consider the state

$$|\psi_6\rangle = \frac{1}{\sqrt{2}}(|00000\rangle + |11111\rangle)$$

corresponding to $c_0 = c_5 = \frac{1}{\sqrt{2}}$ and $c_1 = c_2 = c_3 = c_4 = 0$. The Majorana roots z_k are the solutions of the simple equation

$$z^5 - 1 = 0$$

that gives

$$z_1 = \omega, \quad z_2 = \omega^2, \quad z_3 = \omega^3, \quad z_4 = \omega^4, \quad z_5 = \omega^5 = 1, \quad \omega = \exp\left(\frac{i2\pi}{5}\right)$$

in agreement with (19). The state $|\psi_6\rangle$ can be written

$$|\psi_6\rangle = \mathcal{N}_6 \sum_{\sigma \in S_5} \sigma(|z_1\rangle_M \otimes |z_2\rangle_M \otimes |z_3\rangle_M \otimes |z_4\rangle_M \otimes |z_5\rangle_M)$$

where

$$|z_k\rangle_M = \frac{1}{\sqrt{2}}(|0\rangle + \omega^k|1\rangle)$$

for $k = 1, 2, 3, 4, 5$. From a geometrical point of view, we have

$$\langle z_i | z_j \rangle = \frac{1}{2}(1 + \omega^{j-i})$$

and

$$n_i \cdot n_j = \cos\left[(j-i)\frac{2\pi}{5}\right].$$

Equation (20) can be particularized as

$$\frac{1}{\sqrt{2}} = 5! \left(\frac{1}{\sqrt{2}}\right)^5 \mathcal{N}_6 \Rightarrow \mathcal{N}_6 = \frac{1}{30}$$

and finally (11) gives $P_6 = \frac{1}{16}$.

6. Concluding Remarks and Prospects

The aim of the present paper was to investigate the properties and calculation procedures of the perma-concurrence parameter P_d defined in Ref. [16]. The parameter P_d makes it possible to characterize the entanglement degree of a qudit $|\psi_d\rangle$ (in dimension d) constructed from Dicke states made of N -qubits with $N = d - 1$. It varies from $P_d = \frac{1}{2^{d-2}}$ (for maximally entangled states) to $P_d = 1$ (for non entangled states, i.e., separable states). The perma-concurrence P_d extends to totally symmetric multi-qubit systems the notion of concurrence for two-qubit systems. Numerous examples were given in order to complete, precise and render easier the formalism developed in [16].

Due to the invariance of the Dicke states under the symmetric group S_N , the parameter P_d works only for totally symmetric N -qubit states. This emphasizes the importance of S_N in the construction of P_d . In this respect, it would be interesting to define a parameter similar to P_d for a N -qubit state transforming as an arbitrary irreducible representation of S_N .

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