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A Priori Estimates for a Nonlinear System with Some Essential Symmetrical Structures

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Abstract: In this paper, we are concerned with a nonlinear system containing some essential symmetrical structures (e.g., cross-diffusion) in the two-dimensional setting, which is proposed to model the biological transport networks. We first provide an a priori blow-up criterion of strong solution of the corresponding Cauchy problem. Based on this, we also establish a priori upper bounds to strong solution for all positive times.

Keywords: Cauchy problem; a priori estimates; blow-up criterion; global solution; biological transport networks

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1. Introduction and Main Results

Driven by the need to understand the biological transportation networks (for instance, leaf venation in plants, angiogenesis of blood vessels and neural networks which transport electric charge), biologists and physicists have expressed great interest in investigating the qualitative properties of network structures in the last few decades (see for instance [1–5] and references therein). Recently, Hu and Cai [6] introduced a purely local dynamic adaptation model based on mechanical laws on a graph, which was extended to a continuum one in [7–9] that was subsequently studied in the series of papers [10–15]. This continuum model, posed in spatial domain Ω, can be read as

$$\begin{cases} \partial_t \mathbf{m} - D^2 \Delta \mathbf{m} - c^2 (\mathbf{m} \cdot \nabla p) \nabla p + |\mathbf{m}|^{2(\gamma-1)} \mathbf{m} = \mathbf{0}, & x \in \Omega,\ t > 0, \\ -\nabla \cdot \Big(\big(I + \mathbf{m} \otimes \mathbf{m} \big) \nabla p \Big) = S, & x \in \Omega,\ t > 0. \end{cases} \tag{1}$$

The unknown function $p = p(t, x)$ denotes a scalar pressure of the fluid transported within the network which satisfies Darcy's type equation due to Darcy's law for slow flow in the network being valid, and thus ∇p represents a driving force for the evolution of the vector-valued conductance $\mathbf{m} = \big(m_1(t, x), \cdots, m_n(t, x)\big)$ that describes the dynamics of networks by using a reaction–diffusion equation consisting of three different mechanisms—pressure effect, diffusion (representing microscopic Brownian process) and an algebraic conductance-relaxation; the given function $S = S(x)$ models the sources and sinks. Values of the parameters $D^2 > 0$ (diffusion coefficient), $c^2 > 0$ (activation parameter) and $\gamma \in \mathbb{R}$ (relaxation exponent) are determined by the particular physical applications. For instance, we get from the known experimental studies (see [6,16] and ([8], Section 2) for details) that $\gamma = \frac{1}{2}$ can be used to describe blood vessel systems in the human body and that $\gamma = 1$ corresponds to leaf venation.

From a mathematical perspective, system (1) exhibits two rather peculiar nonlinear structures: $-c^2 (\mathbf{m} \cdot \nabla p) \nabla p$ in system $(1)_1$ and $-\nabla \cdot \big((\mathbf{m} \otimes \mathbf{m}) \nabla p \big)$ in $(1)_2$, which may result in yielding several difficulties in the mathematical analysis. For instance, the absence of a priori L^∞-bound for $\mathbf{m}$ may cause the elliptic coefficients in $(1)_2$ to be singular, and thus the solution of $(1)_2$ is too weak to control the nonlinear term $-c^2(\mathbf{m}\cdot\nabla p)\nabla p$ in $(1)_1$. These challenges make the cross-diffusion system (1) of interest. We would like to mention that, when system (1) is posed in a bounded domain Ω, it should be supplemented with the initial condition

$$\mathbf{m}(0,x)=\mathbf{m}_0(x) \qquad \text{in} \quad \Omega \tag{2}$$

and the homogeneous Dirichlet boundary conditions

$$p(t,x)=0 \quad \text{and} \quad \mathbf{m}(t,x)=\mathbf{0} \qquad \text{on} \quad \partial\Omega. \tag{3}$$

The initial-boundary value problems (1)–(3) has attracted a lot of interest already—see, for instance, [10] for the existence of global weak solution and of local mild solution when $\gamma \geq 1$ that was extended to the case of $\gamma \geq \frac{1}{2}$ in [11], ref. [13] for the partial regularity of weak solution, ref. [14] for the regularity of stationary weak solution in two space dimensions and ref. [15] for the existence of local classical large-data solution and of global classical small-data solution.

To our best knowledge, the Cauchy problems (1) and (2) was only studied in [17] when $\Omega=\mathbb{R}^3$, in which local existence as well as blow-up criterion for large initial data and global existence for small initial data were established based on $\dot{H}(\mathbb{R}^3)$ being a Hilbert space; however, there is no result on the Cauchy problems (1) and (2) on $\mathbb{R}^2$. Our goal is to fill this gap. It is worthwhile pointing out that the important problem of the mathematical theory concerning the Cauchy problems (1) and (2) is whether or not the global in time smooth solution exists for any prescribed smooth initial data. Hence, in the absence of a global well-posedness theory, establishing a priori estimates is of major importance for both theoretical and practical purposes, which motivates us to investigate an a priori blow-up criterion and a priori upper bounds of strong solutions to system (1) and (2) with $\Omega=\mathbb{R}^2$. We would like remark that, in contrast to system (1) and (2) with $\Omega=\mathbb{R}^3$, $\dot{H}(\mathbb{R}^2)$ not being a Hilbert space poses obstacles with adopting the same strategies in [17] to study the Cauchy problems (1) and (2) with $\Omega=\mathbb{R}^2$. To overcome this challenge, we will derive the a priori L^∞-bound for $\mathbf{m}$ to rule out the degeneracy and develop some new dissipation mechanisms hidden in the system (1) and (2) by fully utilizing some essential symmetrical structures of system when $\Omega=\mathbb{R}^2$.

Before stating our main results, we need to layout some notations. $H^k(\mathbb{R}^2)$ and $L^q(\mathbb{R}^2)$ denote the usual Hilbert space and Lebesgue space with the norms $\|\cdot\|_{H^k(\mathbb{R}^2)}$ and $\|\cdot\|_{L^q(\mathbb{R}^2)}$ (or $\|\cdot\|_{H^k}$ and $\|\cdot\|_{L^q}$ for short), respectively. The functions in these spaces are usually understood to be real valued. If no confusion is likely, we shall use the same notation for similar spaces of vector-valued functions and of matrix-valued functions, for instance,

$$\|\mathbf{m}\|_{L^q}=\sum_{i=1}^{2}\|m_i\|_{L^q} \quad \text{and} \quad \|\nabla\mathbf{m}\|_{L^q}=\sum_{i,j=1}^{2}\|\partial_j m_i\|_{L^q}.$$

Throughout this paper, we will use $A\lesssim B$ to denote $A\leq CB$ for some uniform constant $C>0$. Unless specified, the values of the constants may vary line by line according to the context.

With the aforementioned notations, we now state our first result as follows.

Theorem 1. *Let $\gamma\geq 1$, $\mathbf{m}_0\in H^2(\mathbb{R}^2)$ and $w\nabla^i S\in L^2(\mathbb{R}^2)$, $i=0,1,2$, where $w(x):=(1+|x|)(1+\ln(1+|x|))$. Assume that there exist a time $T>0$ and a strong solution $(\mathbf{m},p)$ to systems (1) and (2) on $\mathbb{R}^2\times[0,T]$ satisfying*

$$\mathbf{m}\in C\big([0,T];H^2(\mathbb{R}^2)\big),\ \nabla\mathbf{m}\in L^2\big([0,T];H^2(\mathbb{R}^2)\big) \text{ and } \nabla p\in C\big([0,T];H^2(\mathbb{R}^2)\big).$$

Then, there exists a maximal time $T_{\max}\in(T,\infty]$ such that $(\mathbf{m},p)$ is a strong solution of the Cauchy problem (1) and (2) on $\mathbb{R}^2\times[0,T_{\max})$. Moreover, if $T_{\max}<\infty$, it holds that

$$\limsup_{t\to T_{\max}} \|\mathbf{m}(t)\|_{H^2(\mathbb{R}^2)} = \infty$$

if and only if

$$\int_0^{T_{\max}} \|\nabla \mathbf{m}(t)\|^{\beta}_{L^q(\mathbb{R}^2)} dt = \infty, \tag{4}$$

for any $q \in (2, +\infty]$, *where*

$$\beta = \max\Big\{\frac{4q}{q-2}, \frac{4q(\gamma-2)}{2q-3}\Big\}.$$

Remark 1. *The blow-up criterion* (4) *in Theorem 1 is similar to the blow-up criterion of the strong solution of the Cauchy problem* (1) *and* (2) *on* $\mathbb{R}^3$ *[17] and to the Serrin-type criterion of Leray–Hopf weak solution of the Navier–Stokes equations [18].*

Invoking the blow-up criterion (4) obtained in Theorem 1, we can further present that the local strong solution of the Cauchy problem (1) and (2) can be extended to a global one.

Theorem 2. *Assume that all conditions in Theorem 1 hold. Then, there exists a positive constant* $h(4)$ *(see* (32) *below) such that, if* $D \geq h(4)$, *then the local strong solution of the Cauchy problem* (1) *and* (2) *with* $\Omega = \mathbb{R}^2$ *is global indeed in the sense that*

$$\|\mathbf{m}(t)\|_{H^2(\mathbb{R}^2)} + \|\nabla p\|_{H^2(\mathbb{R}^2)} + \int_0^t \|\nabla \mathbf{m}(s)\|^2_{H^2(\mathbb{R}^2)} ds \leq Ce^{kt} \tag{5}$$

for all $t > 0$ *and for some positive constants* k *and* C *being independent of* t.

Remark 2. *As far as we are concerned, even though there are substantial results regarding the system* (1) *on* $\mathbb{R}^3$ *or a bounded domain* $\Omega \subset \mathbb{R}^n$, *there are not yet any on* $\mathbb{R}^2$. *Theorem 1 and Theorem 2 seem to be the first rigorous theoretical analysis on the initial-value problem* (1) *and* (2) *on* $\mathbb{R}^2$ *and are a first step toward filling this gap.*

Remark 3. *Compared to the initial-value problem* (1) *and* (2) *on* $\mathbb{R}^3$ *in [17], establishing an a priori blow-up criterion* (4) *and a priori upper bounds* (5) *are nontrivial in the sense that the benefit emanating from* $\dot{H}(\mathbb{R}^3)$ *being a Hilbert space will be not granted and that our strategy depends on the weight Hardy inequality over* $\mathbb{R}^2$ *(see* (12)*) as well as seeking some new estimates.*

The rest of this paper is organized as follows. In Section 2, an a priori blow-up criterion is established. In Section 3, invoking this a priori blow-up criterion, we present that the local strong solution of the Cauchy problem (1) and (2) is global indeed in the two-dimensional setting.

2. Blow-Up Criterion. Proof of Theorem 1

Assume that there exist a time $T_0 > 0$ and a unique strong solution $(\mathbf{m}, p)$ to systems (1) and (2) on $[0, T_0]$ such that

$$\mathbf{m} \in C\big([0, T_0]; H^2(\mathbb{R}^2)\big),\ \nabla \mathbf{m} \in L^2\big([0, T_0]; H^2(\mathbb{R}^2)\big) \text{ and } \nabla p \in C\big([0, T_0]; H^2(\mathbb{R}^2)\big). \tag{6}$$

Clearly, by employing a standard bootstrap argument, we can extend this strong solution $(\mathbf{m}, p)$ to the maximal interval of existence $[0, T_{\max})$, where either $T_{\max} = \infty$ or $T_{\max} < \infty$. If $T_{\max} < \infty$, then we have

$$\lim_{t\to T_{\max}} \|\mathbf{m}(\cdot, t)\|_{H^2(\mathbb{R}^2)} = \infty$$

and vice versa. The goal of this section is to further establish a more precise blow-up criterion for such strong solution, which allows us to extend the local strong solution to a global one in the next section. To this end, for any $T \in (0, T_{\max})$, we abbreviate

$$X(0,T) := \Big\{ (\mathbf{m}, p) \,|\, \mathbf{m} \in C\big([0,T]; H^2(\mathbb{R}^2)\big), \;\; \nabla \mathbf{m} \in L^2\big([0,T]; H^2(\mathbb{R}^2)\big), \\ \partial_t \mathbf{m} \in L^2\big([0,T]; H^1(\mathbb{R}^2)\big), \;\; \nabla p \in L^\infty\big([0,T]; H^2(\mathbb{R}^2)\big) \Big\}$$

for simplicity. We now begin with establishing the following a priori estimates to the strong solution of systems (1) and (2) on $\mathbb{R}^2$.

Lemma 1. *Let $(\mathbf{m}, p)$ be a solution to system (1) and (2) in $X(0,T)$. We have*

$$\|\nabla \mathbf{m}(t)\|_{L^2}^2 + \|\mathbf{m}(t)\|_{L^{2\gamma}}^{2\gamma} + \|\nabla p(t)\|_{L^2}^2 + \|\mathbf{m} \cdot \nabla p(t)\|_{L^2}^2 + \int_0^t \|\partial_t \mathbf{m}(s)\|_{L^2}^2 ds \lesssim 1 + \|\mathbf{m}_0\|_{H^2}^{2\gamma} + \|wS\|_{L^2}^2, \tag{7}$$

$$\|\mathbf{m}(t)\|_{L^2}^2 + \int_0^t \Big(\|\nabla \mathbf{m}(s)\|_{L^2}^2 + \|\mathbf{m}(s)\|_{L^{2\gamma}}^{2\gamma} + \|\nabla p(s)\|_{L^2}^2 + \|\mathbf{m} \cdot \nabla p(s)\|_{L^2}^2 \Big) ds \le CT \|wS\|_{L^2}^2 + \|\mathbf{m}_0\|_{L^2}^2 \tag{8}$$

for any $t \in (0,T)$, where $w(x) := (1+|x|)(1+\ln(1+|x|))$ and the positive constant C depend only on D, c and γ.

Proof. Without loss of generality, we may assume that $(\mathbf{m}, p)$ is sufficiently smooth. The general case can be dealt with by taking an approximation procedure.

We proceed along the lines of the proof of ([17], Lemma 3.1) and have

$$\frac{d}{dt} \Big(D^2 \|\nabla \mathbf{m}\|_{L^2}^2 + \frac{1}{\gamma} \|\mathbf{m}\|_{L^{2\gamma}}^{2\gamma} + c^2 \|\nabla p\|_{L^2}^2 + c^2 \|\mathbf{m} \cdot \nabla p\|_{L^2}^2 \Big) + 2\|\partial_t \mathbf{m}\|_{L^2}^2 = 0. \tag{9}$$

For any $t \in (0,T)$, by integrating (9) from 0 to t, we infer that

$$\begin{aligned} & D^2 \|\nabla \mathbf{m}(t)\|_{L^2}^2 + \frac{1}{\gamma} \|\mathbf{m}(t)\|_{L^{2\gamma}}^{2\gamma} + c^2 \|\nabla p(t)\|_{L^2}^2 + c^2 \|\mathbf{m} \cdot \nabla p(t)\|_{L^2}^2 + 2 \int_0^t \|\partial_t \mathbf{m}(s)\|_{L^2}^2 ds \\ & = D^2 \|\nabla \mathbf{m}_0\|_{L^2}^2 + \frac{1}{\gamma} \|\mathbf{m}_0\|_{L^{2\gamma}}^{2\gamma} + c^2 \|\nabla p_0\|_{L^2}^2 + c^2 \|\mathbf{m}_0 \cdot \nabla p_0\|_{L^2}^2, \end{aligned} \tag{10}$$

where the function $p_0 = p(0,x)$ solves the following Poisson equation

$$-\nabla \cdot \Big((I + \mathbf{m}_0 \otimes \mathbf{m}_0) \nabla p_0 \Big) = S, \qquad x \in \mathbb{R}^2. \tag{11}$$

To estimate the terms related to p_0 on the right-hand side of (10), by using p_0 as a test function in (11), we obtain from the integration by parts that

$$\int_{\mathbb{R}^2} |\nabla p_0|^2 dx + \int_{\mathbb{R}^2} (\mathbf{m}_0 \cdot \nabla p_0)^2 dx = \int_{\mathbb{R}^2} p_0 S dx.$$

Recalling the Hardy inequality over $\mathbb{R}^2$ ([19])

$$\int_{\mathbb{R}^2} \left| \frac{f}{w(x)} \right|^2 dx \le C \int_{\mathbb{R}^2} |\nabla f|^2 dx \quad \text{for} \quad w(x) = (1+|x|)(1+\ln(1+|x|)) \tag{12}$$

and employing Hölder's inequality, we arrive at

$$\int_{\mathbb{R}^2} p_0 S dx \le \left\| \frac{p_0}{w} \right\|_{L^2} \|wS\|_{L^2} \lesssim \|\nabla p_0\|_{L^2}\|wS\|_{L^2} \le \frac{1}{2}\|\nabla p_0\|_{L^2}^2 + C\|wS\|_{L^2}^2.$$

From this, we conclude that

$$\|\nabla p_0\|_{L^2}^2 + \|\mathbf{m}_0 \cdot \nabla p_0\|_{L^2}^2 \le C\|wS\|_{L^2}^2.$$

This together with (10) yields that

$$\begin{aligned}
&\|\nabla \mathbf{m}(t)\|_{L^2}^2 + \|\mathbf{m}(t)\|_{L^{2\gamma}}^{2\gamma} + \|\nabla p(t)\|_{L^2}^2 + \|\mathbf{m}\cdot\nabla p(t)\|_{L^2}^2 + \int_0^t \|\partial_t \mathbf{m}(s)\|_{L^2}^2 ds \\
&\lesssim \|\nabla \mathbf{m}_0\|_{L^2}^2 + \|\mathbf{m}_0\|_{L^{2\gamma}}^{2\gamma} + \|wS\|_{L^2}^2 \\
&\lesssim 1 + \|\mathbf{m}_0\|_{H^2}^{2\gamma} + \|wS\|_{L^2}^2,
\end{aligned}$$

where we used Sobolev's embedding $\|\mathbf{m}_0\|_{L^{2\gamma}} \lesssim \|\mathbf{m}_0\|_{H^2}$, Young's inequality and the fact $2\gamma \ge 2$ in the last inequality.

Next, taking the L^2 inner product of $(1)_1$ with $\mathbf{m}$ and using the integration by parts, we can obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\mathbf{m}|^2 dx + D^2 \int_{\mathbb{R}^2} |\nabla \mathbf{m}|^2 dx + \int_{\mathbb{R}^2} |\mathbf{m}|^{2\gamma} dx = c^2 \int_{\mathbb{R}^2} (\mathbf{m}\cdot\nabla p)^2 dx. \tag{13}$$

Taking the L^2 inner product of $(1)_2$ with p, we can obtain from the integration by parts that

$$\int_{\mathbb{R}^2} |\nabla p|^2 dx + \int_{\mathbb{R}^2} (\mathbf{m}\cdot\nabla p)^2 dx = \int_{\mathbb{R}^2} pS dx. \tag{14}$$

Similarly, we conclude from Hölder's inequality, Sobolev's inequality and Hardy's inequality (12) that

$$\int_{\mathbb{R}^2} |\nabla p|^2 dx + 2\int_{\mathbb{R}^2} (\mathbf{m}\cdot\nabla p)^2 dx \le C\|wS\|_{L^2}^2, \tag{15}$$

which together with (13) yields that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\mathbf{m}|^2 dx + \int_{\mathbb{R}^2} \left(D^2|\nabla \mathbf{m}|^2 + |\mathbf{m}|^{2\gamma} + c^2|\nabla p|^2 + c^2(\mathbf{m}\cdot\nabla p)^2\right) dx \le C\|wS\|_{L^2}^2.$$

For any $t \in (0,T)$, we integrate the above inequality from 0 to t and thus obtain

$$\begin{aligned}
&\|\mathbf{m}(t)\|_{L^2}^2 + \int_0^t \left(D^2\|\nabla \mathbf{m}(s)\|_{L^2}^2 + \|\mathbf{m}(s)\|_{L^{2\gamma}}^{2\gamma} + c^2\|\nabla p(s)\|_{L^2}^2 + c^2\|\mathbf{m}\cdot\nabla p(s)\|_{L^2}^2\right) ds \\
&\le CT\|wS\|_{L^2}^2 + \|\mathbf{m}_0\|_{L^2}^2.
\end{aligned}$$

This completes the proof of Lemma 1. □

To obtain the higher-order estimates of the solution component $\mathbf{m}$, we should establish the higher-order estimates of the solution component p.

Lemma 2. *Let $(\mathbf{m}, p)$ be a solution to systems (1) and (2) in $X(0,T)$. Then, there exists a positive constant C depending only on D^2, c^2 and γ such that*

$$\|\nabla^2 p\|_{L^2}^2 + \|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 \le C\left(\|\nabla \mathbf{m}\|_{L^q}^{\frac{2q}{q-2}} + \|w\nabla S\|_{L^2}^2\right), \tag{16}$$

$$\|\nabla^3 p\|_{L^2}^2 + \|\nabla^2(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 \le \frac{1}{2}\|\nabla^3\mathbf{m}\|_{L^2}^2 + C\|\nabla^2\mathbf{m}\|_{L^2}^2\Big(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4\Big) + C\Big(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4 + \|w\nabla^2 S\|_{L^2}^2\Big) \tag{17}$$

for any $q\in(2,+\infty]$ and all $t\in(0,T)$.

Proof. We begin with establishing the estimate (16). Differentiating $(1)_2$ with respect to x_k for $k=1,2$, we have

$$-\Delta\partial_k p - \nabla\cdot\big(\partial_k(\mathbf{m}\cdot\nabla p)\mathbf{m} + (\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m}\big) = \partial_k S. \tag{18}$$

By taking the L^2 inner product of (18) with $\partial_k p$, we obtain from the integration by parts that

$$\|\nabla\partial_k p\|_{L^2}^2 + \int_{\mathbb{R}^2}\partial_k(\mathbf{m}\cdot\nabla p)\,\mathbf{m}\cdot\nabla\partial_k p + \int_{\mathbb{R}^2}(\mathbf{m}\cdot\nabla p)\,\partial_k\mathbf{m}\cdot\nabla\partial_k p = \int_{\mathbb{R}^2}\partial_k S\partial_k p.$$

Hence, it reduces to

$$\|\nabla\partial_k p\|_{L^2}^2 + \|\partial_k(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 = \int_{\mathbb{R}^2}\partial_k(\mathbf{m}\cdot\nabla p)\,\partial_k\mathbf{m}\cdot\nabla p - \int_{\mathbb{R}^2}(\mathbf{m}\cdot\nabla p)\,\partial_k\mathbf{m}\cdot\nabla\partial_k p + \int_{\mathbb{R}^2}\partial_k S\partial_k p. \tag{19}$$

Invoking Hölder's inequality and Hardy's inequality (12), the third term on the right-hand side of (19) can be controlled as follows:

$$\int_{\mathbb{R}^2}\partial_k S\partial_k p dx \le \left\|\frac{\partial_k p}{w}\right\|_{L^2}\|w\partial_k S\|_{L^2} \le C\|\nabla\partial_k p\|_{L^2}\|w\partial_k S\|_{L^2}.$$

Based on this, we get from Hölder's inequality and Young's inequality that

$$\begin{aligned}&\|\nabla\partial_k p\|_{L^2}^2 + \|\partial_k(\mathbf{m}\cdot\nabla p)\|_{L^2}^2\\ &\le \|\partial_k(\mathbf{m}\cdot\nabla p)\|_{L^2}\|\partial_k\mathbf{m}\cdot\nabla p\|_{L^2} + \|(\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m}\|_{L^2}\|\nabla\partial_k p\|_{L^2} + C\|\nabla\partial_k p\|_{L^2}\|w\nabla S\|_{L^2}\\ &\le \frac{1}{2}\Big(\|\partial_k(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 + \|\nabla\partial_k p\|_{L^2}^2\Big) + \|\partial_k\mathbf{m}\cdot\nabla p\|_{L^2}^2 + \|(\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m}\|_{L^2}^2 + C\|w\nabla S\|_{L^2}^2.\end{aligned}$$

From this, one arrives at

$$\|\nabla\partial_k p\|_{L^2}^2 + \|\partial_k(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 \le 2\|\partial_k\mathbf{m}\cdot\nabla p\|_{L^2}^2 + 2\|(\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m}\|_{L^2}^2 + C\|w\nabla S\|_{L^2}^2,$$

which implies that

$$\begin{aligned}&\|\nabla^2 p\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2(\mathbb{R}^2)}^2\\ &\le 2\big(\|\nabla p\cdot\nabla\mathbf{m}\|_{L^2(\mathbb{R}^2)}^2 + \|(\mathbf{m}\cdot\nabla p)\nabla\mathbf{m}\|_{L^2(\mathbb{R}^2)}^2\big) + C\|w\nabla S\|_{L^2}^2.\end{aligned} \tag{20}$$

Note that, for any $q\in(2,+\infty]$, Hölder's inequality, Sobolev's embedding, Young's inequality and Lemma 1 yield that

$$\begin{aligned}\|\nabla p\cdot\nabla\mathbf{m}\|_{L^2(\mathbb{R}^2)}^2 &\le \|\nabla\mathbf{m}\|_{L^q}^2\|\nabla p\|_{L^{\frac{2q}{q-2}}}^2\\ &\lesssim \|\nabla\mathbf{m}\|_{L^q}^2\|\nabla p\|_{L^2}^{2\left(1-\frac{2}{q}\right)}\|\nabla^2 p\|_{L^2}^{\frac{4}{q}}\\ &\lesssim \|\nabla\mathbf{m}\|_{L^q}^2\|\nabla^2 p\|_{L^2}^{\frac{4}{q}}\\ &\le \frac{1}{4}\|\nabla^2 p\|_{L^2}^2 + C\|\nabla\mathbf{m}\|_{L^q}^{\frac{2q}{q-2}},\end{aligned}$$

and similarly that

$$\|(\mathbf{m}\cdot\nabla p)\nabla\mathbf{m}\|_{L^2(\mathbb{R}^2)}^2 \le \frac{1}{4}\|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 + C\|\nabla\mathbf{m}\|_{L^q}^{\frac{2q}{q-2}}.$$

Hence, we can conclude from (20) that, for any $q\in(2,+\infty]$,

$$\|\nabla^2 p\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2(\mathbb{R}^2)}^2 \le C\Big(\|\nabla\mathbf{m}\|_{L^q}^{\frac{2q}{q-2}} + \|w\nabla S\|_{L^2}^2\Big). \tag{21}$$

This indicates that (16) holds.

We now turn to the higher-order estimate (17). To achieve this, differentiating (18) with respect to x_j for $j=1,2$, we obtain

$$-\Delta\partial_k\partial_j p - \nabla\cdot\big((\mathbf{m}\cdot\nabla p)\partial_k\partial_j\mathbf{m} + \partial_j(\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m} + \partial_k(\mathbf{m}\cdot\nabla p)\partial_j\mathbf{m} + \partial_k\partial_j(\mathbf{m}\cdot\nabla p)\mathbf{m}\big) = \partial_k\partial_j S. \tag{22}$$

Then, multiplying (22) by $\partial_{ij}^2 p$ and using the integration by parts, we have

$$\begin{aligned}\|\nabla\partial_k\partial_j p\|_{L^2}^2 + \int(\mathbf{m}\cdot\nabla p)\partial_k\partial_j\mathbf{m}\cdot\nabla\partial_k\partial_j p + \int\big(\partial_j(\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m} + \partial_k(\mathbf{m}\cdot\nabla p)\partial_j\mathbf{m}\big)\cdot\nabla\partial_k\partial_j p\\ + \int\partial_k\partial_j(\mathbf{m}\cdot\nabla p)\,\mathbf{m}\cdot\nabla\partial_k\partial_j p = \int\partial_k\partial_j S\partial_k\partial_j p.\end{aligned}$$

Since

$$\mathbf{m}\cdot\nabla\partial_k\partial_j p = \partial_k\partial_j(\mathbf{m}\cdot\nabla p) - \partial_k\partial_j\mathbf{m}\cdot\nabla p - \partial_j\mathbf{m}\cdot\nabla\partial_k p - \partial_k\mathbf{m}\cdot\nabla\partial_j p,$$

we have

$$\begin{aligned}&\|\nabla\partial_k\partial_j p\|_{L^2}^2 + \|\partial_k\partial_j(\mathbf{m}\cdot\nabla p)\|_{L^2}^2\\ &= -\int(\mathbf{m}\cdot\nabla p)\partial_k\partial_j\mathbf{m}\cdot\nabla\partial_k\partial_j p - \int\big(\partial_j(\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m} + \partial_k(\mathbf{m}\cdot\nabla p)\partial_j\mathbf{m}\big)\cdot\nabla\partial_k\partial_j p\\ &\quad + \int\big(\partial_k\partial_j\mathbf{m}\cdot\nabla p + \partial_j\mathbf{m}\cdot\nabla\partial_k p + \partial_k\mathbf{m}\cdot\nabla\partial_j p\big)\partial_k\partial_j(\mathbf{m}\cdot\nabla p) + \int\partial_k\partial_j S\partial_k\partial_j p.\end{aligned}$$

We now use Hölder's inequality and Hardy's inequality (12) to estimate the rightmost term of the above equation as follows:

$$\int_{\mathbb{R}^2}\partial_k\partial_j S\partial_k\partial_j p \le \left\|\frac{\partial_k\partial_j p}{w}\right\|_{L^2}\|w\partial_k\partial_j S\|_{L^2} \le C\|\nabla\partial_k\partial_j p\|_{L^2}\|w\partial_k\partial_j S\|_{L^2}.$$

One has

$$\begin{aligned}&\|\nabla\partial_k\partial_j p\|_{L^2}^2 + \|\partial_k\partial_j(\mathbf{m}\cdot\nabla p)\|_{L^2}^2\\ &\le \|\nabla\partial_k\partial_j p\|_{L^2}\Big(\|(\mathbf{m}\cdot\nabla p)\partial_k\partial_j\mathbf{m}\|_{L^2} + \|\partial_j(\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m}\|_{L^2} + \|\partial_k(\mathbf{m}\cdot\nabla p)\partial_j\mathbf{m}\|_{L^2}\Big)\\ &\quad + \|\partial_k\partial_j(\mathbf{m}\cdot\nabla p)\|_{L^2}\Big(\|\partial_k\partial_j\mathbf{m}\cdot\nabla p\|_{L^2} + \|\partial_j\mathbf{m}\cdot\nabla\partial_k p\|_{L^2} + \|\partial_k\mathbf{m}\cdot\nabla\partial_j p\|_{L^2}\Big)\\ &\quad + C\|\nabla\partial_k\partial_j p\|_{L^2}\|w\nabla^2 S\|_{L^2}.\end{aligned}$$

By employing Young's inequality, we can deduce that

$$\begin{aligned}&\|\nabla\partial_k\partial_j p\|_{L^2}^2 + \|\partial_k\partial_j(\mathbf{m}\cdot\nabla p)\|_{L^2}^2\\ &\lesssim \|(\mathbf{m}\cdot\nabla p)\partial_k\partial_j\mathbf{m}\|_{L^2}^2 + \|\partial_j(\mathbf{m}\cdot\nabla p)\partial_k\mathbf{m}\|_{L^2}^2 + \|\partial_k(\mathbf{m}\cdot\nabla p)\partial_j\mathbf{m}\|_{L^2}^2\end{aligned}$$

$$+\|\partial_k\partial_j\mathbf{m}\cdot\nabla p\|_{L^2}^2+\|\partial_j\mathbf{m}\cdot\nabla\partial_k p\|_{L^2}^2+\|\partial_k\mathbf{m}\cdot\nabla\partial_j p\|_{L^2}^2+\|w\nabla^2 S\|_{L^2}^2,$$

and thus that

$$\begin{aligned}\|\nabla^3 p\|_{L^2}^2+\|\nabla^2(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 \lesssim &\|(\mathbf{m}\cdot\nabla p)\nabla^2\mathbf{m}\|_{L^2}^2+\|\nabla(\mathbf{m}\cdot\nabla p)\cdot\nabla\mathbf{m}\|_{L^2}^2\\ &+\sum_{j=1}^{2}\|\nabla p\cdot\nabla\partial_j\mathbf{m}\|_{L^2}^2+\|\nabla\mathbf{m}\nabla^2 p\|_{L^2}^2+\|w\nabla^2 S\|_{L^2}^2. \end{aligned}\tag{23}$$

We estimate the terms on the right-hand side of (23) one by one. Firstly, we conclude from Hölder's inequality, Sobolev's embedding, the interpolation, Young's inequality and Lemma 1 that

$$\begin{aligned}\|(\mathbf{m}\cdot\nabla p)\nabla^2\mathbf{m}\|_{L^2}^2 &\le \|(\mathbf{m}\cdot\nabla p)\|_{L^6}^2\|\nabla^2\mathbf{m}\|_{L^3}^2\\ &\lesssim \|\mathbf{m}\cdot\nabla p\|_{L^2}^{\frac{2}{3}}\|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^{\frac{4}{3}}\|\nabla^2\mathbf{m}\|_{L^2}^{\frac{4}{3}}\|\nabla^3\mathbf{m}\|_{L^2}^{\frac{2}{3}}\\ &\le \frac{1}{8}\|\nabla^3\mathbf{m}\|_{L^2}^2+C\|\nabla^2\mathbf{m}\|_{L^2}^2\|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^2.\end{aligned}$$

Secondly, we get from a similar procedure that, for any $q\in(n,+\infty]$,

$$\begin{aligned}\|\nabla(\mathbf{m}\cdot\nabla p)\cdot\nabla\mathbf{m}\|_{L^2}^2 &\le \|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^{\frac{2q}{q-2}}}^2\|\nabla\mathbf{m}\|_{L^q}^2\\ &\lesssim \|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^{2\left(1-\frac{2}{q}\right)}\|\nabla^2(\mathbf{m}\cdot\nabla p)\|_{L^2}^{\frac{4}{q}}\|\nabla\mathbf{m}\|_{L^q}^2\\ &\le \frac{1}{2}\|\nabla^2(\mathbf{m}\cdot\nabla p)\|_{L^2}^2+C\|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^2\|\nabla\mathbf{m}\|_{L^q}^{\frac{2q}{q-2}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\nabla p\cdot\nabla\partial_j\mathbf{m}\|_{L^2}^2 &\le \frac{1}{8}\|\nabla^3\mathbf{m}\|_{L^2}^2+C\|\nabla^2\mathbf{m}\|_{L^2}^2\|\nabla^2 p\|_{L^2}^2,\\ \|\nabla\mathbf{m}\nabla^2 p\|_{L^2}^2 &\le \frac{1}{2}\|\nabla^3 p\|_{L^2}^2+C\|\nabla^2 p\|_{L^2}^2\|\nabla\mathbf{m}\|_{L^q}^{\frac{2q}{q-2}}\end{aligned}$$

for any $q\in(2,+\infty]$. It follows from inserting the above estimates into (23) that

$$\begin{aligned}\|\nabla^3 p\|_{L^2}^2+\|\nabla^2(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 \le &\frac{1}{2}\|\nabla^3\mathbf{m}\|_{L^2}^2+C\|\nabla^2\mathbf{m}\|_{L^2}^2(\|\nabla^2 p\|_{L^2}^2+\|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^2)\\ &+C(\|\nabla^2 p\|_{L^2}^2+\|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^2)\|\nabla\mathbf{m}\|_{L^q}^{\frac{2q}{q-2}}+C\|w\nabla^2 S\|_{L^2}^2\end{aligned}$$

for any $q\in(2,+\infty]$. This implies (17) and thus completes the proof of Lemma 2. □

With the improved a priori estimates on p at hand, we can directly derive the higher-order estimates of the solution component $\mathbf{m}$ under the same assumptions as above.

Lemma 3. *Let $(\mathbf{m},p)$ be a solution to systems (1) and (2) in $X(0,T)$. Then, there exists a positive constant C depending only on D, c and γ such that for all $0<t<T$,*

$$\begin{aligned}&\frac{d}{dt}\|\nabla^2\mathbf{m}\|_{L^2}^2+\frac{D^2}{2}\|\nabla^3\mathbf{m}\|_{L^2}^2\\ &\lesssim \|\nabla^2\mathbf{m}\|_{L^2}^2\left(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}}+\|w\nabla S\|_{L^2}^4\right)+\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}}+\|\mathbf{m}\|_{L^2}^{\frac{8(\gamma-1)}{9-2\gamma}}+\|w\nabla S\|_{L^2}^4+\|w\nabla^2 S\|_{L^2}^2+1\end{aligned}$$

for $1 \le \gamma < 3$, *and*

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla^2\mathbf{m}\|_{L^2}^2 + \frac{D^2}{2}\|\nabla^3\mathbf{m}\|_{L^2}^2 \\
&\lesssim \|\nabla^2\mathbf{m}\|_{L^2}^2\Big(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4\Big) + \|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4 + \|w\nabla^2 S\|_{L^2}^2 + 1 \\
&+ \|\mathbf{m}\|_{L^2}^{1+\frac{1}{2q-3}}\|\nabla\mathbf{m}\|_{L^q}^{\frac{2q(\gamma-2)}{2q-3}}\|\nabla^2\mathbf{m}\|_{L^2}
\end{aligned}
$$

for $\gamma \ge 3$.

Proof. Similar to ([17], Lemma 3.3), we get that there exists a positive constant C_0 independent of t such that

$$
\frac{d}{dt}\|\nabla^2\mathbf{m}\|_{L^2}^2 + D^2\|\nabla^3\mathbf{m}\|_{L^2}^2 \le C_0\Big(\|\nabla\big((\mathbf{m}\cdot\nabla p)\nabla p\big)\|_{L^2}^2 + \|\nabla(|\mathbf{m}|^{2(\gamma-1)}\mathbf{m})\|_{L^2}^2\Big). \tag{24}
$$

We estimate the terms on the right-hand side of (24) one by one. For the first term on the right-hand side of (24), we obtain from Hölder's inequality, the interpolation inequality, Young's inequality and Lemma 1 that

$$
\begin{aligned}
&\big\|\nabla\big((\mathbf{m}\cdot\nabla p)\nabla p\big)\big\|_{L^2}^2 \\
&\le \big\|\nabla(\mathbf{m}\cdot\nabla p)\nabla p\big\|_{L^2}^2 + \big\|(\mathbf{m}\cdot\nabla p)\nabla^2 p\big\|_{L^2}^2 \\
&\le \|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^3}^2\|\nabla p\|_{L^6}^2 + \|\mathbf{m}\cdot\nabla p\|_{L^6}^2\|\nabla^2 p\|_{L^3}^2 \\
&\lesssim \|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^{\frac{4}{3}}\|\nabla^2(\mathbf{m}\cdot\nabla p)\|_{L^2}^{\frac{2}{3}}\|\nabla p\|_{L^2}^{\frac{2}{3}}\|\nabla^2 p\|_{L^2}^{\frac{4}{3}} + \|\mathbf{m}\cdot\nabla p\|_{L^2}^{\frac{2}{3}}\|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^{\frac{4}{3}}\|\nabla^2 p\|_{L^2}^{\frac{4}{3}}\|\nabla^3 p\|_{L^2}^{\frac{2}{3}} \\
&\le \frac{D^2}{2C_0}\Big(\|\nabla^2(\mathbf{m}\cdot\nabla p)\|_{L^2}^2 + \|\nabla^3 p\|_{L^2}^2\Big) + C\Big(\|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^2\|\nabla^2 p\|_{L^2}^2 + \|\nabla(\mathbf{m}\cdot\nabla p)\|_{L^2}^2\|\nabla^2 p\|_{L^2}^2\Big),
\end{aligned}
$$

which together with Lemma 2 yields that

$$
\begin{aligned}
\big\|\nabla\big((\mathbf{m}\cdot\nabla p)\nabla p\big)\big\|_{L^2}^2 \le{}& \frac{D^2}{4C_0}\|\nabla^3\mathbf{m}\|_{L^2}^2 + C\|\nabla^2\mathbf{m}\|_{L^2}^2\Big(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4\Big) \\
&+ C\Big(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4 + \|w\nabla^2 S\|_{L^2}^2 + 1\Big).
\end{aligned} \tag{25}
$$

To estimate the second term on the right-hand side of (24), by using the Leibniz's product rule and Hölder's inequality, we obtain

$$
\|\nabla(|\mathbf{m}|^{2(\gamma-1)}\mathbf{m})\|_{L^2}^2 \lesssim \||\mathbf{m}|^{2(\gamma-1)}\nabla\mathbf{m}\|_{L^2}^2 \lesssim \|\mathbf{m}\|_{L^\infty}^{4(\gamma-1)}\|\nabla\mathbf{m}\|_{L^2}^2.
$$

For $1 \le \gamma < 3$, a straightforward application of the interpolation inequality yields that

$$
\begin{aligned}
\|\mathbf{m}\|_{L^\infty}^{4(\gamma-1)} &\lesssim \|\mathbf{m}\|_{L^6}^{\frac{24(\gamma-1)}{7}}\|\nabla^3\mathbf{m}\|_{L^2}^{\frac{4(\gamma-1)}{7}} \\
&\lesssim \|\mathbf{m}\|_{L^2}^{\frac{8(\gamma-1)}{7}}\|\nabla\mathbf{m}\|_{L^2}^{\frac{16(\gamma-1)}{7}}\|\nabla^3\mathbf{m}\|_{L^2}^{\frac{4(\gamma-1)}{7}}
\end{aligned}
$$

and thus that

$$
\begin{aligned}
\|\nabla(|\mathbf{m}|^{2(\gamma-1)}\mathbf{m})\|_{L^2}^2 &\lesssim \|\mathbf{m}\|_{L^2}^{\frac{8(\gamma-1)}{7}}\|\nabla\mathbf{m}\|_{L^2}^{2+\frac{16(\gamma-1)}{7}}\|\nabla^3\mathbf{m}\|_{L^2}^{\frac{4(\gamma-1)}{7}} \\
&\le \frac{D^2}{4C_0}\|\nabla^3\mathbf{m}\|_{L^2}^2 + C\|\mathbf{m}\|_{L^2}^{\frac{8(\gamma-1)}{9-2\gamma}}\|\nabla\mathbf{m}\|_{L^2}^{\frac{14+16(\gamma-1)}{9-2\gamma}}.
\end{aligned} \tag{26}
$$

On the other hand, for $\gamma \geq 3$, we have from Hölder's inequality and the interpolation inequality that

$$\begin{aligned}
\|\mathbf{m}\|_{L^\infty}^{4(\gamma-1)} &\leq \|\mathbf{m}\|_{L^\infty}^4 \|\mathbf{m}\|_{L^\infty}^{4(\gamma-2)} \\
&\lesssim \|\mathbf{m}\|_{L^6}^3 \|\nabla^2\mathbf{m}\|_{L^2} \left(\|\mathbf{m}\|_{L^6}^{\frac{3(q-2)}{2(2q-3)}} \|\nabla\mathbf{m}\|_{L^q}^{\frac{q}{2(2q-3)}} \right)^{4(\gamma-2)} \\
&\lesssim \|\mathbf{m}\|_{L^2} \|\nabla\mathbf{m}\|_{L^2}^2 \|\nabla^2\mathbf{m}\|_{L^2} \left(\|\mathbf{m}\|_{L^2}^{\frac{q-2}{2(2q-3)}} \|\nabla\mathbf{m}\|_{L^2}^{\frac{q-2}{2q-3}} \|\nabla\mathbf{m}\|_{L^q}^{\frac{q}{2(2q-3)}} \right)^{4(\gamma-2)} \\
&\lesssim \|\mathbf{m}\|_{L^2}^{1+\frac{1}{2q-3}} \|\nabla\mathbf{m}\|_{L^2}^{\frac{4(q-2)(\gamma-2)}{2q-3}+2} \|\nabla\mathbf{m}\|_{L^q}^{\frac{2q(\gamma-2)}{2q-3}} \|\nabla^2\mathbf{m}\|_{L^2}
\end{aligned}$$

for any $q \in (2, +\infty]$ and thus

$$\|\nabla(|\mathbf{m}|^{2(\gamma-1)}\mathbf{m})\|_{L^2}^2 \lesssim \|\mathbf{m}\|_{L^2}^{1+\frac{1}{2q-3}} \|\nabla\mathbf{m}\|_{L^2}^{\frac{4(q-2)(\gamma-2)}{2q-3}+4} \|\nabla\mathbf{m}\|_{L^q}^{\frac{2q(\gamma-2)}{2q-3}} \|\nabla^2\mathbf{m}\|_{L^2}. \tag{27}$$

Substituting (25), (26) and (27) into (24), we deduce from Lemma 1 that, for $1 \leq \gamma < 3$,

$$\begin{aligned}
\frac{d}{dt}\|\nabla^2\mathbf{m}\|_{L^2}^2 + \frac{D^2}{2}\|\nabla^3\mathbf{m}\|_{L^2}^2 \lesssim& \|\nabla^2\mathbf{m}\|_{L^2}^2 \left(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4 \right) \\
&+ \|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{2q-3}} + \|\mathbf{m}\|_{L^2}^{\frac{8(\gamma-1)}{9-2\gamma}} + \|w\nabla S\|_{L^2}^4 + \|w\nabla^2 S\|_{L^2}^2 + 1,
\end{aligned}$$

and that, for $\gamma \geq 3$,

$$\begin{aligned}
&\frac{d}{dt}\|\nabla^2\mathbf{m}\|_{L^2}^2 + \frac{D^2}{2}\|\nabla^3\mathbf{m}\|_{L^2}^2 \\
&\lesssim \|\nabla^2\mathbf{m}\|_{L^2}^2 \left(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4 \right) + \|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|w\nabla S\|_{L^2}^4 + \|w\nabla^2 S\|_{L^2}^2 + 1 \\
&\quad + \|\mathbf{m}\|_{L^2}^{1+\frac{1}{2q-3}} \|\nabla\mathbf{m}\|_{L^q}^{\frac{2q(\gamma-2)}{2q-3}} \|\nabla^2\mathbf{m}\|_{L^2}.
\end{aligned}$$

This completes the proof of Lemma 3. □

Invoking Lemmas 1–3, we now establish a more precise blow-up criterion for the strong solution as follows.

Lemma 4. *Let $(\mathbf{m}, p)$ be a solution to systems (1) and (2) in $X(0, T)$ and $T_{\max}$ be the maximal existence time. If $T_{\max} < \infty$, it holds that*

$$\limsup_{t\to T_{\max}} \|\mathbf{m}(t)\|_{H^2(\mathbb{R}^2)} = \infty$$

if and only if

$$\int_0^{T_{\max}} \|\nabla\mathbf{m}(t)\|_{L^q(\mathbb{R}^2)}^{\beta} dt = \infty$$

for any $q \in (2, +\infty]$, where

$$\beta = \max\left\{\frac{4q}{q-2}, \frac{4q(\gamma-2)}{2q-3}\right\}.$$

Proof. We can first deduce from Lemma 3 that, for $1 \leq \gamma < 3$ and all $0 < t < T_{\max}$,

$$\frac{d}{dt}\|\nabla^2\mathbf{m}\|_{L^2}^2 + \frac{D^2}{2}\|\nabla^3\mathbf{m}\|_{L^2}^2 \lesssim \|\nabla^2\mathbf{m}\|_{L^2}^2 \left(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + 1 \right) + \|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|\mathbf{m}\|_{L^2}^{\frac{8(\gamma-1)}{9-2\gamma}} + 1,$$

and that for $\gamma \geq 3$ and all $0 < t < T_{\max}$,

$$\frac{d}{dt}\|\nabla^2\mathbf{m}\|_{L^2}^2 + \frac{D^2}{2}\|\nabla^3\mathbf{m}\|_{L^2}^2 \lesssim \|\nabla^2\mathbf{m}\|_{L^2}^2\left(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}}+1\right) + \|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|\nabla\mathbf{m}\|_{L^q}^{\frac{4q(\gamma-2)}{2q-3}}\|\mathbf{m}\|_{L^2}^{\frac{4(q-1)}{2q-3}} + 1.$$

Hence, Gronwall's inequality implies that, for $1 \leq \gamma < 3$,

$$\begin{aligned}&\|\nabla^2\mathbf{m}(t)\|_{L^2}^2\\ &\leq e^{C\int_0^t\left(\|\nabla\mathbf{m}(s)\|_{L^q}^{\frac{4q}{q-2}}+1\right)ds}\left(\|\nabla^2\mathbf{m}_0\|_{L^2}^2 + C\int_0^t\left(\|\nabla\mathbf{m}(s)\|_{L^q}^{\frac{4q}{q-2}} + \|\mathbf{m}(s)\|_{L^2}^{\frac{8(\gamma-1)}{9-2\gamma}}+1\right)ds\right),\end{aligned}$$

and that, for $\gamma \geq 3$,

$$\begin{aligned}&\|\nabla^2\mathbf{m}(t)\|_{L^2}^2\\ &\leq e^{C\int_0^t\left(\|\nabla\mathbf{m}(s)\|_{L^q}^{\frac{4q}{q-2}}+1\right)ds}\left(\|\nabla^2\mathbf{m}_0\|_{L^2}^2 + C\int_0^t\left(\|\nabla\mathbf{m}\|_{L^q}^{\frac{4q}{q-2}} + \|\nabla\mathbf{m}(s)\|_{L^q}^{\frac{4q(\gamma-2)}{2q-3}}\|\mathbf{m}(s)\|_{L^2}^{\frac{4(q-1)}{2q-3}}+1\right)ds\right).\end{aligned}$$

By employing Lemma 1 again, we conclude that

$$\limsup_{t\to T_{\max}}\|\mathbf{m}(t)\|_{H^2} = \infty$$

if and only if

$$\int_0^{T_{\max}}\|\nabla\mathbf{m}(s)\|_{L^q}^{\frac{4q}{q-2}}ds = \infty$$

for $1 \leq \gamma < 3$, and

$$\int_0^{T_{\max}}\|\nabla\mathbf{m}(s)\|_{L^q}^{\frac{4q}{q-2}}ds + \int_0^{T_{\max}}\|\nabla\mathbf{m}(s)\|_{L^q}^{\frac{4q(\gamma-2)}{2q-3}}ds = \infty$$

for $\gamma \geq 3$, where $q \in (2, +\infty]$. This completes the proof of Lemma 4. □

Based on Lemma 4, we now prove Theorem 1.

Proof of Theorem 1. The blow-up criterion is a by-product of Lemma 4. Thus, we have completed the proof of Theorem 1. □

3. Global Existence. Proof of Theorem 2

In this section, invoking blow-up criterion (4), we show that the local strong solution of the Cauchy problem (1) and (2) is global indeed in the two-dimensional setting.

Lemma 5. *Assume that all conditions in Theorem 2 hold and the systems (1) and (2) possess a solution $(\mathbf{m}, p)$ in $X(0, T)$. For fixed $r \in (2, \infty)$, then there exists a positive constant $h(r)$ (see (32) below) such that, for all $D \geq h(r)$,*

$$\|\nabla\mathbf{m}(t)\|_{L^r}^r \leq C(1+t) \tag{28}$$

for any $0 \leq t \leq T$ and certain positive constant C independent of t.

Proof. We begin with $r \geq 4$. We now apply the operator ∂_i $(i = 1, 2)$ to $(1)_1$:

$$\partial_t\partial_i\mathbf{m} - D^2\Delta\partial_i\mathbf{m} + \partial_i\left(|\mathbf{m}|^{2(\gamma-1)}\mathbf{m}\right) = c^2\partial_i\left((\mathbf{m}\cdot\nabla p)\nabla p\right). \tag{29}$$

Then, taking the L^2 inner product of (29) with $\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}$ and using the integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^2} |\partial_i \mathbf{m}|^r dx + D^2 \int_{\mathbb{R}^2} \nabla \partial_i \mathbf{m} \cdot \nabla\left(\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}\right) dx + \int_{\mathbb{R}^2} \partial_i\left(|\mathbf{m}|^{2(\gamma-1)}\mathbf{m}\right) \cdot \left(\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}\right) dx \\
&= \int_{\mathbb{R}^2} \partial_i\left(\mathbf{m}\cdot\nabla p\right)\nabla p \cdot \left(\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}\right) dx + \int_{\mathbb{R}^2} \left(\mathbf{m}\cdot\nabla p\right)\nabla\partial_i p \cdot \left(\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}\right) dx. \qquad (30)
\end{aligned}$$

The second and third terms on the left-hand side of (30) can be calculated as follows:

$$\begin{aligned}
\int_{\mathbb{R}^2} \nabla \partial_i \mathbf{m} \cdot \nabla\left(\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}\right) dx &= \int_{\mathbb{R}^2} |\nabla \partial_i \mathbf{m}|^2 |\partial_i \mathbf{m}|^{r-2} dx + (r-2)\int_{\mathbb{R}^2} \left(\partial_i \mathbf{m}\cdot\nabla\partial_i \mathbf{m}\right)^2 |\partial_i \mathbf{m}|^{r-4} dx \\
&\geq \frac{4(r-1)}{r^2}\int_{\mathbb{R}^2} \left|\nabla\left(|\partial_i \mathbf{m}|^{\frac{r}{2}}\right)\right|^2 dx,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^2} \partial_i\left(|\mathbf{m}|^{2(\gamma-1)}\mathbf{m}\right) \cdot \left(\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}\right) dx &= \int_{\mathbb{R}^2} |\mathbf{m}|^{2(\gamma-1)} |\partial_i \mathbf{m}|^r dx \\
&\quad + 2(\gamma-1)\int_{\mathbb{R}^2} \left(\mathbf{m}\cdot\partial_i \mathbf{m}\right)^2 |\mathbf{m}|^{2(\gamma-2)} |\partial_i \mathbf{m}|^{r-2} dx.
\end{aligned}$$

Due to $r-2>0$ and $\gamma-1\geq 0$, we then get from Hölder's inequality and (30) that

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|\partial_i \mathbf{m}\|_{L^r}^r + \frac{4(r-1)D^2}{r^2}\left\|\nabla\left(|\partial_i \mathbf{m}|^{\frac{r}{2}}\right)\right\|_{L^2}^2 \\
&\leq \left\|\partial_i\left(\mathbf{m}\cdot\nabla p\right)\right\|_{L^2}\left\|\nabla p\cdot\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}\right\|_{L^2} + \|\nabla\partial_i p\|_{L^2}\left\|\left(\mathbf{m}\cdot\nabla p\right)\partial_i \mathbf{m}|\partial_i \mathbf{m}|^{r-2}\right\|_{L^2} \\
&\leq \left\|\partial_i\left(\mathbf{m}\cdot\nabla p\right)\right\|_{L^2}\|\nabla p\|_{L^3}\|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1} + \|\nabla\partial_i p\|_{L^2}\left\|\mathbf{m}\cdot\nabla p\right\|_{L^3}\|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1}.
\end{aligned}$$

We use the interpolation inequality and Lemma 1 to show

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\|\partial_i \mathbf{m}\|_{L^r}^r + \frac{4(r-1)D^2}{r^2}\left\|\nabla\left(|\partial_i \mathbf{m}|^{\frac{r}{2}}\right)\right\|_{L^2}^2 &\lesssim \left\|\partial_i\left(\mathbf{m}\cdot\nabla p\right)\right\|_{L^2}\|\nabla p\|_{L^2}^{\frac{2}{3}}\|\nabla^2 p\|_{L^2}^{\frac{1}{3}}\|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1} \\
&\quad + \|\nabla\partial_i p\|_{L^2}\|\mathbf{m}\cdot\nabla p\|_{L^2}^{\frac{2}{3}}\left\|\nabla\left(\mathbf{m}\cdot\nabla p\right)\right\|_{L^2}^{\frac{1}{3}}\|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1} \\
&\lesssim \left\|\partial_i\left(\mathbf{m}\cdot\nabla p\right)\right\|_{L^2}\|\nabla^2 p\|_{L^2}^{\frac{1}{3}}\|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1} \\
&\quad + \|\nabla\partial_i p\|_{L^2}\left\|\nabla\left(\mathbf{m}\cdot\nabla p\right)\right\|_{L^2}^{\frac{1}{3}}\|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1}.
\end{aligned}$$

Hence, by employing (16) with $q=4$, it arrives at

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\|\partial_i \mathbf{m}\|_{L^r}^r + \frac{4D^2(r-1)}{r^2}\left\|\nabla\left(|\partial_i \mathbf{m}|^{\frac{r}{2}}\right)\right\|_{L^2}^2 &\lesssim \|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1}\left(\|\nabla \mathbf{m}\|_{L^4}^4 + 1\right)^{\frac{2}{3}} \\
&\lesssim \|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1}\|\nabla \mathbf{m}\|_{L^4}^{\frac{8}{3}} + \|\partial_i \mathbf{m}\|_{L^{6(r-1)}}^{r-1}. \qquad (31)
\end{aligned}$$

On the other hand, a direct application of the interpolation inequality yields that for $r\geq 4$

$$\begin{aligned}
\|\nabla \mathbf{m}\|_{L^r}^{\frac{r}{2}} &= \left\||\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2} \\
&\lesssim \left\||\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^1}^{\frac{1}{2}}\left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\nabla \mathbf{m}\|_{L^{\frac{r}{2}}}^{\frac{r}{4}}\left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\nabla \mathbf{m}\|_{L^2}^{\frac{r}{2(r-2)}}\|\nabla \mathbf{m}\|_{L^r}^{\frac{r(r-4)}{4(r-2)}}\left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{1}{2}},
\end{aligned}$$

which leads to

$$\|\nabla \mathbf{m}\|_{L^r}^{\frac{r^2}{4(r-2)}} \lesssim \|\nabla \mathbf{m}\|_{L^2}^{\frac{r}{2(r-2)}} \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{1}{2}}.$$

Here, we used the fact that $\frac{r}{2} \geq 2$, due to $r \geq 4$. From this, one then has

$$\|\nabla \mathbf{m}\|_{L^4} \leq \|\nabla \mathbf{m}\|_{L^2}^{\frac{r-4}{2(r-2)}} \|\nabla \mathbf{m}\|_{L^r}^{\frac{r}{2(r-2)}} \lesssim \|\nabla \mathbf{m}\|_{L^2}^{\frac{r-4}{2(r-2)}+\frac{1}{r-2}} \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{1}{r}}.$$

Similarly, we also have

$$\begin{aligned}
\|\nabla \mathbf{m}\|_{L^{6(r-1)}}^{\frac{r}{2}} &= \left\||\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^{\frac{12(r-1)}{r}}} \\
&\lesssim \left\||\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{r}{6(r-1)}} \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{5r-6}{6(r-1)}} \\
&\lesssim \|\nabla \mathbf{m}\|_{L^r}^{\frac{r^2}{12(r-1)}} \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{5r-6}{6(r-1)}} \\
&\lesssim \|\nabla \mathbf{m}\|_{L^2}^{\frac{r}{6(r-1)}} \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{r-2}{6(r-1)}+\frac{5r-6}{6(r-1)}}.
\end{aligned}$$

Here, we used the fact that $\frac{12(r-1)}{r} > 2$ due to $r \geq 4$. Whereupon, invoking the above inequalities, we infer from (31) and Lemma 1 that

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|\nabla \mathbf{m}\|_{L^r}^r + \frac{4D^2(r-1)}{r^2}\left\|\nabla\left(|\nabla \mathbf{m}|^{\frac{r}{2}}\right)\right\|_{L^2}^2 \\
&\lesssim \|\nabla \mathbf{m}\|_{L^2}^{\frac{1}{3}} \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{2(3r-4)}{3r}} \|\nabla \mathbf{m}\|_{L^2}^{\frac{4}{3}} \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{8}{3r}} + \|\nabla \mathbf{m}\|_{L^2}^{\frac{1}{3}} \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{2(3r-4)}{3r}} \\
&\lesssim \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{2(3r-4)}{3r}+\frac{8}{3r}} + \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^{\frac{2(3r-4)}{3r}}.
\end{aligned}$$

Note that $0 < \frac{2(3r-4)}{3r} < 2$ and $\frac{2(3r-4)}{3r} + \frac{8}{3r} = 2$, due to $r \geq 4$. Thus, Young's inequality implies from the above inequality that

$$\frac{1}{r}\frac{d}{dt}\|\nabla \mathbf{m}\|_{L^r}^r + \frac{4D^2(r-1)}{r^2}\left\|\nabla\left(|\nabla \mathbf{m}|^{\frac{r}{2}}\right)\right\|_{L^2}^2 \leq C_1 \left\|\nabla|\nabla \mathbf{m}|^{\frac{r}{2}}\right\|_{L^2}^2 + C_2$$

for two positive constants C_1, C_2 independent of t. Furthermore, taking

$$h(r) = \left(\frac{C_1 r^2}{4(r-1)}\right)^{\frac{1}{2}}, \tag{32}$$

under our assumptions, then one arrives at

$$\frac{1}{r}\frac{d}{dt}\|\nabla \mathbf{m}\|_{L^r}^r \leq C_2.$$

Finally, a time integration of the above inequality shows that, for all $r \geq 4$,

$$\|\nabla \mathbf{m}(t)\|_{L^r}^r \leq \|\nabla \mathbf{m}_0\|_{L^r}^r + C_2 t. \tag{33}$$

Next, we turn to the case of $r \in (2,4)$. Recalling the interpolation inequality

$$\|\nabla \mathbf{m}\|_{L^r} \lesssim \|\nabla \mathbf{m}\|_{L^2}^{\frac{4-r}{r}} \|\nabla \mathbf{m}\|_{L^4}^{\frac{2(r-2)}{r}} \quad \text{for all } r \in (2,4),$$

we can get from (7), (33) and Young's inequality that

$$\|\nabla \mathbf{m}(t)\|_{L^r}^r \lesssim \left(\|\nabla \mathbf{m}_0\|_{L^4}^4 + C_2 t\right)^{\frac{2(r-2)}{4}} \lesssim 1 + t. \tag{34}$$

Combining with (33) and (34) yields (28). This finishes the proof of Lemma 5. □

By means of Lemma 5 and blow-up criterion (4), we can proceed to prove Theorem 2.

Proof of Theorem 2. By taking $r = 4$ in (32), we can get from Lemma 5 that

$$\|\nabla \mathbf{m}(t,\cdot)\|_{L^4}^4 \le C(1+t) \quad \text{for all } t > 0.$$

Based on this, the local strong solution $(\mathbf{m}, p)$ can be extended to a global one by employing the blow-up criterion (4) obtained in Theorem 2. Moreover, the boundedness (5) directly follows from Lemma 1, Lemma 2, Lemma 3 and Lemma 5. This completes the proof of Theorem 2. □

4. Discussion

In this section, let us begin with presenting a short summary of our main results for the nonlinear PDE system (1) and (2) in the two-dimensional setting:

- By developing some new dissipation mechanisms hidden in the system (1), we provide an a priori blow-up criterion of a strong solution of the Cauchy problems (1) and (2);
- Invoking an a priori blow-up criterion and seeking some suitable estimates, we establish a priori upper bounds to a strong solution for all positive times.

We conclude this section by providing two unsolved problems which will be the subject of our future research:

- How to establish the local existence of a strong solution of the Cauchy problems (1) and (2);
- We establish an a priori blow-up criterion and a priori upper bounds to a strong solution of systems (1) and (2) when $\gamma \ge 1$. Can these be generalized to $\gamma > \frac{1}{2}$?

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References

1. Nancy, D.; Julie, K. Vascular patterning and leaf shape. *Curr. Opin. Plant Biol.* **2001**, *4*, 50–56.
2. Malinowski, R. Understanding of leaf development-the science of complexity. *Plants* **2013**, *2*, 396–415. [CrossRef] [PubMed]
3. Sedmera, D. Function and form in the developing cardiovascular system. *Cardiovasc. Res.* **2011**, *91*, 252–259. [CrossRef] [PubMed]
4. Anne, E.; Ferdinand, L.N.; Monica, A.; Peter, C. Guidance of vascular and neural network formation. *Curr. Opin. Neurobiol.* **2005**, *15*, 108–115.
5. Michel, O.; Biondi, J. Morphogenesis of neural networks. *Neural Process. Lett.* **1995**, *2*, 9–12. [CrossRef]
6. Hu, D.; Cai, D. Adaptation and optimization of biological transport networks. *Phys. Rev. Lett.* **2013**, *111*, 138701. [CrossRef] [PubMed]
7. Albi, G.; Artina, M.; Foransier, M.; Markowich, P. Biological transportation networks: Modeling and simulation. *Anal. Appl.* **2016**, *14*, 185–206. [CrossRef]

8. Albi, G.; Burger, M.; Haskovec, J.; Markowich, P.; Schlottbom, M. Continuum Modeling of Biological Network Formation. In *Active Particles, Volume 1. Modeling and Simulation in Science and Technology*; Bellomo, N., Degond, P., Tamdor, T., Eds.; Birkhäuser: Boston, MA, USA, 2017; pp. 1–48.
9. Hu, D. Optimization, Adaptation, and Initialization of Biological Transport Networks. *Notes Lect.* **2014**.
10. Haskovec, J.; Markowich, P.; Perthame, B. Mathematical analysis of a PDE system for biological network formation. *Commun. Part. Differ. Equ.* **2015**, *40*, 918–956. [CrossRef]
11. Haskovec, J.; Markowich, P.; Perthame, B.; Schlottbom, M. Notes on a PDE system for biological network formation. *Nonlinear Anal.* **2016**, *138*, 127–155. [CrossRef]
12. Li, B. Long time behavior of the solution to a parabolic-elliptic system. *Comput. Math. Appl.* **2019**. [CrossRef]
13. Liu, J.; Xu, X. Partial regularity of weak solutions to a PDE system with cubic nonlinearity. *J. Differ. Equ.* **2018**, *264*, 5489–5526. [CrossRef]
14. Xu, X. Regularity theorems for a biological network formulation model in two space dimensions. *Kinet. Relat. Models* **2018**, *11*, 397–408. [CrossRef]
15. Xu, X. Life-span of smooth solutions to a PDE system with cubic nonlinearity. *arXiv* **2017**, arXiv:1706.06057.
16. Katifori, E.; Szöllosi, G.J.; Magnasco, M.O. Damage and fluctuations induce loops in optimal transport networks. *Phys. Rev. Lett.* **2010**, *104*, 048704. [CrossRef] [PubMed]
17. Li, B. On the blow-up criterion and global existence of a nonlinear PDE system in biological transport networks. *Kinet. Relat. Models* **2019**, in press.
18. Serrin, J. On the interior regularity of weak solutions of Navier-Stokes equations. *Arch. Ration. Mech. Anal.* **1962**, *9*, 187–195. [CrossRef]
19. Duan, R.; Lorz, A.; Markowich, P. Global solutions to the coupled chemotaxis-fluid equations. *Commun. Part Differ. Equ.* **2010**, *35*, 1635–1673. [CrossRef]