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Asymptotic Semicircular Laws Induced by p-Adic Number Fields $\mathbb{Q}_p$ and C^*-Algebras over Primes p

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Abstract: In this paper, we study asymptotic semicircular laws induced both by arbitrarily fixed C^*-probability spaces, and p-adic number fields $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$, as $p \to \infty$ in the set $\mathcal{P}$ of all primes.

Keywords: free probability; p-adic number fields $\mathbb{Q}_p$; Banach $*$-probability spaces; C^*-algebras; semicircular elements; the semicircular law; asymptotic semicircular laws

1. Introduction

The main purposes of this paper are (i) to establish *tensor product C^*-probability spaces*

$$(A \otimes_{\mathbb{C}} \mathfrak{S}_p, \psi \otimes \varphi_j^p)$$

induced both by arbitrary unital C^*-probability spaces (A, ψ), and by analytic structures $(\mathfrak{S}_p, \varphi_j^p)$ acting on *p-adic number fields* $\mathbb{Q}_p$ for all primes p in the set $\mathcal{P}$ of all *primes*, where $j \in \mathbb{Z}$, (ii) to consider free-probabilistic structures of (i) affected both by the free probability on (A, ψ), and by the number theory on $\mathbb{Q}_p$ for all $p \in \mathcal{P}$, (iii) to study *asymptotic behaviors* on the structures of (i) as $p \to \infty$ in $\mathcal{P}$, based on the results of (ii), and (iv), and then investigate *asymptotic semicircular laws* from the free-distributional data of (iii).

Our main results illustrate cross-connections among *number theory*, *representation theory*, *operator theory*, *operator algebra theory*, and *stochastic analysis*, via *free probability theory*.

1.1. Preview and Motivation

Relations between primes and *operators* have been studied in various different approaches. In [1], we studied how primes act on *operator algebras* induced by *dynamical systems* on *p-adic*, and *Adelic* objects. Meanwhile, in [2], primes are acting as *linear functionals* on *arithmetic functions*, characterized by *Krein-space operators*.

For number theory and free probability theory, see [3–22], respectively.

In [23], *weighted-semicircular elements*, and *semicircular elements* induced by *p-adic number fields* $\mathbb{Q}_p$ are considered by the author and Jorgensen, for each $p \in \mathcal{P}$, statistically. In [24], the author extended the constructions of *weighted-semicircular elements* of [23] under *free product* of [15,22]. The main results of [24] demonstrate that the (weighted-)semicircular law(s) of [23] is (are) well-determined free-probability-theoretically. As an application, the *free stochastic calculus* was considered in [6].

Independent from the above series of works, we considered *asymptotic semicircular laws* induced by $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$ in [1]. The constructions of [1] are highly motivated by those of [6,23,24], but they are totally different not only conceptually, but also theoretically. Thus, even though the main results of [1] seem similar to those of [6,24], they indicate-and-emphasize "asymptotic" semicircularity induced by $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$, as $p \to \infty$. For example, they show that our analyses on $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$ not only provide natural semicircularity but also asymptotic semicircularity under free probability theory.

In this paper, we study *asymptotic-semicircular laws* over "both" primes and *unital C^*-probability spaces*. Since we generalize the asymptotic semicircularity of [25] up to C^*-algebra-tensor, the patterns and results of this paper would be similar to those of [25], but generalize-or-universalize them.

1.2. Overview

In Section 2, fundamental concepts and backgrounds are introduced. In Sections 3–6, suitable free-probabilistic models are considered, where they contain p-adic number-theoretic information, for our purposes.

In Section 7, we establish-and-study C^*-probability spaces containing both analytic data from $\mathbb{Q}_p$, and free-probabilistic information of fixed unital C^*-probability spaces. Then, our free-probabilistic structure $\mathfrak{LS}_A$, a free product Banach $*$-probability space, is constructed, and the free probability on $\mathfrak{LS}_A$ is investigated in Section 8.

In Section 9, asymptotic behaviors on $\mathfrak{LS}_A$ are considered over $\mathcal{P}$, and they analyze the asymptotic semicircular laws on $\mathfrak{LS}_A$ over $\mathcal{P}$ in Section 10.

2. Preliminaries

In this section, we briefly mention backgrounds of our proceeding works.

2.1. Free Probability

See [15,22] (and the cited papers therein) for basic free probability theory. Roughly speaking, *free probability* is the noncommutative operator-algebraic extension of measure theory (containing probability theory) and statistical analysis. As an independent branch of operator algebra theory, it is applied not only to mathematical analysis (e.g., [5,12–14,26]), but also to related fields (e.g., [18,27–31]).

Here, combinatorial free probability is used (e.g., [15–17]). In the text, *free moments*, *free cumulants*, and the *free product of $*$-probability spaces* are considered without detailed introduction.

2.2. Analysis on $\mathbb{Q}_p$

For *p-adic analysis* and *Adelic analysis*, see [21,22]. We use definitions, concepts, and notations from there. Let $p \in \mathcal{P}$ be a prime, and let $\mathbb{Q}$ be the set of all *rational numbers*. Define a *non-Archimedean norm* $|.|_p$, called the *p-norm on* $\mathbb{Q}$ by

$$|x|_p = \left|p^k \tfrac{a}{b}\right|_p = \tfrac{1}{p^k},$$

for all $x = p^k \frac{a}{b} \in \mathbb{Q}$, where $k, a \in \mathbb{Z}$, and $b \in \mathbb{Z} \setminus \{0\}$.

The normed space $\mathbb{Q}_p$ is the maximal p-norm closures in $\mathbb{Q}$, i.e., the set $\mathbb{Q}_p$ forms a *Banach space*, for $p \in \mathcal{P}$ (e.g., [22]). Each element x of $\mathbb{Q}_p$ is uniquely expressed by

$$x = \textstyle\sum_{k=-N}^{\infty} x_k p^k,\ x_k \in \{0, 1, ..., p-1\},$$

for $N \in \mathbb{N}$, decomposed by

$$x = \textstyle\sum_{l=-N}^{-1} x_l p^l + \sum_{k=0}^{\infty} x_k p^k.$$

If $x = \sum_{k=0}^{\infty} x_k p^k$ in $\mathbb{Q}_p$, then x is said to be a *p-adic integer*, and it satisfies $|x|_p \leq 1$. Thus, one can define the *unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$*,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

For the *p-adic addition* and the *p-adic multiplication* in the sense of [22], the algebraic structure $\mathbb{Q}_p$ forms a *field*, and hence, $\mathbb{Q}_p$ is a *Banach field*.

Note that $\mathbb{Q}_p$ is also a *measure space*,

$$\mathbb{Q}_p = \left(\mathbb{Q}_p,\ \sigma(\mathbb{Q}_p),\ \mu_p\right),$$

equipped with the *σ-algebra* $\sigma(\mathbb{Q}_p)$ *of* $\mathbb{Q}_p$, and a left-and-right additive invariant *Haar measure* on μ_p, satisfying

$$\mu_p(\mathbb{Z}_p) = 1.$$

If we take

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in \mathbb{Q}_p : x \in \mathbb{Z}_p\}, \tag{1}$$

in $\sigma\left(\mathbb{Q}_p\right)$, for all $k \in \mathbb{Z}$, then these subsets U_k's of (1) satisfy

$$\mathbb{Q}_p = \underset{k\in\mathbb{Z}}{\cup} U_k,$$

and

$$\mu_p\left(U_k\right) = \frac{1}{p^k} = \mu_p\left(x + U_k\right), \tag{2}$$

for all $x \in \mathbb{Q}_p$, and

$$\cdots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_{-1} \subset U_{-2} \subset \cdots,$$

i.e., the family $\{U_k\}_{k\in\mathbb{Z}}$ of (1) is a *topological basis element of* $\mathbb{Q}_p$ (e.g., [22]).

Define subsets $\partial_k \in \sigma(\mathbb{Q}_p)$ by

$$\partial_k = U_k \setminus U_{k+1}, \tag{3}$$

for all $k \in \mathbb{Z}$.

Such μ_p-measurable subsets ∂_k of (3) are called the *k-th boundaries* (*of* U_k) in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. By (2) and (3),

$$\mathbb{Q}_p = \underset{k\in\mathbb{Z}}{\sqcup} \partial_k,$$

$$\mu_p\left(\partial_k\right) = \mu_p\left(U_k\right) - \mu_p\left(U_{k+1}\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \tag{4}$$

where $\sqcup$ is the *disjoint union*, for all $k \in \mathbb{Z}$,

Let $\mathcal{M}_p$ be an algebraic *algebra*,

$$\mathcal{M}_p = \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p)\}\right], \tag{5a}$$

where χ_S are the usual *characteristic functions* of μ_p-measurable subsets S of $\mathbb{Q}_p$. Thus, $f \in \mathcal{M}_p$, if and only if

$$f = \sum_{S\in\sigma\left(\mathbb{Q}_p\right)} t_S \chi_S; \, t_S \in \mathbb{C}, \tag{5b}$$

where $\sum$ is the *finite sum*. Note that the algebra $\mathcal{M}_p$ of (5a) is a *∗-algebra over* $\mathbb{C}$, with its well-defined *adjoint*,

$$\left(\sum_{S\in\sigma(G_p)} t_S \chi_S\right)^* \overset{def}{=} \sum_{S\in\sigma(G_p)} \overline{t_S}\, \chi_S,$$

for $t_S \in \mathbb{C}$ with their *conjugates* $\overline{t_S}$ in $\mathbb{C}$.

If $f \in \mathcal{M}_p$ is given as in (5b), then one defines the *integral of* f by

$$\int_{\mathbb{Q}_p} f \, d\mu_p = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S \, \mu_p(S). \tag{6a}$$

Remark that, by (5a), the integral (6a) is unbounded on $\mathcal{M}_p$, i.e.,

$$\int_{\mathbb{Q}_p} \chi_{\mathbb{Q}_p} d\mu_p = \mu_p\left(\mathbb{Q}_p\right) = \infty, \tag{6b}$$

by (2).

Note that, by (4), for each $S \in \sigma(\mathbb{Q}_p)$, there exists a corresponding subset Λ_S of $\mathbb{Z}$,

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}, \tag{7}$$

satisfying

$$\begin{aligned} \int_{\mathbb{Q}_p} \chi_S \, d\mu_p &= \int_{\mathbb{Q}_p} \sum_{j \in \Lambda_S} \chi_{S \cap \partial_j} \, d\mu_p \\ &= \sum_{j \in \Lambda_S} \mu_p \left(S \cap \partial_j\right) \end{aligned}$$

by (6a)

$$\leq \sum_{j \in \Lambda_S} \mu_p \left(\partial_j\right) = \sum_{j \in \Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{8}$$

by (4), for the set Λ_S of (7).

Remark again that the right-hand side of (8) can be ∞; for instance, $\Lambda_{\mathbb{Q}_p} = \mathbb{Z}$, e.g., see (4), (6a) and (6b). By (8), one obtains the following proposition.

Proposition 1. *Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then, there exists $r_j \in \mathbb{R}$, such that*

$$\begin{gathered} 0 \leq r_j = \frac{\mu_p(S \cap \partial_j)}{\mu_p(\partial_j)} \leq 1, \forall j \in \Lambda_S; \\ \int_{\mathbb{Q}_p} \chi_S \, d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right). \end{gathered} \tag{9}$$

3. Statistical Models on $\mathcal{M}_p$

In this section, fix $p \in \mathcal{P}$, and let $\mathbb{Q}_p$ be the p-adic number field, and let $\mathcal{M}_p$ be the $*$-algebra (5a). We here establish a suitable statistical model on $\mathcal{M}_p$ with free-probabilistic language.

Let U_k be the basis elements (1), and ∂_k, their boundaries (3) of $\mathbb{Q}_p$, i.e.,

$$U_k = p^k \mathbb{Z}_p,$$

for all $k \in \mathbb{Z}$, and

$$\partial_k = U_k \setminus U_{k+1}; k \in \mathbb{Z}. \tag{10}$$

Define a linear functional $\varphi_p : \mathcal{M}_p \to \mathbb{C}$ by the *integration* (6a), i.e.,

$$\varphi_p\left(f\right) = \int_{\mathbb{Q}_p} f \, d\mu_p, \tag{11}$$

for all $f \in \mathcal{M}_p$.

Then, by (9), one obtains that $\varphi_p\left(\chi_{U_j}\right) = \frac{1}{p^j}$, and $\varphi_p\left(\chi_{\partial_j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}}$, since $\Lambda_{U_j} = \{k \in \mathbb{Z} : k \geq j\}$, and $\Lambda_{\partial_j} = \{j\}$, for all $j \in \mathbb{Z}$, where Λ_S are in the sense of (7) for all $S \in \sigma(\mathbb{Q}_p)$.

Definition 1. *The pair $\left(\mathcal{M}_p, \ \varphi_p\right)$ is called the p-adic (unbounded-)measure space for $p \in \mathcal{P}$, where φ_p is the linear functional (11) on $\mathcal{M}_p$.*

Let ∂_k be the k-th boundaries (10) of $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then, for $k_1, k_2 \in \mathbb{Z}$, one obtains that

$$\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} = \chi_{\partial_{k_1} \cap \partial_{k_2}} = \delta_{k_1, k_2} \chi_{\partial_{k_1}},$$

and hence,

$$\begin{aligned} \varphi_p\left(\chi_{\partial_{k_1}} \chi_{\partial_{k_2}}\right) &= \delta_{k_1,k_2} \varphi_p\left(\chi_{\partial_{k_1}}\right) \\ &= \delta_{k_1,k_2} \left(\frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}}\right). \end{aligned} \tag{12}$$

Proposition 2. *Let* $(j_1, ..., j_N) \in \mathbb{Z}^N$*, for* $N \in \mathbb{N}$*. Then,*

$$\prod_{l=1}^{N} \chi_{\partial_{j_l}} = \delta_{(j_1,...,j_N)} \chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p,$$

and hence,

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{\partial_{j_l}}\right) = \delta_{(j_1,...,j_N)}\left(\frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}}\right), \tag{13}$$

where

$$\delta_{(j_1,...,j_N)} = \left(\prod_{l=1}^{N-1} \delta_{j_l,j_{l+1}}\right)\left(\delta_{j_N,j_1}\right).$$

Proof. The computation (13) is shown by the induction on (12). □

Recall that, for any $S \in \sigma\left(\mathbb{Q}_p\right)$,

$$\varphi_p\left(\chi_S\right) = \sum_{j \in \Lambda_S} r_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{14}$$

for some $0 \leq r_j \leq 1$, for $j \in \Lambda_S$, by (9). Thus, by (14), if $S_1, S_2 \in \sigma\left(\mathbb{Q}_p\right)$, then

$$\begin{aligned} \chi_{S_1}\chi_{S_2} &= \left(\sum_{k \in \Lambda_{S_1}} \chi_{S_1 \cap \partial_k}\right)\left(\sum_{j \in \Lambda_{S_2}} \chi_{S_2 \cap \partial_j}\right) \\ &= \sum_{(k,j) \in \Lambda_{S_1} \times \Lambda_{S_2}} \left(\chi_{S_1 \cap \partial_k}\chi_{S_2 \cap \partial_j}\right) \\ &= \sum_{(k,j) \in \Lambda_{S_1} \times \Lambda_{S_2}} \delta_{k,j}\, \chi_{(S_1 \cap S_2) \cap \partial_j} \\ &= \sum_{j \in \Lambda_{S_1,S_2}} \chi_{(S_1 \cap S_2) \cap \partial_j}, \end{aligned} \tag{15}$$

where

$$\Lambda_{S_1,S_2} = \Lambda_{S_1} \cap \Lambda_{S_2},$$

by (4).

Proposition 3. *Let* $S_l \in \sigma(\mathbb{Q}_p)$*, and let* $\chi_{S_l} \in \left(\mathcal{M}_p,\ \varphi_p\right)$*, for* $l = 1, ..., N$*, for* $N \in \mathbb{N}$*. Let*

$$\Lambda_{S_1,...,S_N} = \bigcap_{l=1}^{N} \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} *are in the sense of (7), for* $l = 1, ..., N$*. Then, there exists* $r_j \in \mathbb{R}$*, such that*

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R},$$

for all $j \in \Lambda_{S_1,...,S_N}$*, and*

$$\varphi_p\left(\prod_{l=1}^{N} \chi_{S_l}\right) = \sum_{j \in \Lambda_{S_1,...,S_N}} r_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right). \tag{16}$$

Proof. The proof of (16) is done by the induction on (15), and by (13). □

4. Representation of $\left(\mathcal{M}_p,\ \varphi_p\right)$

Fix a prime $p \in \mathcal{P}$. Let $\left(\mathcal{M}_p,\ \varphi_p\right)$ be the p-adic measure space. By understanding $\mathbb{Q}_p$ as a measure space, construct the L^2-*space*,

$$H_p \overset{def}{=} L^2\left(\mathbb{Q}_p,\ \sigma(\mathbb{Q}_p),\ \mu_p\right) = L^2\left(\mathbb{Q}_p\right), \tag{17}$$

over $\mathbb{C}$. Then, this *Hilbert space* H_p of (17) consists of all square-integrable elements of $\mathcal{M}_p$, equipped with its *inner product* $<,>_2$,

$$\langle f_1,\ f_2\rangle_2 \overset{def}{=} \int_{\mathbb{Q}_p} f_1\, f_2^*\, d\mu_p, \tag{18a}$$

for all $f_1, f_2 \in H_p$. Naturally, H_p is has its L^2-*norm* $\|.\|_2$ on $\mathcal{M}_p$,

$$\|f\|_2 \overset{def}{=} \sqrt{\langle f,\ f\rangle_2}, \tag{18b}$$

for all $f \in H_p$, where $<,>_2$ is the inner product (18a) on H_p.

Definition 2. *The Hilbert space H_p of (17) is called the p-adic Hilbert space.*

Our $*$-algebra $\mathcal{M}_p$ acts on the p-adic Hilbert space H_p, via an action α^p,

$$\alpha^p(f)\,(h) = fh, \text{ for all } h \in H_p, \tag{19a}$$

for all $f \in \mathcal{M}_p$. i.e., the morphism α^p of (19a) is a $*$-homomorphism from $\mathcal{M}_p$ to the *operator algebra* $B(H_p)$, consisting of all Hilbert-space operators on H_p. For instance,

$$\begin{aligned}\alpha^p\left(\chi_{\mathbb{Q}_p}\right)\left(\sum_{S\in\sigma(\mathbb{Q}_p)} t_S\chi_S\right) &= \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\chi_{\mathbb{Q}_p\cap S} \\ &= \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\chi_S,\end{aligned} \tag{19b}$$

for all $h = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\chi_S \in H_p$, with $\|h\|_2 < \infty$, for $\chi_{\mathbb{Q}_p} \in \mathcal{M}_p$, even though $\chi_{\mathbb{Q}_p} \notin H_p$.

Indeed, It is not difficult to check that

$$\begin{gathered}\alpha^p(f_1f_2) = \alpha^p(f_1)\alpha^p(f_2) \text{ on } H_p, \forall f_1, f_2 \in \mathcal{M}_p, \\ \\ (\alpha^p(f))^* = \alpha(f^*) \text{ on } H_p, \forall f \in \mathcal{M}_p.\end{gathered} \tag{20a}$$

Notation 1. Denote $\alpha^p(f)$ by α_f^p, for all $f \in \mathcal{M}_p$. In addition, for convenience, denote $\alpha_{\chi_S}^p$ simply by α_S^p, for all $S \in \sigma\left(\mathbb{Q}_p\right)$.

Note that, by (19b), one can have a well-defined operator $\alpha_{\mathbb{Q}_p}^p = \alpha_{\chi_{\mathbb{Q}_p}}^p$ in $B(H_p)$, and it satisfies that

$$\alpha_{\mathbb{Q}_p}^p\left(h\right) = h = 1_{H_p}\left(h\right), \forall h \in H_p, \tag{20b}$$

where $1_{H_p} \in B(H_p)$ is the identity operator on H_p.

Proposition 4. *The pair (H_p, α^p) is a Hilbert-space representation of $\mathcal{M}_p$.*

Proof. It suffices to show that α^p is an algebra-action of $\mathcal{M}_p$ on H_p. However, this morphism α^p is a $*$-homomorphism from $\mathcal{M}_p$ into $B(H_p)$, by (20a). □

Definition 3. *The Hilbert-space representation $\left(H_p,\ \alpha^p\right)$ is called the p-adic representation of $\mathcal{M}_p$.*

Depending on the p-adic representation (H_p, α^p) of $\mathcal{M}_p$, one can define the C^*-subalgebra M_p of $B(H_p)$ as follows.

Definition 4. *Let M_p be the operator-norm closure of $\mathcal{M}_p$,*

$$M_p \overset{def}{=} \overline{\alpha^p(\mathcal{M}_p)} = \overline{\mathbb{C}\left[\alpha_f^p : f \in \mathcal{M}_p\right]} \tag{21}$$

in $B(H_p)$, where $\overline{X}$ are the operator-norm closures of subsets X of $B(H_p)$. This C^*-algebra M_p is said to be the p-adic C^*-algebra of $(\mathcal{M}_p, \varphi_p)$.

By (21), the p-adic C^*-algebra M_p is a unital C^*-algebra contains its *unity* (or the unit, or the multiplication-identity) $1_{H_p} = \alpha^p_{\mathbb{Q}_p}$, by (20b).

5. Statistics on M_p

In this section, fix $p \in \mathcal{P}$, and let M_p be the corresponding p-adic C^*-algebra of (21). Define a linear functional $\varphi_j^p : M_p \to \mathbb{C}$ by

$$\varphi_j^p(a) \overset{def}{=} \left\langle a(\chi_{\partial_j}),\ \chi_{\partial_j}\right\rangle_2, \forall a \in M_p, \tag{22a}$$

for $\chi_{\partial_j} \in H_p$, where $<, >_2$ is the inner product (4.2) on the p-adic Hilbert space H_p of (4.1), and ∂_j are the boundaries (3.1) of $\mathbb{Q}_p$, for all $j \in \mathbb{Z}$. It is not hard to check such a linear functional φ_j^p on M_p is bounded, since

$$\begin{aligned}\varphi_j^p\left(\alpha_S^p\right) &= \left\langle \alpha_S^p\left(\chi_{\partial_j}\right),\ \chi_{\partial_j}\right\rangle_2 = \left\langle \chi_S\chi_{\partial_j},\ \chi_{\partial_j}\right\rangle_2 \\ &= \left\langle \chi_{S\cap\partial_j},\ \chi_{\partial_j}\right\rangle_2 = \int_{\mathbb{Q}_p}\chi_{S\cap\partial_j}d\mu_p \\ &\leq \int_{\mathbb{Q}_p}\chi_{\partial_j}d\mu_p = \mu_p\left(\partial_j\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}},\end{aligned} \tag{22b}$$

for all $S \in \sigma(\mathbb{Q}_p)$, for any fixed $j \in \mathbb{Z}$.

Definition 5. *Let φ_j^p be bounded linear functionals (22a) on the p-adic C^*-algebra M_p, for all $j \in \mathbb{Z}$. Then, the pairs $\left(M_p,\ \varphi_j^p\right)$ are said to be the j-th p-adic C^*-measure spaces, for all $j \in \mathbb{Z}$.*

Thus, one can get the system

$$\{(M_p, \varphi_j^p) : j \in \mathbb{Z}\}$$

of the j-th p-adic C^*-measure spaces (M_p, φ_j^p)'s.

Note that, for any fixed $j \in \mathbb{Z}$, and (M_p, φ_j^p), the unity

$$1_{M_p} \overset{denote}{=} 1_{H_p} = \alpha^p_{\mathbb{Q}_p} \text{ of } M_p$$

satisfies that

$$\begin{aligned}\varphi_j^p\left(1_{M_p}\right) &= \left\langle \chi_{\mathbb{Q}_p\cap\partial_j},\ \chi_{\partial_j}\right\rangle_2 \\ &= \left\|\chi_{\partial_j}\right\|^2 = \frac{1}{p^j} - \frac{1}{p^{j+1}}.\end{aligned} \tag{23}$$

Thus, the j-th p-adic C^*-measure space (M_p, φ_j^p) is a bounded-measure space, but not a probability space, in general.

Proposition 5. *Let $S \in \sigma(\mathbb{Q}_p)$, and $\alpha_S^p \in \left(M_p,\ \varphi_j^p\right)$, for a fixed $j \in \mathbb{Z}$. Then, there exists $r_S \in \mathbb{R}$, such that*

$$0 \leq r_S \leq 1 \text{ in } \mathbb{R},$$

and

$$\varphi_j^p \left(\left(\alpha_S^p \right)^n \right) = r_S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right); n \in \mathbb{N}. \tag{24}$$

Proof. Remark that the element α_S^p is a projection in M_p, in the sense that:

$$\left(\alpha_S^p\right)^* = \alpha^p_{(\chi_S^*)} = \alpha_S^p = \alpha^p_{(\chi_S \cap \chi_S)} = \left(\alpha_S^p\right)^2, \text{ in } M_p,$$

and hence,

$$\left(\alpha_S^p\right)^n = \alpha_S^p,$$

for all $n \in \mathbb{N}$. Thus, we obtain the formula (24) by (22b). □

As a corollary of (24), one obtains that, if ∂_k is a k-th boundaries of $\mathbb{Q}_p$, then

$$\varphi_j^p \left(\left(\alpha_{\partial_k}^p \right)^n \right) = \delta_{j,k} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \tag{25}$$

for all $n \in \mathbb{N}$, for $k \in \mathbb{Z}$.

6. The C^*-Subalgebra $\mathfrak{S}_p$ of M_p

Let M_p be the p-adic C^*-algebra for $p \in \mathcal{P}$. Let

$$P_{p,j} = \alpha_{\partial_j}^p \in M_p, \tag{26}$$

for all $j \in \mathbb{Z}$. By (24) and (25), these operators $P_{p,j}$ of (26) are *projections* on the p-adic Hilbert space H_p, in M_p, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Definition 6. *Let $p \in \mathcal{P}$, and let $\mathfrak{S}_p$ be the C^*-subalgebra*

$$\mathfrak{S}_p = C^* \left(\{P_{p,j}\}_{j \in \mathbb{Z}} \right) = \overline{\mathbb{C} \left[\{P_{p,j}\}_{j \in \mathbb{Z}} \right]} \text{ of } M_p, \tag{27}$$

where $P_{p,j}$ are in the sense of ((26)), for all $j \in \mathbb{Z}$. We call $\mathfrak{S}_p$, the p-adic boundary (C^*-)subalgebra of M_p.

Proposition 6. *If $\mathfrak{S}_p$ is the p-adic boundary subalgebra (27), then*

$$\mathfrak{S}_p \overset{*\text{-iso}}{=} \underset{j \in \mathbb{Z}}{\oplus} \left(\mathbb{C} \cdot P_{p,j} \right) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus |\mathbb{Z}|}, \tag{28}$$

in the p-adic C^*-algebra M_p.

Proof. It is enough to show that the generating operators $\{P_{p,j}\}_{j \in \mathbb{Z}}$ of $\mathfrak{S}_p$ are mutually orthogonal from each other. It is not hard to check that

$$P_{p,j_1} P_{p,j_2} = \alpha^p \left(\chi_{\partial_{j_1}^p \cap \partial_{j_2}^p} \right) = \delta_{j_1,j_2} \alpha^p_{\partial_{j_1}^p} = \delta_{j_1,j_2} P_{p,j_1},$$

in $\mathfrak{S}_p$, for all $j_1, j_2 \in \mathbb{Z}$. Therefore, the structure theorem (28) is shown. □

By (27), one can define the measure spaces,

$$\mathfrak{S}_p(j) \overset{denote}{=} \left(\mathfrak{S}_p, \varphi_j^p \right), \forall j \in \mathbb{Z}, \tag{29}$$

for $p \in \mathcal{P}$, where the linear functionals φ_j^p of (29) are the restrictions $\varphi_j^p \mid_{\mathfrak{S}_p}$ of (22a), for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

7. On the Tensor Product C^*-Probability Spaces $\left(A \otimes_{\mathbb{C}} \mathfrak{S}_p,\ \psi \otimes \varphi_j^p\right)$

In this section, we define and study our main objects of this paper. Let (A, ψ) be an arbitrary unital C^*-probability space (e.g., [22]), satisfying

$$\psi(1_A) = 1,$$

where 1_A is the unity of a C^*-algebra A. In addition, let

$$\mathfrak{S}_p(j) = \left(\mathfrak{S}_p,\ \varphi_j^p\right) \tag{30}$$

be the p-adic C^*-measure spaces (29), for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Fix now a unital C^*-probability space (A, ψ), and $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Define a tensor product C^*-algebra

$$\mathfrak{S}_p^A \overset{def}{=} A \otimes_{\mathbb{C}} \mathfrak{S}_p, \tag{31}$$

and a linear functional ψ_j^p on $\mathfrak{S}_p^A$ by a linear morphism satisfying

$$\psi_j^p \left(a \otimes P_{p,k}\right) = \varphi_j^p \left(\psi(a) P_{p,k}\right), \tag{32}$$

for all $a \in (A, \psi)$, and $k \in \mathbb{Z}$.

Note that, by the structure theorem (28) of the p-adic boundary subalgebra $\mathfrak{S}_p$,

$$\mathfrak{S}_p^A \overset{*\text{-iso}}{=} A \otimes_{\mathbb{C}} \left(\mathbb{C}^{\oplus|\mathbb{Z}|}\right) \overset{*\text{-iso}}{=} A^{\oplus|\mathbb{Z}|}, \tag{33}$$

by (31).

By (33), one can verify that a morphism ψ_j^p of (32) is indeed a well-defined bounded linear functional on $\mathfrak{S}_p^A$.

Definition 7. *For any arbitrarily fixed $p \in \mathcal{P}$, $j \in \mathbb{Z}$, let $\mathfrak{S}_p^A$ be the tensor product C^*-algebra (31), and ψ_j^p, the linear functional (32) on $\mathfrak{S}_p^A$. Then, we call $\mathfrak{S}_p^A$, the A-tensor p-adic boundary algebra. The corresponding structure,*

$$\mathfrak{S}_p^A(j) \overset{denote}{=} \left(\mathfrak{S}_p^A,\ \psi_j^p\right) \tag{34}$$

is said to be the j-th p-adic A-(tensor C^*-probability-)space.

Note that, by (22a), (22b) and (32), the j-th p-adic A-space $\mathfrak{S}_p^A(j)$ of (34) is not a "unital" C^*-probability space, even though (A, ψ) is. Indeed, the C^*-algebra $\mathfrak{S}_p^A$ of (31) has its unity $1_A \otimes 1_{M_p}$, satisfying

$$\begin{aligned} \psi_j^p \left(1_A \otimes 1_{M_p}\right) &= \varphi_j^p \left(\psi(1_A) 1_{M_p}\right) \\ &= 1 \cdot \varphi_j^p(1_{M_p}) = \tfrac{1}{p^j} - \tfrac{1}{p^{j+1}}, \end{aligned}$$

for $j \in \mathbb{Z}$.

Remark that, by (32),

$$\psi_j^p \left(a \otimes P_{p,k}\right) = \psi(a)\ \varphi_j^p \left(P_{p,k}\right), \tag{35a}$$

for all $a \in (A, \psi)$, and $k \in \mathbb{Z}$. Thus, by abusing notation, one may write the definition (32) by

$$\psi_j^p = \psi \otimes \varphi_j^p \text{ on } A \otimes_{\mathbb{C}} \mathfrak{S}_p = \mathfrak{S}_p^A, \tag{35b}$$

in the sense of (35a), for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Proposition 7. *Let $a \in (A, \psi)$, and $P_{p,k}$, the k-th generating projection of $\mathfrak{S}_p$, for all $k \in \mathbb{Z}$, and let $a \otimes P_{p,k}$ be the corresponding free random variable of the j-th p-adic A-space $\mathfrak{S}_p^A(j)$, for $j \in \mathbb{Z}$. Then,*

$$\psi_j^p \left(\left(a \otimes P_{p,k} \right)^n \right) = \delta_{j,k} \, \psi(a^n) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \tag{36}$$

for all $n \in \mathbb{N}$.

Proof. Let $T_{p,k}^a = a \otimes P_{p,k}$ be a given free random variable of $\mathfrak{S}_p^A(j)$. Then,

$$\left(T_{p,k}^a \right)^n = \left(a \otimes P_{p,k} \right)^n = a^n \otimes P_{p,k} = T_{p,k}^{a^n},$$

and hence

$$\psi_j^p \left(\left(T_{p,k}^a \right)^n \right) = \psi_j^p \left(T_{p,k}^{a^n} \right)$$
$$= \psi(a^n) \, \varphi_j^p \left(P_{p,k} \right) = \psi(a^n) \left(\delta_{j,k} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right)$$

by (35a)

$$= \delta_{j,k} \psi(a^n) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right),$$

for all $n \in \mathbb{N}$. Therefore, the free-distributional data (36) holds. □

Suppose a is a "self-adjoint" free random variable in (A, ψ) in the above proposition. Then, formula (36) completely characterizes the free distribution of $a \otimes P_{p,k}$ in the j-th p-adic A-space $\mathfrak{S}_p^A(j)$ of (34), i.e., the free distribution of $a \otimes P_{p,k}$ is characterized by the sequence,

$$\left(\delta_{j,k} \psi(a^n) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right)_{n=1}^{\infty}$$

for all $p \in \mathcal{P}$, and $j, k \in \mathbb{Z}$ because $a \otimes P_{p,k}$ is self-adjoint in $\mathfrak{S}_p^A$ too.

It illustrates that the free probability on $\mathfrak{S}_p^A(j)$ is determined both by the free probability on (A, ψ), and by the statistical data on $\mathfrak{S}_p(j)$ of (30) (implying p-adic analytic information), for $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Notation. From below, for convenience, let's denote the free random variables $a \otimes P_{p,k}$ of $\mathfrak{S}_p^A(j)$, with $a \in (A, \psi)$ and $k \in \mathbb{Z}$, by $T_{p,k}^a$, i.e.,

$$T_{p,k}^a \overset{denote}{=} a \otimes P_{p,k},$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

In the proof of (36), it is observed that

$$\left(T_{p,k}^a \right)^n = T_{p,k}^{a^n} \in \mathfrak{S}_p^A(j) \tag{37}$$

for all $n \in \mathbb{N}$. More generally, the following free-distributional data is obtained.

Theorem 1. *Fix $p \in \mathcal{P}$, and $j \in \mathbb{Z}$, and let $\mathfrak{S}_p^A(j)$ be the j-th p-adic A-space (34). Let $T_{p,k_l}^{a_l} \in \mathfrak{S}_p^A(j)$, for $l = 1, ..., N$, for $N \in \mathbb{N}$. Then,*

$$\psi_j^p \left(\prod_{l=1}^{N} \left(T_{p,k_l}^{a_l} \right)^{n_l} \right) = \left(\prod_{l=1}^{N} \delta_{j,k_l} \right) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \psi \left(\prod_{l=1}^{N} a_l^{n_l} \right), \tag{38}$$

for all $n_1, ..., n_N \in \mathbb{N}$.

Proof. Let $T_{p,k_l}^{a_l} = a_l \otimes P_{p,k_l}$ be free random variables of $\mathfrak{S}_p^A(j)$, for $l = 1, ..., N$. Then, by (37),

$$\left(T_{p,k_l}^{a_l} \right)^{n_l} = T_{p,k_l}^{a_l^{n_l}} \in \mathfrak{S}_p^A(j), \text{ for } n_l \in \mathbb{N},$$

for all $l = 1, ..., N$. Thus,

$$T = \prod_{l=1}^{N} \left(T_{p,k_l}^{a_l} \right)^{n_l} = \left(\prod_{l=1}^{N} a_l^{n_l} \right) \otimes \left(\delta_{j:k_1,...,k_N} P_{p,j} \right)$$

in $\mathfrak{S}_p^A(j)$, with

$$\delta_{j:k_1,...,k_N} = \prod_{l=1}^{N} \delta_{j,k_l} \in \{0, 1\}.$$

Therefore,

$$\begin{aligned} \psi_j^p(T) &= \delta_{j:k_1,...,k_N} \psi \left(\prod_{l=1}^{N} a_l^{n_l} \right) \varphi_j^p(P_{p,j}) \\ &= \delta_{j:k_1,...,k_N} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \psi \left(\prod_{l=1}^{N} a_l^{n_l} \right), \end{aligned}$$

by (35a). Thus, the joint free-distributional data (38) holds. □

Definitely, if $N = 1$ in (38), one obtains the formula (36).

8. On the Banach $*$-Probability Spaces $\mathfrak{L}\mathfrak{S}_{p,j}^A$

Let (A, ψ) be an arbitrarily fixed unital C^*-probability space, and let $\mathfrak{S}_p(j)$ be in the sense of (30), for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. Then, one can construct the tensor product C^*-probability spaces, the j-th p-adic A-space,

$$\mathfrak{S}_p^A(j) = \left(\mathfrak{S}_p^A, \ \psi_j^p \right) = \left(A \otimes_{\mathbb{C}} \mathfrak{S}_p, \ \psi \otimes \varphi_j^p \right)$$

of (34), for $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Throughout this section, we fix $p \in \mathcal{P}$, $j \in \mathbb{Z}$, and the corresponding j-th p-adic A-space $\mathfrak{S}_p^A(j)$. In addition, we keep using our notation $T_{p,k}^a$ for the free random variables $a \otimes P_{p,k}$ of $\mathfrak{S}^A(j)$, for all $a \in (A, \psi)$ and $k \in \mathbb{Z}$, where $P_{p,k}$ are the generating projections (26) of the p-adic boundary subalgebra $\mathfrak{S}_p$.

Recall that, by (36) and (38),

$$\psi_j^p \left(T_{p,k}^a \right) = \delta_{j,k} \psi(a) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \forall k \in \mathbb{Z}. \tag{39}$$

Now, let ϕ be the *Euler totient function*,

$$\phi : \mathbb{N} \to \mathbb{C},$$

defined by

$$\phi(n) = |\{k \in \mathbb{N} : k \leq n, \ \gcd(n,k) = 1\}|, \tag{40}$$

for all $n \in \mathbb{N}$, where $|X|$ are the *cardinalities of sets* X, and gcd is the *greatest common divisor*.

By the definition (40),

$$\phi(n) = n \left(\prod_{q \in \mathcal{P}, \, q|n} \left(1 - \frac{1}{q} \right) \right), \tag{41}$$

for all $n \in \mathbb{N}$, where "$q \mid n$" means "*q divides n*." Thus,

$$\phi(q) = q - 1 = q \left(1 - \frac{1}{q} \right), \forall q \in \mathcal{P}, \tag{42}$$

by (40) and (41).

By (42), we have

$$\begin{aligned} \varphi_j^p \left(P_{p,k} \right) &= \frac{\delta_{j,k}}{p^j} \left(1 - \frac{1}{p} \right) \\ &= \frac{\delta_{j,k} \phi(p)}{p^{j+1}}, \end{aligned}$$

for $P_{p,k} \in \mathfrak{S}_p$, and hence,

$$\psi_j^p\left(T_{p,k}^a\right) = \delta_{j,k}\left(\frac{\phi(p)}{p^{j+1}}\right)\psi(a), \tag{43}$$

for all $T_{p,k}^a \in \mathfrak{S}_p^A(j)$, by (39).

Let's consider the following estimates.

Lemma 1. *Let ϕ be the Euler totient function (40). Then,*

$$\lim_{p\to\infty} \frac{\phi(p)}{p^{j+1}} = \begin{cases} 0, & \text{if } j > 0, \\ 1, & \text{if } j = 0, \\ \infty, \text{ Undefined}, & \text{if } j < 0, \end{cases} \tag{44}$$

for all $j \in \mathbb{Z}$, where "$p \to \infty$" means "p is getting bigger and bigger in $\mathcal{P}$."

Proof. Observe that

$$\lim_{p\to\infty} \frac{\phi(p)}{p} = \lim_{p\to\infty}\left(1 - \frac{1}{p}\right) = 1,$$

by (42). Thus, one can get that

$$\lim_{p\to\infty} \frac{\phi(p)}{p^{j+1}} = \lim_{p\to\infty}\left(\frac{\phi(p)}{p}\right)\left(\frac{1}{p^j}\right) = \lim_{p\to\infty}\frac{1}{p^j},$$

for $j \in \mathbb{Z}$. Thus,

$$\lim_{p\to\infty} \frac{\phi(p)}{p^{j+1}} = \lim_{p\to\infty}\frac{1}{p^j} = \begin{cases} 0, & \text{if } j > 0, \\ 1, & \text{if } j = 0, \\ \lim\limits_{p\to\infty} p^{|j|} = \infty, & \text{if } j < 0, \end{cases}$$

where $|j|$ are the absolute values of $j \in \mathbb{Z}$. Thus, the estimation (44) holds. □

8.1. Semicircular Elements

Let (B, φ) be an arbitrary *topological $*$-probability space* (C^*-probability space, or W^*-probability space, or Banach $*$-probability space, etc.) equipped with a topological $*$-algebra B (C^*-algebra, resp., W^*-algebra, resp., Banach $*$-algebra), and a linear functional φ on B.

Definition 8. *A self-adjoint operator $a \in B$ is said to be semicircular in (B, φ), if*

$$\varphi\left(a^n\right) = \omega_n c_{\frac{n}{2}};\, n \in \mathbb{N},\, \omega_n = \begin{cases} 1, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}, \end{cases} \tag{45}$$

and c_k are the k-th Catalan numbers,

$$c_k = \frac{1}{k+1}\begin{pmatrix} 2k \\ k \end{pmatrix} = \frac{(2k)!}{k!(k+1)!},$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

By [15–17], if $k_n(...)$ is the *free cumulant on B in terms of φ*, then a self-adjoint operator a is *semicircular* in (B, φ), if and only if

$$k_n\left(\underbrace{a,\, a,\, \ldots\ldots,\, a}_{n\text{-times}}\right) = \begin{cases} 1, & \text{if } n = 2, \\ 0, & \text{otherwise}, \end{cases} \tag{46}$$

for all $n \in \mathbb{N}$. The above characterization (46) of the semicircularity (45) holds by the *Möbius inversion of* [15]. For example, definition (45) and the characterization (46) give equivalent free distributions, *the semicircular law*.

If a_l are semicircular elements in topological $*$-probability spaces (B_l, φ_l), for $l = 1, 2$, then the free distributions of a_l are completely characterized by the free-moment sequences,

$$\left(\varphi_l(a_l^n)\right)_{n=1}^{\infty}, \text{ for } l = 1, 2,$$

by the self-adjointness of a_1 and a_2; and by (45), one obtains that

$$\begin{aligned} \left(\varphi_1(a_1^n)\right)_{n=1}^{\infty} &= \left(\omega_n c_{\frac{n}{2}}\right)_{n=1}^{\infty} \\ &= (0,\ c_1,\ 0,\ c_2,\ 0,\ c_3,\ \ldots) \\ &= \left(\varphi_2(a_2^n)\right)_{n=1}^{\infty}. \end{aligned}$$

Equivalently, the free distributions of the semicircular elements a_1 and a_2 are characterized by the free-cumulant sequences,

$$\left(k_n^1(a_1, \ldots, a_1)\right)_{n=1}^{\infty} = (0,\ 1,\ 0,\ 0,\ 0,\ \ldots) = \left(k_n^2(a_2, \ldots, a_2)\right)_{n=1}^{\infty},$$

by (46), where $k_n^l(\ldots)$ are the free cumulants on B_l in terms of φ_l, for all $l = 1, 2$.

It shows the universality of free distributions of semicircular elements. For example, the free distributions of any semicircular elements are universally characterized by either the free-moment sequence

$$\left(\omega_n c_{\frac{n}{2}}\right)_{n=1}^{\infty}, \tag{47}$$

or the free-cumulant sequence

$$(0, 1, 0, 0, \ldots).$$

Definition 9. *Let a be a semicircular element of a topological $*$-probability space (B, φ). The free distribution of a is called "the" semicircular law.*

8.2. Tensor Product Banach $*$-Algebra $\mathfrak{L}\mathfrak{S}_p^A$

Let $\mathfrak{S}_p^A(k) = \left(\mathfrak{S}_p^A,\ \psi_k^p\right)$ be the k-th p-adic A-space (34), for all $p \in \mathcal{P}$, $k \in \mathbb{Z}$. Throughout this section, we fix $p \in \mathcal{P}$, $k \in \mathbb{Z}$, and $\mathfrak{S}_p^A(k)$. In addition, denote $a \otimes P_{p,j}$ by $T_{p,j}^a$ in $\mathfrak{S}_p^A(k)$, for all $a \in (A, \psi)$ and $j \in \mathbb{Z}$.

Define now bounded linear transformations $\mathbf{c}_p^A$ and $\mathbf{a}_p^A$ "acting on the tensor product C^*-algebra $\mathfrak{S}_p^A$," by linear morphisms satisfying,

$$\begin{aligned} \mathbf{c}_p^A\left(T_{p,j}^a\right) &= T_{p,j+1}^a, \\ \mathbf{a}_p^A\left(T_{p,j}^a\right) &= T_{p,j-1}^a, \end{aligned} \tag{48}$$

on $\mathfrak{S}_p$, for all $j \in \mathbb{Z}$.

By the definitions (27) and (31), and by the structure theorem (33), the above linear morphisms $\mathbf{c}_p^A$ and $\mathbf{a}_p^A$ of (48) are well-defined on $\mathfrak{S}_p^A$.

By (48), one can understand $\mathbf{c}_p^A$ and $\mathbf{a}_p^A$ as bounded linear transformations contained in the *operator space* $B(\mathfrak{S}_p^A)$ consisting of all bounded linear operators acting on $\mathfrak{S}_p^A$, by regarding the C^*-algebra $\mathfrak{S}_p^A$ as a *Banach space* equipped with its C^*-norm (e.g., [32]). Under this sense, the operators $\mathbf{c}_p^A$ and $\mathbf{a}_p^A$ of (48) are well-defined *Banach-space operators on* $\mathfrak{S}_p^A$.

Definition 10. *The Banach-space operators $\mathbf{c}_p^A$ and $\mathbf{a}_p^A$ on $\mathfrak{S}_p^A$, in the sense of (48), are called the A-tensor p-creation, respectively, the A-tensor p-annihilation on $\mathfrak{S}_p^A$. Define a new Banach-space operator l_p^A by*

$$\mathbf{l}_p^A = \mathbf{c}_p^A + \mathbf{a}_p^A \text{ on } \mathfrak{S}_p^A. \tag{49}$$

We call this operator $\mathbf{l}_p^A$, the A-tensor p-radial operator on $\mathfrak{S}_p^A$.

Let $\mathbf{l}_p^A$ be the A-tensor p-radial operator $\mathbf{c}_p^A + \mathbf{a}_p^A$ of (49) in $B(\mathfrak{S}_p^A)$. Construct a *closed subspace* $\mathfrak{L}_p^A$ of $B(\mathfrak{S}_p^A)$ by

$$\mathfrak{L}_p^A = \overline{\mathbb{C}[\{\mathbf{l}_p^A\}]} \subset B(\mathfrak{S}_p^A), \tag{50}$$

equipped with the inherited *operator-norm* $\|.\|$ from the operator space $B(\mathfrak{S}_p^A)$, defined by

$$\|T\| = \sup\{\|Tx\|_{\mathfrak{S}_p^A} : x \in \mathfrak{S}_p^A \text{ s.t., } \|x\|_{\mathfrak{S}_p^A} = 1\},$$

where $\|.\|_{\mathfrak{S}_p^A}$ is the C^*-norm on the A-tensor p-adic algebra $\mathfrak{S}_p^A$ (e.g., [32]).

By the definition (50), the set $\mathfrak{L}_p^A$ is not only a closed subspace of $B(\mathfrak{S}_p^A)$, but also an algebra over $\mathbb{C}$. Thus, the subspace $\mathfrak{L}_p^A$ is a Banach algebra embedded in $B(\mathfrak{S}_p^A)$.

On the Banach algebra $\mathfrak{L}_p^A$ of (50), define a unary operation $(*)$ by

$$\left(\sum_{k=0}^{\infty} s_k \left(\mathbf{l}_p^A\right)^k\right)^* = \sum_{k=0}^{\infty} \overline{s_k} \left(\mathbf{l}_p^A\right)^k \text{ in } \mathfrak{L}_p^A, \tag{51}$$

where $s_k \in \mathbb{C}$, with their conjugates $\overline{s_k} \in \mathbb{C}$.

Then, the operation (51) is a well-defined *adjoint on* $\mathfrak{L}_p^A$. Thus, equipped with the adjoint (51), this Banach algebra $\mathfrak{L}_p^A$ of (50) forms a *Banach $*$-algebra* in $B(\mathfrak{S}_p^A)$. For example, all elements of $\mathfrak{L}_p^A$ are adjointable (in the sense of [32]) in $B(\mathfrak{S}_p^A)$.

Let $\mathfrak{L}_p^A$ be in the sense of (50). Construct now the tensor product Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p^A$ by

$$\mathfrak{L}\mathfrak{S}_p^A \overset{def}{=} \mathfrak{L}_p^A \otimes_{\mathbb{C}} \mathfrak{S}_p^A = \mathfrak{L}_p^A \otimes_{\mathbb{C}} \left(A \otimes_{\mathbb{C}} \mathfrak{S}_p\right), \tag{52}$$

where $\otimes_{\mathbb{C}}$ is the *tensor product* of Banach $*$-algebras. Since $\mathfrak{S}_p^A$ is a C^*-algebra, it is a Banach $*$-algebra too.

Take now a generating element $\left(\mathbf{l}_p^A\right)^n \otimes T_{p,j}^a$, for some $n \in \mathbb{N}_0$, and $j \in \mathbb{Z}$, where $T_{p,j}^a = a \otimes P_{p,j}$ are in the sense of (37) in $\mathfrak{S}_p^A$, with axiomatization:

$$\left(\mathbf{l}_p^A\right)^0 = 1_{\mathfrak{S}_p^A},$$

the *identity operator on* $\mathfrak{S}_p^A$ *in* $B\left(\mathfrak{S}_p^A\right)$, satisfying

$$1_{\mathfrak{S}_p^A}(T) = T,$$

for all $T \in \mathfrak{S}_p^A$. Define now a bounded linear morphism $E_p^A : \mathfrak{L}\mathfrak{S}_p^A \to \mathfrak{S}_p^A$ by a linear transformation satisfying that:

$$E_p^A\left(\left(\mathbf{l}_p^A\right)^k \otimes T_{p,j}^a\right) = \frac{1}{\left[\frac{k}{2}\right]+1}\left(\mathbf{l}_p^A\right)^k (T_{p,j}^a), \tag{53}$$

for all $k \in \mathbb{N}_0$, $j \in \mathbb{Z}$, where $\left[\frac{k}{2}\right]$ is the *minimal integer greater than or equal to* $\frac{k}{2}$, for all $k \in \mathbb{N}_0$, for example,

$$\left[\tfrac{3}{2}\right] = 2 = \left[\tfrac{4}{2}\right].$$

By the cyclicity (50) of the tensor factor $\mathfrak{L}_p^A$ of $\mathfrak{L}\mathfrak{S}_p^A$, and by the structure theorem (33) of the other tensor factor $\mathfrak{S}_p^A$ of $\mathfrak{L}\mathfrak{S}_p^A$, the above morphism E_p^A of (53) is a well-defined bounded linear transformation from $\mathfrak{L}\mathfrak{S}_p^A$ onto $\mathfrak{S}_p^A$.

Now, consider how our A-tensor p-radial operator $\mathbf{l}_p^A = \mathbf{c}_p^A + \mathbf{a}_p^A$ acts on $\mathfrak{S}_p^A$. First, observe that: if $\mathbf{c}_p^A$ and $\mathbf{a}_p^A$ are the A-tensor p-creation, respectively, the A-tensor p-annihilation on $\mathfrak{S}_p^A$, then

$$\mathbf{c}_p^A \mathbf{a}_p^A \left(T_{p,j}^a\right) = T_{p,j}^a = \mathbf{a}_p^A \mathbf{c}_p^A \left(T_{p,j}^a\right),$$

for all $a \in (A, \psi)$, and for all $j \in \mathbb{Z}$, $p \in \mathcal{P}$, and, hence,

$$\mathbf{c}_p^A \mathbf{a}_p^A = 1_{\mathfrak{S}_p^A} = \mathbf{a}_p^A \mathbf{c}_p^A \text{ on } \mathfrak{S}_p^A. \tag{54}$$

Lemma 2. *Let $\mathbf{c}_p^A$, $\mathbf{a}_p^A$ be the A-tensor p-creation, respectively, the A-tensor p-annihilation on $\mathfrak{S}_p^A$. Then,*

$$\left(\mathbf{c}_p^A\right)^n \left(\mathbf{a}_p^A\right)^n = 1_{\mathfrak{S}_p^A} = \left(\mathbf{a}_p^A\right)^n \left(\mathbf{c}_p^A\right)^n,$$

$$\left(\mathbf{c}_p^A\right)^{n_1} \left(\mathbf{a}_p^A\right)^{n_2} = \left(\mathbf{a}_p^A\right)^{n_2} \left(\mathbf{c}_p^A\right)^{n_1}, \tag{55}$$

on $\mathfrak{S}_p^A$, for all $n, n_1, n_2 \in \mathbb{N}$.

Proof. The formulas in (55) hold by induction on (54). □

By (55), one can get that

$$\left(\mathbf{l}_p^A\right)^n = \left(\mathbf{c}_p^A + \mathbf{a}_p^A\right)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \left(\mathbf{c}_p^A\right)^k \left(\mathbf{a}_p^A\right)^{n-k}, \tag{56}$$

with identity:

$$\left(\mathbf{c}_p^A\right)^0 = 1_{\mathfrak{S}_p^A} = \left(\mathbf{a}_p^A\right)^0,$$

for all $n \in \mathbb{N}$, where

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!},$$

for all $k \leq n \in \mathbb{N}_0$. By (56), one obtains the following proposition.

Proposition 8. *Let $\mathbf{l}_p^A \in \mathfrak{L}_p^A$ be the A-tensor p-radial operator on $\mathfrak{S}_p^A$. Then,*

$$\left(\mathbf{l}_p^A\right)^{2m-1} \text{ does not contain } 1_{\mathfrak{S}_p^A}\text{-term, and} \tag{57}$$

$$\left(\mathbf{l}_p^A\right)^{2m} \text{ contains its } 1_{\mathfrak{S}_p^A}\text{-term, } \begin{pmatrix} 2m \\ m \end{pmatrix} \cdot 1_{\mathfrak{S}_p^A}, \tag{58}$$

for all $m \in \mathbb{N}$.

Proof. The proofs of (57) and (58) are done by straightforward computations of (56) with the help of (55). □

8.3. Free-Probabilistic Information of $Q_{p,j}^{a}$ in $\mathfrak{LS}_{p}^{A}$

Fix $p \in \mathcal{P}$, and a unital C^{*}-probability space (A, ψ), and let $\mathfrak{LS}_{p}^{A}$ be the Banach $*$-algebra (52). Let $E_{p}^{A} : \mathfrak{LS}_{p}^{A} \to \mathfrak{S}_{p}^{A}$ be the linear transformation (53). Throughout this section, let

$$Q_{p,j}^{a} \overset{denote}{=} \mathbf{1}_{p}^{A} \otimes T_{p,j}^{a} \in \mathfrak{LS}_{p}^{A}, \tag{59}$$

for all $j \in \mathbb{Z}$, where $T_{p,j}^{a} = a \otimes P_{p,j} \in \mathfrak{S}_{p}^{A}$ are in the sense of (37) generating $\mathfrak{S}_{p}^{A}$, for $a \in (A, \psi)$, and $j \in \mathbb{Z}$. Observe that

$$\begin{aligned} \left(Q_{p,j}^{a}\right)^{n} &= \left(\mathbf{1}_{p}^{A} \otimes T_{p,j}^{a}\right)^{n} \\ &= \left(\mathbf{1}_{p}^{A}\right)^{n} \otimes \left(T_{p,j}^{a}\right)^{n} = \left(\mathbf{1}_{p}^{A}\right)^{n} \otimes T_{p,j}^{a^{n}}, \end{aligned} \tag{60}$$

by (37), for all $n \in \mathbb{N}$, for all $j \in \mathbb{Z}$.

If $Q_{p,j}^{a} \in \mathfrak{LS}_{p}^{A}$ is in the sense of (59) for $j \in \mathbb{Z}$, then

$$E_{p}^{A}\left(\left(Q_{p,j}^{a}\right)^{n}\right) = \frac{1}{\left[\frac{n}{2}\right]+1}\left(\mathbf{1}_{p}^{A}\right)^{n}\left(T_{p,j}^{a^{n}}\right), \tag{61}$$

by (53) and (60), for all $n \in \mathbb{N}$.

For any fixed $j \in \mathbb{Z}$, define a linear functional τ_{j}^{p} on $\mathfrak{LS}_{p}^{A}$ by

$$\tau_{j}^{p} = \psi_{j}^{p} \circ E_{p}^{A} \text{ on } \mathfrak{LS}_{p}^{A}, \tag{62}$$

where $\psi_{j}^{p} = \psi \otimes \varphi_{j}^{p}$ is a linear functional (35a), or (35b) on $\mathfrak{S}_{p}^{A}$.

By the linearity of both ψ_{j}^{p} and E_{p}^{A}, the morphism τ_{j}^{p} of (62) is a well-defined linear functional on $\mathfrak{LS}_{p}^{A}$ for $j \in \mathbb{Z}$. Thus, the pair $\left(\mathfrak{LS}_{p}^{A}, \tau_{j}^{p}\right)$ forms a *Banach $*$-probability space* (e.g., [22]).

Definition 11. *The Banach $*$-probability spaces*

$$\mathfrak{LS}_{p,j}^{A} \overset{denote}{=} \left(\mathfrak{LS}_{p}^{A}, \tau_{j}^{p}\right) \tag{63}$$

are called the A-tensor j-th p-adic (free-)filters, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, where τ_{j}^{p} are in the sense of (62).

By (61) and (62), if $Q_{p,j}^{a}$ is in the sense of (59) in $\mathfrak{LS}_{p,j}^{A}$, then

$$\tau_{j}^{p}\left(\left(Q_{p,j}^{a}\right)^{n}\right) = \frac{1}{\left[\frac{n}{2}\right]+1}\psi_{j}^{p}\left((\mathbf{1}_{p}^{A})^{n}\left(T_{p,j}^{a^{n}}\right)\right), \tag{64}$$

for all $n \in \mathbb{N}$.

Theorem 2. *Let $Q_{p,k}^{a} = \mathbf{1}_{p}^{A} \otimes T_{p,k}^{a} = \mathbf{1}_{p}^{A} \otimes \left(a \otimes P_{p,k}\right)$ be a free random variable (59) of the A-tensor j-th p-adic filter $\mathfrak{LS}_{p,j}^{A}$ of (63), for $p \in \mathcal{P}$, $j \in \mathbb{Z}$, for all $k \in \mathbb{Z}$. Then,*

$$\tau_{j}^{p}\left(\left(Q_{p,k}^{a}\right)^{n}\right) = \delta_{j,k}\omega_{n}\psi(a^{n})c_{\frac{n}{2}}\left(\frac{\phi(p)}{p^{j+1}}\right), \tag{65}$$

where ω_{n} are in the sense of (45), for all $n \in \mathbb{N}$.

Proof. Let $Q_{p,j}^{a}$ be in the sense of (59) in $\mathfrak{LS}_{p,j}^{A}$, for the fixed $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Then,

$$\tau_{j}^{p}\left(\left(Q_{p,j}^{a}\right)^{2n-1}\right) = \psi_{j}^{p}\left(E_{p}^{A}\left(\left(Q_{p,j}^{a}\right)^{2n-1}\right)\right)$$

by (62)

$$= \left(\frac{1}{\left[\frac{2n-1}{2}\right]+1}\right)\psi_{j}^{p}\left((\mathbf{1}_{p}^{A})^{2n-1}\left(T_{p,j}^{a^{2n-1}}\right)\right)$$

by (64)

$$= \left(\frac{1}{\left[\frac{2n-1}{2}\right]+1}\right) \psi_j^p \left(\left(\sum_{k=0}^{n} \begin{pmatrix} 2n-1 \\ k \end{pmatrix} (\mathbf{c}_p^A)^k (\mathbf{a}_p^A)^{2n-1-k} \right) \left(T_{p,j}^{a^{2n-1}} \right) \right)$$

by (56)

$$= 0,$$

by (57), for all $n \in \mathbb{N}$.

Observe now that, for any $n \in \mathbb{N}$,

$$\tau_j^p \left(\left(Q_{p,j}^a \right)^{2n} \right) = \left(\frac{1}{\left[\frac{2n}{2}\right]+1} \right) \psi_j^p \left((\mathbf{1}_p^A)^{2n} \left(T_{p,j}^{a^{2n}} \right) \right)$$

by (64)

$$= \left(\frac{1}{n+1}\right) \psi_j^p \left(\left(\sum_{k=0}^{2n} \begin{pmatrix} 2n \\ k \end{pmatrix} (\mathbf{c}_p^A)^k (\mathbf{a}_p^A)^{2n-k} \right) \left(T_{p,j}^{a^{2n}} \right) \right)$$

by (56)

$$= \left(\frac{1}{n+1}\right) \psi_j^p \left(\begin{pmatrix} 2n \\ n \end{pmatrix} T_{p,j}^{a^{2n}} + [\text{Rest terms}] \right)$$

by (58)

$$= \frac{1}{n+1} \begin{pmatrix} 2n \\ n \end{pmatrix} \psi_j^p \left(T_{p,j}^{a^{2n}} \right) = \frac{1}{n+1} \begin{pmatrix} 2n \\ n \end{pmatrix} \psi(a^{2n}) \left(\frac{\phi(p)}{p^{j+1}} \right)$$

by (39) and (43)

$$= c_n \psi(a^{2n}) \left(\frac{\phi(p)}{p^{j+1}} \right),$$

where c_n are the n-th Catalan numbers.

If $k \neq j$ in $\mathbb{Z}$, and if $Q_{p,k}^a$ are in the sense of (59) in $\mathfrak{LS}_{p,j}^A$, then

$$\tau_j^p \left(\left(Q_{p,k}^a \right)^n \right) = 0,$$

for all $n \in \mathbb{N}$, by the definition (22a) of the linear functional φ_j^p on $\mathfrak{S}_p$, inducing the linear functional $\psi_j^p = \psi \otimes \varphi_j^p$ on the tensor factor $\mathfrak{S}_p^A$ of $\mathfrak{LS}_{p,j}^A$.

Therefore, the free-distributional data (65) holds true. □

Note that, if a is self-adjoint in (A, ψ), then the generating operators $Q_{p,k}^a$ of the A-tensor j-th p-adic filter $\mathfrak{LS}_{p,j}^A$ are self-adjoint in $\mathfrak{LS}_p^A$, since

$$\begin{aligned} \left(Q_{p,k}^a \right)^* &= \left(\mathbf{1}_p^A \otimes T_{p,k}^a \right)^* = (\mathbf{1}_p^A)^* \otimes \left(T_{p,k}^a \right)^* \\ &= \mathbf{1}_p^A \otimes T_{p,k}^{a^*} = Q_{p,k}^a, \end{aligned}$$

for all $k \in \mathbb{Z}$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$, by (51).

Thus, if a is a self-adjoint free random variable of (A, ψ), then the above formula (65) fully characterizes the free distributions (up to τ_j^p) of the generating operators $Q_{p,k}^a$ of $\mathfrak{LS}_p^A$, for all $k, j \in \mathbb{Z}$, for $p \in \mathcal{P}$.

The free-distributional data (65) can be refined as follows: if $p \in \mathcal{P}$, $j \in \mathbb{Z}$, and if $\mathfrak{LS}_{p,j}^A$ is the corresponding A-tensor j-th p-adic filter (63), then

$$\tau_j^p \left(\left(Q_{p,j}^a \right)^n \right) = \omega_n c_{\frac{n}{2}} \psi(a^n) \left(\frac{\phi(p)}{p^{j+1}} \right), \tag{66}$$

for all $n \in \mathbb{N}$, and

$$\tau_j^p\left(\left(Q_{p,k}^a\right)^n\right)=0, \tag{67}$$

for all $n\in\mathbb{N}$, whenever $k\neq j$ in $\mathbb{Z}$, for all $n\in\mathbb{N}$.

Before we focus on non-zero free-distributional data (66) of $Q_{p,j}^a$, let's conclude the following result for $\{Q_{p,k}^a\}_{k\neq j\in\mathbb{Z}}$.

Corollary 1. *Let $p\in\mathcal{P}$, $j\in\mathbb{Z}$, and let $\mathfrak{L}\mathfrak{S}_{p,j}^A$ be the A-tensor j-th p-adic filter (63). Then, the generating operators*

$$Q_{p,k}^a=\mathbf{1}_p^A\otimes T_{p,j}^a=\mathbf{1}_p^A\otimes\left(a\otimes P_{p,j}\right)\in\mathfrak{L}\mathfrak{S}_{p,j}^A$$

have the zero free distribution, whenever $k\neq j$ in $\mathbb{Z}$.

Proof. It is proven by (65) and (67). □

By the above corollary, we now restrict our interests to the "j-th" generating operators $Q_{p,j}^a$ of (59) in the A-tensor "j-th" p-adic filter $\mathfrak{L}\mathfrak{S}_{p,j}^A$, for all $p\in\mathcal{P}$, $j\in\mathbb{Z}$, having non-zero free distributions determined by (66).

9. On the Free Product Banach $*$-Probability Space $\mathfrak{L}\mathfrak{S}_A$

Throughout this section, let $(A,\ \psi)$ be a fixed unital C^*-probability space, and let

$$\mathfrak{L}\mathfrak{S}_{p,j}^A=\left(\mathfrak{L}\mathfrak{S}_p^A,\ \tau_j^p\right) \tag{68}$$

be A-tensor j-th p-adic filters, where

$$\mathfrak{L}\mathfrak{S}_p^A=\mathfrak{L}_p^A\otimes_{\mathbb{C}}\mathfrak{S}_p^A=\mathfrak{L}_p^A\otimes_{\mathbb{C}}\left(A\otimes_{\mathbb{C}}\mathfrak{S}_p\right),$$

are in the sense of (52), and τ_j^p are the linear functionals (62) on $\mathfrak{L}\mathfrak{S}_p^A$, for all $p\in\mathcal{P}$, $j\in\mathbb{Z}$.

Let $Q_{p,k}^a=\mathbf{1}_p^A\otimes T_{p,k}^a=\mathbf{1}_p^A\otimes\left(a\otimes P_{p,k}\right)$ be the generating elements (59) of $\mathfrak{L}\mathfrak{S}_{p,j}^A$ of (68), for $a\in(A,\ \psi)$, $p\in\mathcal{P}$, and $k,\ j\in\mathbb{Z}$. Then, these operators $Q_{p,k}^a$ of $\mathfrak{L}\mathfrak{S}_{p,j}^A$ have their free-distributional data,

$$\tau_j^p\left(\left(Q_{p,k}^a\right)^n\right)=\delta_{j,k}\omega_n\psi(a^n)c_{\frac{n}{2}}\left(\frac{\phi(p)}{p^{j+1}}\right), \tag{69}$$

for all $n\in\mathbb{N}$, by (65).

By (66) and (67), we here concentrate on the "j-th" generating operators of $\mathfrak{L}\mathfrak{S}_{p,j}^A$ having non-zero free distributions (69) for all $j\in\mathbb{Z}$, for all $p\in\mathcal{P}$.

9.1. Free Product Banach $*$-Probability Space $(\mathfrak{L}\mathfrak{S}_A,\ \tau)$

By (68), we have the family

$$\left\{\mathfrak{L}\mathfrak{S}_{p,j}^A:p\in\mathcal{P},\ j\in\mathbb{Z}\right\}$$

of Banach $*$-probability spaces, consisting of the A-tensor j-th p-adic filters $\mathfrak{L}\mathfrak{S}_{p,j}^A$.

Define the *free product Banach $*$-probability space*,

$$\begin{aligned}(\mathfrak{L}\mathfrak{S}_A,\ \tau)&\overset{def}{=}\underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star}\mathfrak{L}\mathfrak{S}_{p,j}^A,\\&=\left(\underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star}\mathfrak{L}\mathfrak{S}_p^A,\ \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star}\tau_j^p\right)\end{aligned} \tag{70}$$

in the sense of [15,22].

By (70), the A-tensor j-th p-adic filters $\mathfrak{LS}_{p,j}$ of (68) are the *free blocks* of the Banach $*$-probability space $(\mathfrak{LS}_A, \tau)$ of (70).

All operators of the Banach $*$-algebra $\mathfrak{LS}_A$ in (70) are the Banach-topology limits of linear combinations of noncommutative free reduced words (under operator-multiplication) in

$$\underset{p \in \mathcal{P},\, j \in \mathbb{Z}}{\sqcup} \mathfrak{LS}_{p,j}^A.$$

More precisely, since each free block $\mathfrak{LS}_{p,j}^A$ is generated by $\{Q_{p,k}^a\}_{a \in A, k \in \mathbb{Z}}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, all elements of $\mathfrak{LS}_A$ are the Banach-topology limits of linear combinations of free words in

$$\underset{p \in \mathcal{P}, j \in \mathbb{Z}}{\sqcup} \{Q_{p,k}^a \in \mathfrak{LS}_{p,j} : a \in A, k \in \mathbb{Z}\}.$$

In particular, all noncommutative free words have their unique free "reduced" words (as operators of $\mathfrak{LS}_A$ under operator-multiplication) formed by

$$\prod_{l=1}^{N} \left(Q_{p_l,k_l}^{a_l}\right)^{n_l}, \text{ where } Q_{p_l,k_l}^{a_l} \in \mathfrak{LS}_{p_l,j_l}^A$$

in $\mathfrak{LS}_A$, for all $a_1, ..., a_N \in (A, \psi)$, and $n_1, ..., n_N \in \mathbb{N}$, where either the N-tuple

$$(p_1, \ ..., \ p_N), \text{ or } (j_1, \ ..., \ j_N)$$

is alternating in $\mathcal{P}$, respectively, in $\mathbb{Z}$, in the sense that:

$$p_1 \neq p_2, p_2 \neq p_3, ..., p_{N-1} \neq p_N \text{ in } \mathcal{P},$$

respectively,

$$j_1 \neq j_2, j_2 \neq j_3, ..., j_{N-1} \neq j_N \text{ in } \mathbb{Z}$$

(e.g., see [22]).

For example, a 5-tuple

$$(2, \ 2, \ 3, \ 7, \ 2)$$

is not alternating in $\mathcal{P}$, while a 5-tuple

$$(2, \ 3, \ 2, \ 7, \ 2)$$

is alternating in $\mathcal{P}$, etc.

By (70), if $Q_{p,j}^a$ are the j-th a-tensor generating operators of a free block $\mathfrak{LS}_{p,j}^A$ of the Banach $*$-probability space $(\mathfrak{LS}_A, \tau)$, for all $j \in \mathbb{Z}$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$, then $\left(Q_{p,j}^a\right)^n$ are contained in the same free block $\mathfrak{LS}_{p,j}^A$ of $(\mathfrak{LS}_A, \ \tau)$, and, hence, they are free reduced words with their lengths-1, for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \tau\left(\left(Q_{p,j}^a\right)^n\right) &= \tau_j^p\left(\left(Q_{p,j}^a\right)^n\right) \\ &= \omega_n c_{\frac{n}{2}} \psi(a^n)\left(\frac{\phi(p)}{p^{j+1}}\right), \end{aligned} \tag{71}$$

for all $n \in \mathbb{N}$, by (69).

Definition 12. *The Banach $*$-probability space $\mathfrak{LS}_A \overset{denote}{=} (\mathfrak{LS}_A, \ \tau)$ of (70) is called the A-tensor (free-)Adelic filterization of $\{\mathfrak{LS}_{p,j}^A\}_{p \in \mathcal{P}, j \in \mathbb{Z}}$.*

As we discussed at the beginning of Section 9, we now focus on studying free random variables of the A-tensor Adelic filterization $\mathfrak{LS}_A$ of (70) having "non-zero" free distributions.

Define a subset $\mathcal{U}$ of $\mathfrak{LS}_A$ by

$$\mathcal{U} = \left\{ Q_{p,j}^{1_A} \in \mathfrak{LS}_{p,j}^{A} \,|\forall p \in \mathcal{P},\ j \in \mathbb{Z} \right\} \tag{72}$$

in $\mathfrak{LS}_A$, where 1_A is the unity of A, and $Q_{p,j}^{1_A}$ are the "j-th" 1_A-tensor generating operators of $\mathfrak{LS}_A$, in the free blocks $\mathfrak{LS}_{p,j}^{A}$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Then, the elements $Q_{p,j}^{1_A}$ of $\mathcal{U}$ have their non-zero free distributions,

$$\left(\omega_n c_{\frac{n}{2}} \psi(1_A^n) \left(\frac{\phi(p)}{p^{j+1}} \right) \right)_{n=1}^{\infty} = \left(\omega_n c_{\frac{n}{2}} \left(\frac{\phi(p)}{p^{j+1}} \right) \right)_{n=1}^{\infty},$$

by (71), since

$$\psi(1_A^n) = \psi(1_A) = 1,$$

for all $n \in \mathbb{N}$. Now, define a Cartesian product set

$$\mathcal{U}_A \overset{def}{=} A \times \mathcal{U}, \tag{73a}$$

set-theoretically, where $\mathcal{U}$ is in the sense of (72).

Define a function $\Omega : \mathcal{U}_A \to \mathfrak{LS}_A$ by

$$\Omega\left((a,\ Q_{p,j}^{1_A}) \right) \overset{def}{=} Q_{p,j}^{a} \text{ in } \mathfrak{LS}_A, \tag{73b}$$

for all $(a,\ Q_{p,j}^{1_A}) \in \mathcal{U}_A$, where $\mathcal{U}_A$ is in the sense of (73a).

It is not difficult to check that this function Ω of (73b) is a well-defined injective map. Moreover, it induces all j-th a-tensor generating elements $Q_{p,j}^{a}$ of $\mathfrak{LS}_{p,j}^{a}$ in $\mathfrak{LS}_A$, for all $p \in \mathcal{P}$, and $j \in \mathbb{Z}$.

Define a Banach $*$-subalgebra $\mathbb{LS}_A$ of the A-tensor Adelic filterization $\mathfrak{LS}_A$ of (70) by

$$\mathbb{LS}_A \overset{def}{=} \overline{\mathbb{C}\left[\Omega\left(\mathcal{U}_A\right)\right]} \text{ in } \mathfrak{LS}_A, \tag{74a}$$

where $\Omega(\mathcal{U}_A)$ is the subset of $\mathfrak{LS}_A$, induced by (73a) and (73b), and $\overline{Y}$ mean the Banach-topology closures of subsets Y of $\mathfrak{LS}_A$.

Then, this Banach $*$-subalgebra $\mathbb{LS}_A$ of (74a) has a sub-structure,

$$\mathbb{LS}_A \overset{denote}{=} (\mathbb{LS}_A,\ \tau = \tau \mid_{\mathbb{LS}_A}) \tag{74b}$$

in the A-tensor Adelic filterization $\mathfrak{LS}_A$.

Theorem 3. *Let $\mathbb{LS}_A$ be the Banach $*$-algebra (74a) in the A-tensor Adelic filterization $\mathfrak{LS}_A$. Then,*

$$\begin{aligned} \mathbb{LS}_A &\overset{*\text{-iso}}{=} \underset{p \in \mathcal{P},\ j \in \mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{p,j}^{a} : a \in (A,\ \psi\}\right]} \\ &\overset{*\text{-iso}}{=} \overline{\mathbb{C}\left[\underset{p \in \mathcal{P},\ j \in \mathbb{Z}}{\star} \{Q_{p,j}^{a} : a \in (A,\ \psi\}\right]}, \end{aligned} \tag{75}$$

where $Q_{p,j}^{a} \in \Omega(\mathcal{U}_A)$ of (73b). Here, $(\star)$ in the first $*$-isomorphic relation in (75) is the free-probability-theoretic free product determined by the linear functional τ of (70), or of (74b) (e.g., [15,22]), and $(\star)$ in the second $*$-isomorphic relation in (75) is the pure-algebraic free product generating noncommutative free words in $\Omega(\mathcal{U}_A)$.

Proof. Let $\mathbb{LS}_A$ be the Banach $*$-subalgebra (74a) in $\mathfrak{LS}_A$. Then,

$$\mathbb{LS}_A = \overline{\mathbb{C}\left[\{Q_{p,j}^{a} \in \mathfrak{LS}_{p,j}^{A} : a \in (A,\ \psi)\}_{p \in \mathcal{P},\ j \in \mathbb{Z}}\right]}$$

by (73a), (73b) and (74a)

$$\overset{*\text{-iso}}{=} \underset{p\in\mathcal{P},\, j\in\mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{p,j}^{a} : a \in (A,\ \psi)\}\right]}$$

in $\mathfrak{LS}_A$, since all elements $Q_{p,j}^a \in \Omega\left(\mathcal{U}_A\right)$ are chosen from mutually distinct free blocks $\mathfrak{LS}_{p,j}^A$ of the A-tensor Adelic filterization $\mathfrak{LS}_A$, and, hence, the operators $\{Q_{p,j}^a,\ Q_{p,j}^{a^*}\}_{p\in\mathcal{P},\ j\in\mathbb{Z}}$ are free from each other in $\mathfrak{LS}_A$, for any $a \in (A, \psi)$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, moreover,

$$\overset{*\text{-iso}}{=} \overline{\mathbb{C}\left[\underset{p\in\mathcal{P},\, j\in\mathbb{Z}}{\star}\{Q_{p,j}^{a} : a \in (A,\ \psi)\}\right]},$$

because all elements of $\mathbb{LS}_A$ are the (Banach-topology limits of) linear combinations of free words in $\Omega(\mathcal{U}_A)$, by the very above $*$-isomorphic relation. Indeed, for any noncommutative (pure-algebraic) free words in

$$\underset{p\in\mathcal{P},\, j\in\mathbb{Z}}{\cup}\{Q_{p,j}^{a} : a \in (A, \psi)\}$$

have their unique free "reduced" words under operator-multiplication on $\mathfrak{LS}_A$, as operators of $\mathbb{LS}_A$.

Therefore, the structure theorem (75) holds. □

The above theorem characterizes the free-probabilistic structure of the Banach $*$-algebra $\mathbb{LS}_A$ of (74a) in the A-tensor Adelic filterization $\mathfrak{LS}_A$. This structure theorem (75) demonstrates that the Banach $*$-probability space $(\mathbb{LS}_A,\ \tau)$ of (74b) is well-determined, having its natural inherited free probability from that on $\mathfrak{LS}_A$.

Definition 13. *Let $(\mathbb{LS}_A, \tau)$ be the Banach $*$-probability space (74b). Then, we call*

$$\mathbb{LS}_A \overset{denote}{=} (\mathbb{LS}_A, \tau),$$

the A-tensor (Adelic) sub-filterization of the A-tensor Adelic filterization $\mathfrak{LS}_A$.

By (69), (71), (72) and (75), one can verify that the free probability on the A-tensor sub-filterization $\mathbb{LS}_A$ provide "possible" non-zero free distributions on the A-tensor Adelic filterization $\mathfrak{LS}_A$, up to free probability on $(A,\ \psi)$. i.e., if $a \in (A,\ \psi)$ have their non-zero free distributions, then $Q_{p,j}^a \in \mathbb{LS}_A$ have non-zero free distributions, and, hence, they have their non-zero free distributions on $\mathfrak{LS}_A$.

Theorem 4. *Let $Q_{p,j}^a \in \Omega(\mathcal{U}_A)$ be free random variables of the A-tensor sub-filterization $\mathbb{LS}_A$, for $a \in (A,\ \psi)$, and $p \in \mathcal{P}$, and $j \in \mathbb{Z}$. Then,*

$$\tau\left(\left(Q_{p,j}^a\right)^n\right) = \omega_n c_{\frac{n}{2}} \psi(a^n) \left(\frac{\phi(p)}{p^{j+1}}\right),$$
$$\tau\left(\left(\left(Q_{p,j}^a\right)^*\right)^n\right) = \omega_n c_{\frac{n}{2}} \overline{\psi(a^n)} \left(\frac{\phi(p)}{p^{j+1}}\right), \tag{76}$$

for all $n \in \mathbb{N}$.

Proof. The first formula of (76) is shown by (71). Thus, it suffices to prove the second formula of (76) holds. Note that

$$\begin{aligned}\left(Q_{p,j}^a\right)^* &= \left(\mathbf{1}_p^A \otimes T_{p,j}^a\right)^* = \left(\mathbf{1}_p^A \otimes \left(a \otimes P_{p,j}\right)\right)^* \\ &= \left(\mathbf{1}_p^A\right)^* \otimes \left(a \otimes P_{p,j}\right)^* = \mathbf{1}_p^A \otimes \left(a^* \otimes P_{p,j}\right),\end{aligned}$$

and, hence,

$$\left(Q_{p,j}^a\right)^* = Q_{p,j}^{a^*} \text{ in } \mathbb{LS}_A, \tag{77}$$

for all $Q^a_{p,j} \in \Omega\left(\mathcal{U}_A\right)$. Thus, one has

$$\left(\left(Q^a_{p,j}\right)^*\right)^n = \left(Q^{a^*}_{p,j}\right)^n = Q^{(a^*)^n}_{p,j} = Q^{(a^n)^*}_{p,j} \text{ in } \mathbb{LS}_A,$$

by (77).

Thus, one has

$$\begin{aligned}\tau\left(\left(\left(Q^a_{p,j}\right)^*\right)^n\right) &= \omega_n c_{\frac{n}{2}} \psi\left((a^n)^*\right)\left(\frac{\phi(p)}{p^{j+1}}\right) \\ &= \omega_n c_{\frac{n}{2}} \overline{\psi(a^n)}\left(\frac{\phi(p)}{p^{j+1}}\right),\end{aligned}$$

by (71), for all $n \in \mathbb{N}$. Therefore, the second formula of (76) holds too. $\square$

9.2. Prime-Shifts on $\mathbb{LS}_A$

Let $\mathbb{LS}_A$ be the A-tensor sub-filterization (70) of the A-tensor Adelic filterization $\mathfrak{LS}_A$. In this section, we define a certain $*$-homomorphism on $\mathbb{LS}_A$, and study asymptotic free-distributional data on $\mathbb{LS}_A$ (and hence those on $\mathfrak{LS}_A$) over primes.

Let $\mathcal{P}$ be the set of all primes in $\mathbb{N}$, regarded as a *totally ordered set* (in short, a TOset) for the usual ordering $(\leq)$, i.e.,

$$\mathcal{P} = \{q_1 < q_2 < q_3 < q_4 < \cdots\}, \tag{78}$$

with

$$q_1 = 2, q_2 = 3, q_3 = 5, q_4 = 7, q_5 = 11, ..., \text{etc.}$$

Define an injective function $h : \mathcal{P} \to \mathcal{P}$ by

$$h\left(q_k\right) = q_{k+1}; k \in \mathbb{N}, \tag{79}$$

where q_k are primes of (78), for all $k \in \mathbb{N}$.

Definition 14. *Let h be an injective function (79) on the TOset $\mathcal{P}$ of (78). We call h the shift on $\mathcal{P}$.*

Let h be the shift (79) on the TOset $\mathcal{P}$, and let

$$h^{(n)} \overset{def}{=} \underbrace{h \circ h \circ h \circ \cdots \circ h}_{n\text{-times}}, \text{ on } \mathcal{P}, \tag{80}$$

for all $n \in \mathbb{N}$, where $(\circ)$ is the usual functional *composition*.

By the definitions (79) and (80),

$$h^{(n)}\left(q_k\right) = q_{k+n}, \tag{81}$$

for all $n \in \mathbb{N}$, in $\mathcal{P}$. For instance, $h^{(3)}(2) = 7$, and $h^{(4)}(5) = 17$, etc.

These injective functions $h^{(n)}$ of (80) are called the *n-shifts* on $\mathcal{P}$, for all $n \in \mathbb{N}$.

For the shift h on $\mathcal{P}$, one can define a *$*$-homomorphism* π_h on the A-tensor sub-filterization $\mathbb{LS}_A$ by a bounded "multiplicative" linear transformation, satisfying that

$$\pi_h\left(Q^a_{q_k,j}\right) = Q^a_{h(q_k),j} = Q^a_{q_{k+1},j}, \tag{82}$$

for all $Q_{q_k,j} \in \Omega(\mathcal{U}_A)$, for all $q_k \in \mathcal{P}$, for all $j \in \mathbb{Z}$, where h is the shift (79) on $\mathcal{P}$.

By (82), we have

$$\pi_h\left(\prod_{l=1}^{N}\left(Q^{a_l}_{q_{k_l},j_l}\right)^{n_l}\right) = \prod_{l=1}^{N}\left(Q^{a_l}_{h(q_{k_l}),\, j_l}\right)^{n_l} = \prod_{l=1}^{N}\left(Q^{a_l}_{q_{k_l+1},j_l}\right)^{n_l}, \tag{83}$$

in $\mathbb{LS}_A$, for all $Q^{a}_{q_{k_l},j_l} \in \Omega(\mathcal{U}_A)$, for $q_{k_l} \in \mathcal{P}$, $j_l \in \mathbb{Z}$, for $l = 1, ..., N$, for $N \in \mathbb{N}$, where $n_1, ..., n_N \in \mathbb{N}$.

Remark 1. *Note that the multiplicative linear transformation π_h of (82) is indeed a $*$-homomorphism satisfying*

$$\pi_h\left(T^*\right) = \left(\pi_h(T)\right)^*,$$

for all $T \in \mathbb{LS}_A$, because

$$\begin{aligned}\pi_h\left(\left(Q^{a}_{p,j}\right)^*\right) &= \pi_h\left(Q^{a^*}_{p,j}\right) = Q^{a^*}_{h(p),j} \\ &= \left(Q^{a}_{h(p),j}\right)^* = \left(\pi_h\left(Q^{a}_{p,j}\right)\right)^*,\end{aligned}$$

for all $Q^{a}_{p,j} \in \Omega\left(\mathcal{U}_A\right)$.

In addition, by (82), we obtain the $*$-homomorphisms,

$$\pi_h^n = \underbrace{\pi_h \pi_h \pi_h \cdots \pi_h}_{n\text{-times}}, \text{ on } \mathbb{LS}_A, \tag{84}$$

the products (or compositions) of the n-copies of the $*$-homomorphism π_h of (82), acting on $\mathbb{LS}_A$. It is not difficult to check that

$$\begin{aligned}\pi_h^n\left(Q^{a}_{p,j}\right) &= \pi_h^{n-1}\left(Q^{a}_{h(p),j}\right) = \pi_h^{n-2}\left(Q^{a}_{h^{(2)}(p),j}\right) \\ &= \cdots = \pi_h\left(Q^{a}_{h^{(n-1)}(p),\,j}\right) = Q^{a}_{h^{(n)}(p),\,j},\end{aligned} \tag{85}$$

for all $Q^{a}_{p,j} \in \Omega(\mathcal{U}_A)$ in $\mathbb{LS}_A$, where $h^{(k)}$ are the k-shifts (80) on $\mathcal{P}$, for all $k \in \mathbb{N}$.

Definition 15. *Let π_h be the $*$-homomorphism (82) on the A-tensor sub-filterization $\mathbb{LS}_A$, and let π_h^n be the products (84) acting on $\mathbb{LS}_A$, for all $n \in \mathbb{N}$, with $\pi_h^1 = \pi_h$. Then, we call π_h^n, the n-prime-shift ($*$-homomorphism) on $\mathbb{LS}_A$, for all $n \in \mathbb{N}$. In particular, the 1-prime-shift π_h is simply said to be the prime-shift ($*$-homomorphism) on $\mathbb{LS}_A$.*

Thus, for any $Q^{a}_{q_k,j} \in \Omega(\mathcal{U}_A)$ in $\mathbb{LS}_A$, for $q_k \in \mathcal{P}$ (in the sense of (78) with $k \in \mathbb{N}$), the n-prime-shift π_h^n satisfies

$$\pi_h^n\left(Q^{a}_{q_k,j}\right) = Q^{a}_{h^{(n)}(q_k),j} = Q^{a}_{q_{k+n},j}, \tag{86}$$

by (81) and (85), and, hence,

$$\pi_h^n\left(\prod_{l=1}^{N}\left(Q^{a_l}_{q_{k_l},j_l}\right)^{n_l}\right) = \prod_{l=1}^{N}\left(Q^{a_l}_{q_{k_l+n},j_l}\right)^{n_l}, \tag{87}$$

by (83) and (86), for all $n \in \mathbb{N}$.

By (86) and (87), one may write as follows;

$$\pi_h^n = \pi_{h^{(n)}} \text{ on } \mathbb{LS}_A, \text{ for all } n \in \mathbb{N},$$

where $h^{(n)}$ are the n-shifts (81) on the TOset $\mathcal{P}$.

Consider now the sequence

$$\Pi = \left(\pi_h^n\right)_{n=1}^{\infty} \tag{88}$$

of the n-prime-shifts on $\mathbb{LS}_A$.

For any fixed $T \in \mathbb{LS}_A$, the sequence Π of (88) induces the sequence of operators,

$$\Pi(T) = \left(\pi_h^n(T)\right)_{n=1}^{\infty} = \left(\pi_h(T),\ \pi_h^2(T),\ \pi_h^3(T),\ \cdots\right)$$

in $\mathbb{LS}_A$, and this sequence $\Pi(T)$ has its corresponding free-distributional data, represented by the following $\mathbb{C}$-sequence:

$$\tau(\Pi(T)) = \left(\tau\left(\pi_h^n(T)\right)\right)_{n=1}^{\infty}. \tag{89}$$

We are interested in the convergence of the $\mathbb{C}$-sequence $\tau(\Pi(T))$ of (89), as $n \to \infty$.

Either convergent or divergent, the $\mathbb{C}$-sequence $\tau(\Pi(T))$ of (89), induced by any fixed operator $T \in \mathbb{LS}_A$, shows the asymptotic free distributional data of the family $\{\pi_h^n(T)\}_{n=1}^{\infty} \subset \mathbb{LS}_A$, as $n \to \infty$ in $\mathbb{N}$, equivalently, as $q_n \to \infty$ in $\mathcal{P}$.

9.3. Asymptotic Behaviors in $\mathbb{LS}_A$ over $\mathcal{P}$

Recall that, by (44), we have

$$\lim_{p \to \infty} \frac{\phi(p)}{p^{j+1}} = \begin{cases} 0, & \text{if } j > 0, \\ 1, & \text{if } j = 0, \\ \infty, \text{ Undefined}, & \text{if } j < 0, \end{cases} \tag{90}$$

for $j \in \mathbb{Z}$.

Recall also that there are bounded $*$-homomorphisms

$$\Pi = \left(\pi_h^n\right)_{n=1}^{\infty}, \text{ acting on } \mathbb{LS}_A,$$

of (88), where π_h^n are the *n-prime shifts* of (84), where h is the shift (79) on the TOset $\mathcal{P}$ of (78). Then, these $*$-homomorphisms of Π satisfies

$$\lim_{n \to \infty} \left(\pi_h^n\left(Q_{p,j}^a\right)\right) = \lim_{n \to \infty} \left(Q_{h^{(n)}(p),\, j}^a\right), \tag{91}$$

for all $Q_{p,j}^a \in \Omega(\mathcal{U}_A)$ in $\mathbb{LS}_A$, where $h^{(n)}$ are the n-shifts (80) on $\mathcal{P}$, for all $n \in \mathbb{N}$.

Thus, one can get that: if $\prod_{l=1}^{N} \left(Q_{p_l,j_l}^{a_l}\right)^{n_l}$ is a free reduced words of $\mathbb{LS}_A$ in $\Omega\left(\mathcal{U}_A\right)$, then

$$\begin{aligned} \lim_{n \to \infty} \pi_h^n \left(\prod_{l=1}^{N} \left(Q_{p_l,j_l}^{a_l}\right)^{n_l}\right) &= \lim_{n \to \infty} \left(\prod_{l=1}^{N} \pi_h^n \left(\left(Q_{p_l,j_l}^{a_l}\right)^{n_l}\right)\right) \\ &= \lim_{n \to \infty} \left(\prod_{l=1}^{N} \left(\pi_h^n\left(Q_{p_l,j_l}^{a_l}\right)\right)^{n_l}\right) \end{aligned}$$

since π_h^n are $*$-homomorphisms on $\mathbb{LS}_A$

$$= \lim_{n \to \infty} \left(\prod_{l=1}^{N} \left(Q_{h^{(n)}(p_l),j_l}^{a_l}\right)^{n_l}\right)$$

by (91)

$$= \prod_{l=1}^{N} \left(\lim_{n \to \infty} \left(Q_{h^{(n)}(p_l),\, j_l}^{a_l}\right)^{n_l}\right), \tag{92}$$

under the Banach-topology for $\mathbb{LS}_A$, for all $Q_{p_l,j_l}^{a_l} \in \Omega(\mathcal{U}_A)$, for $a_l \in (A, \psi)$, $p_l \in \mathcal{P}$, $j_l \in \mathbb{Z}$, for $l = 1, \ldots, N$, for all $N \in \mathbb{N}$.

Notation 2. (in short, **N 2** from below) For convenience, we denote $\lim_{n \to \infty} \pi_h^n$ symbolically by π, for the sequence $\Pi = \left(\pi_h^n\right)_{n=1}^{\infty}$ of (88).

Lemma 3. *Let $Q_{p_l,j_l}^{a_l} \in \Omega(\mathcal{U}_A)$ be generators of the A-tensor sub-filterization $\mathbb{LS}_A$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}$. In addition, let Π be the sequence (88) acting on $\mathbb{LS}_A$. If π is in the sense of* **N 2**, *then*

$$\pi\left(Q_{p_1,j_1}^{a_1}\right) = \lim_{n \to \infty} \left(Q_{\left(h^{(n)}(p_1)\right),\, j_1}^{a_1}\right), \tag{93}$$

$$\pi\left(\prod_{l=1}^{N}\left(Q_{p_l,j_l}^{a_l}\right)^{n_l}\right)=\lim_{n\to\infty}\left(\prod_{l=1}^{N}\left(Q_{h^{(n)}(p_l),\,j_l}^{a_l}\right)^{n_l}\right),$$

for all $n_1, ..., n_N \in \mathbb{N}$, *where* $h^{(n)}$ *are the n-shifts (80) on* $\mathcal{P}$.

Proof. The proof of (93) is done by (91) and (92). □

By abusing notation, one may/can understand the above formula (93) as follows

$$\pi\left(Q_{p_1,j_1}^{a_1}\right)=\lim_{p_1\to\infty}Q_{p_1,\,j_1}^{a_1},$$

$$\pi\left(\prod_{l=1}^{N}Q_{p_l,j_l}^{n_l}\right)=\prod_{l=1}^{N}\left(\lim_{p_l\to\infty}\left(Q_{p_l,j_l}^{n_l}\right)\right), \tag{94a}$$

respectively, where " $\lim\limits_{q\to\infty}$ " for $q \in \mathcal{P}$ is in the sense of (44).

Such an understanding (94a) of the formula (93) is meaningful by the constructions (80) of n-shifts $h^{(n)}$ on $\mathcal{P}$. For example,

$$\lim_{n\to\infty}h^{(n)}(q)=\lim_{p\to\infty}p, \text{ for } q\in\mathcal{P}, \tag{94b}$$

where the right-hand side of (94b) means that: starting with q, take bigger primes again and again in the TOset $\mathcal{P}$ of (78).

Assumption and Notation: From below, for convenience, the notations in (94a) are used for (93), if there is no confusion.

We now define a new (unbounded) linear functional τ_0 on $\mathbb{LS}_A$ with respect to the linear functional τ of (74a), by

$$\tau_0 \overset{def}{=} \tau\circ\pi \text{ on } \mathbb{LS}_A, \tag{95}$$

where π is in the sense of **N 2**.

Theorem 5. *Let* $\mathbb{LS}_A = (\mathbb{LS}_A, \tau)$ *be the A-tensor sub-filterization (74b), and let* $\tau_0 = \tau \circ \pi$ *be the new linear functional (95) on the Banach* $*$*-algebra* $\mathbb{LS}_A$ *of (74a). Then, for the generators*

$$\{Q_{p,j}^a\}_{p\in\mathcal{P}} \subset \Omega(\mathcal{U}_A) \text{ of } \mathbb{LS}_A,$$

for an arbitrarily fixed $a \in (A, \psi)$ *and* $j \in \mathbb{Z}$, *we have that*

$$\tau_0\left(\left(Q_{p,j}^a\right)^n\right)=\begin{cases}0, & \text{if } j>0,\\ \omega_n c_{\frac{n}{2}}\psi(a^n), & \text{if } j=0,\\ \infty,\ \text{Undefined}, & \text{if } j<0,\end{cases} \tag{96}$$

for all $n \in \mathbb{N}$.

Proof. Let $\{Q_{p,j}^a\}_{p\in\mathcal{P}} \subset \Omega(\mathcal{U}_A)$ in $\mathbb{LS}_A$, for fixed $a \in (A, \psi)$ and $j \in \mathbb{Z}$. Then,

$$\tau_0\left(\left(Q_{p,j}^a\right)^n\right)=(\tau\circ\pi)\left(\left(Q_{p,j}^a\right)^n\right)=\tau\left(\lim_{p\to\infty}\left(Q_{p,\,j}^a\right)^n\right)$$

by (93) and (94a)

$$=\lim_{p\to\infty}\tau\left(\left(Q_{p,j}^a\right)^n\right)$$

by the boundedness of τ for the (norm, or strong) topology for $\mathbb{LS}_A$

$$= \lim_{p \to \infty} \tau_j^p \left(\left(Q_{p,j}^a \right)^n \right) = \lim_{p \to \infty} \left(\omega_n c_{\frac{n}{2}} \psi(a^n) \left(\frac{\phi(p)}{p^{j+1}} \right) \right)$$

by (70), (75) and (77)

$$= \left(\omega_n c_{\frac{n}{2}} \psi(a^n) \right) \left(\lim_{p \to \infty} \frac{\phi(p)}{p^{j+1}} \right)$$

$$= \begin{cases} 0, & \text{if } j > 0, \\ \omega_n c_{\frac{n}{2}} \psi(a^n), & \text{if } j = 0, \\ \infty, \text{ Undefined}, & \text{if } j < 0, \end{cases}$$

by (90), for each $n \in \mathbb{N}$. Therefore, the free-distributional data (96) holds for τ_0. □

By (96), we obtain the following corollary.

Corollary 2. *Let $Q_{p,0}^{1_A} \in \Omega(\mathcal{U}_A)$ be free random variables of the A-tensor sub-filterization $\mathbb{LS}_A$, for all $p \in \mathcal{P}$, where 1_A is the unity of (A, ψ). Then, the asymptotic free distribution of the family*

$$\mathcal{Q}_0^{1_A} = \{Q_{p,0}^{1_A} \in \Omega(\mathcal{U}_A)\}_{p \in \mathcal{P}}$$

follows the semicircular law asymptotically as $p \to \infty$ in $\mathcal{P}$.

Proof. Let $\mathcal{Q}_0^{1_A} = \{Q_{p,0}^{1_A}\}_{p \in \mathcal{P}} \subset \Omega(\mathcal{U}_A)$ in $\mathbb{LS}_A$. Then, for the linear functional τ_0 of (95) on $\mathbb{LS}_A$,

$$\tau_0 \left(\left(Q_{p,0}^{1_A} \right)^n \right) = \omega_n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, by (96), since

$$\psi(1_A^n) = \psi(1_A) = 1; n \in \mathbb{N}.$$

If $p \to \infty$ in $\mathcal{P}$, then the asymptotic free distribution of the family $\mathcal{Q}_0^{1_A}$ is the semicircular law by the self-adjointness of all $Q_{p,0}^{1_A}$'s, and by the semicircularity (45) and (47). □

Independent from (96), we obtain the following asymptotic free-distributional data on $\mathbb{LS}_A$.

Theorem 6. *Let $j_1, ..., j_N$ be "mutually distinct" in $\mathbb{Z}$, for $N > 1$ in $\mathbb{N}$, and hence the N-tuple*

$$[j] = (j_1, ..., j_N) \in \mathbb{Z}^N$$

is alternating in $\mathbb{Z}$. In addition, let

$$[a] = (a_1, ..., a_N)$$

be an arbitrarily fixed N-tuple of free random variables $a_1, ..., a_N$ of the unital C^*-probability space (A, ψ), and let's fix

$$[n] = (n_1, ..., n_N) \in \mathbb{N}^N.$$

Now, define a family $\mathcal{T}_{[j]}^{[a],[n]}$ of free reduced words with their lengths-N,

$$\mathcal{T}_{[j]}^{[a],[n]} = \left\{ T = \prod_{l=1}^{N} \left(Q_{p_l,j_l}^{a_l} \right)^{n_l} : p_1, ..., p_N \in \mathcal{P} \right\}, \tag{97}$$

in $\mathbb{LS}_A$, for $Q_{p_l,j_l}^{a_l} \in \Omega\left(\mathcal{U}_A\right)$, for all $p_l \in \mathcal{P}$, where $a_l \in [a]$, $j_l \in [j]$, for $l = 1, ..., N$.

For any free reduced words $T \in \mathcal{T}_{[j]}^{[a],[n]}$, if τ_0 is the linear functional (95) on $\mathbb{LS}_A$, then

$$\tau_0\left(T\right) = \begin{cases} 0, & \text{if } \sum_{l=1}^{N} j_l > 1 - N, \\ \prod_{l=1}^{N} \left(\omega_{n_l} c_{\frac{n_l}{2}} \psi(a^{n_l}) \right), & \text{if } \sum_{l=1}^{N} j_l = 1 - N, \\ \infty, \text{ Undefined}, & \text{if } \sum_{l=1}^{N} j_l < 1 - N, \end{cases} \tag{98}$$

for all $n \in \mathbb{N}$.

Proof. Let $T \in \mathcal{T}_{[j]}^{[a],[n]}$ be in the sense of (97) in the A-tensor sub-filterization $\mathbb{LS}_A$. Then, these operators T form free reduced words with their lengths-N in $\mathbb{LS}_A$, since $[j]$ is an alternating N-tuple of "mutually distinct" integers. Observe that

$$\tau_0\,(T) = \tau\,(\pi(T)) = \tau\left(\prod_{l=1}^{N}\left(\lim_{p_l\to\infty}\left(Q_{p_l,\,j_l}^{a_l}\right)^{n_l}\right)\right)$$

by (93) and (94a)

$$= \tau\left(\prod_{l=1}^{N}\left(\lim_{p\to\infty}\left(Q_{p,\,j_l}^{a_l}\right)^{n_l}\right)\right)$$

because

$$\lim_{p\to\infty} p = \lim_{n\to\infty} h^{(n)}\,(p_l) = \lim_{p_l\to\infty} p_l, \text{ in } \mathcal{P},$$

in the sense of (44), for all $l = 1, ..., N$, and, hence, it goes to

$$= \lim_{p\to\infty}\left(\tau\left(\left(\prod_{l=1}^{N} Q_{p,j_l}^{a_l}\right)^{n_l}\right)\right)$$

by the boundedness of τ for the (norm, or strong) topology for $\mathbb{LS}_A$

$$= \lim_{p\to\infty}\left(\prod_{l=1}^{N}\left(\omega_{n_l} c_{\frac{n_l}{2}}\,\psi(a_l^{n_l})\left(\frac{\phi(p)}{p^{j_l+1}}\right)\right)\right)$$

since $[j]$ consists of "mutually-distinct" integers, by the Möbius inversion

$$\begin{aligned}
&= \left(\prod_{l=1}^{N}\omega_{n_l} c_{\frac{n_l}{2}}\,\psi(a_l^{n_l})\right)\left(\lim_{p\to\infty}\left(\prod_{l=1}^{N}\left(\frac{\phi(p)}{p^{j_l+1}}\right)\right)\right)\\
&= \left(\prod_{l=1}^{N}\omega_{n_l} c_{\frac{n_l}{2}}\,\psi(a_l^{n_l})\right)\left(\lim_{p\to\infty}\left(\frac{\phi(p)}{p^{N+\Sigma_{l=1}^{N} j_l}}\right)\right)\\
&= \left(\prod_{l=1}^{N}\omega_{n_l} c_{\frac{n_l}{2}}\,\psi(a_l^{n_l})\right)\left(\lim_{p\to\infty}\left(\frac{\phi(p)}{p^{\left(N-1+\Sigma_{l=1}^{N} j_l\right)+1}}\right)\right)\\
&= \left(\prod_{l=1}^{N}\omega_{n_l} c_{\frac{n_l}{2}}\,\psi(a_l^{n_l})\right)\left(\lim_{p\to\infty}\left(\frac{\phi(p)}{p^{\left(N-1+\Sigma_{l=1}^{N} j_l\right)+1}}\right)\right)\\
&= \begin{cases} 0 & \text{if } N-1+\sum_{l=1}^{N} j_l > 0\\ \prod_{l=1}^{N}\left(\omega_{n_l} c_{\frac{n_l}{2}}\,\psi(a_l^{n_l})\right) & \text{if } N-1+\sum_{l=1}^{N} j_l = 0\\ \infty & \text{if } N-1+\sum_{l=1}^{N} j_l < 0,\end{cases}
\end{aligned}$$

by (90), for all $n \in \mathbb{N}$. Therefore, the family $\mathcal{T}_{[j]}^{[a],[n]}$ of (97) satisfies the asymptotic free-distributional data (98) in the A-tensor sub-filterization $\mathbb{LS}_A$ over $\mathcal{P}$. □

The above two theorems illustrate the asymptotic free-probabilistic behaviors on the A-tensor sub-filterization $\mathbb{LS}_A$ over $\mathcal{P}$, by (96) and (98).

As a corollary of (96), we showed that the family

$$\mathcal{Q}_0^{1_A} = \{Q_{p,0}^{1_A}\}_{p\in\mathcal{P}} \subset \mathbb{LS}_A$$

has its asymptotic free distribution, the semicircular law in $\mathbb{LS}_A$, as $p \to \infty$. More generally, the following theorem is obtained.

Theorem 7. *Let a be a self-adjoint free random variable of our unital C^*-probability space (A, ψ). Assume that it satisfies*

(i) $\psi(a) \in \mathbb{R}^{\times} = \mathbb{R} \setminus \{0\}$ *in* $\mathbb{C}$,
(ii) $\psi(a^{2n}) = \psi(a)^{2n}$, *for all* $n \in \mathbb{N}$.

Then, the family

$$\mathcal{X}_0^a = \left\{ X_{p,0}^a = \tfrac{1}{\psi(a)} Q_{p,0}^a : p \in \mathcal{P} \right\} \tag{99}$$

follows the asymptotic semicircular law, in $\mathbb{LS}_A$ *over* $\mathcal{P}$.

Proof. Let $a \in (A, \psi)$ be a self-adjoint free random variable satisfying two conditions (i) and (ii), and let $\mathcal{X}_0^a$ be the family (99) of the A-tensor sub-filterization $\mathbb{LS}_A$. Then, all elements

$$X_{p,0}^a = \tfrac{1}{\psi(a)} Q_{p,0}^a = \mathbf{1}_p^A \otimes \left(\left(\tfrac{1}{\psi(a)} a \right) \otimes P_{p,0} \right) \text{ of } \mathcal{X}_0^a$$

are self-adjoint in $\mathbb{LS}_A$, by the self-adjointness of $Q_{p,0}^a$, and by the condition (i).

For any $X_{p,0}^a \in \mathcal{X}_0^a$, observe that

$$\tau_0 \left(\left(X_{p,0}^a \right)^n \right) = \tfrac{1}{\psi(a)^n} \, \tau_0 \left(\left(Q_{p,0}^a \right)^n \right) \\ = \tfrac{1}{\psi(a)^n} \left(\omega_n c_{\frac{n}{2}} \psi(a^n) \right)$$

by (96)

$$= \left(\omega_n c_{\frac{n}{2}} \left(\tfrac{\psi(a^n)}{\psi(a^n)} \right) \right)$$

by the condition (ii)

$$= \omega_n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$. Therefore, the family $\mathcal{X}_0^a$ has its asymptotic semicircular law over $\mathcal{P}$, by (45). □

Similar to the construction of $\mathcal{X}_0^a$ of (99), if we construct the families $\mathcal{X}_j^a$,

$$\mathcal{X}_j^a = \left\{ \tfrac{1}{\psi(a)} Q_{p,j}^a : Q_{p,j}^a \in \Omega\left(\mathcal{U}_A\right) \right\}_{p \in \mathcal{P}}, \tag{100}$$

for a fixed $a \in (A, \psi)$ satisfying the conditions (i) and (ii) of the above theorem, and, for a fixed $j \in \mathbb{Z}$, then one obtains the following corollary.

Corollary 3. *Fix* $a \in (A, \psi)$ *satisfying the conditions (i) and (ii) of the above theorem. Let's fix* $j \in \mathbb{Z}$, *and let* $\mathcal{X}_j^a$ *be the corresponding family (100) in the A-tensor sub-filterization* $\mathbb{LS}_A = (\mathbb{LS}_A, \tau)$.

$$\text{If } j = 0, \text{ then } \mathcal{X}_0^a \text{ has the asymptotic semicircular law in } \mathbb{LS}_A. \tag{101}$$

$$\text{If } j > 0, \text{ then } \mathcal{X}_j^a \text{ has its asymptotic free distribution, the zero free distribution, in } \mathbb{LS}_A. \tag{102}$$

$$\text{If } j < 0, \text{ then the asymptotic free distribution of } \mathcal{X}_j^a \text{ is undefined in } \mathbb{LS}_A. \tag{103}$$

Proof. The proof of (101) is done by (99).

By (96), if $j > 0$, then, for any $T = \frac{1}{\psi(a)} Q_{p,j}^a \in \mathcal{X}_j^a$, one has that

$$\tau_0 \left(T^n \right) = \tfrac{1}{\psi(a^n)} \tau_0 \left(\left(Q_{p,j}^a \right)^n \right) = 0,$$

for all $n \in \mathbb{N}$. Thus, the asymptotic free distribution of $\mathcal{X}_j^a$ is the zero free distribution in $\mathbb{LS}_A$, as $p \to \infty$ in $\mathcal{P}$. Thus, the statement (102) holds.

Similarly, by (96), if $j < 0$, then the asymptotic free distribution $\mathcal{X}_j^a$ is undefined in $\mathbb{LS}_A$ over $\mathcal{P}$, equivalently, the statement (103) is shown. □

Motivated by (101), (102) and (103), we study the asymptotic semicircular law (over $\mathcal{P}$) on $\mathbb{LS}_A$ more in detail in Section 10 below.

10. Asymptotic Semicircular Laws on $\mathbb{LS}_A$ over $\mathcal{P}$

We here consider asymptotic semicircular laws on the A-tensor sub-filterization $\mathbb{LS}_A = (\mathbb{LS}_A, \tau)$. In Section 9.3, we showed that the asymptotic free distribution of a family

$$\mathcal{X}_0^a = \{\tfrac{1}{\psi(a)} Q_{p,0}^a : p \in \mathcal{P}\} \tag{104}$$

is the semicircular law in $\mathbb{LS}_A$ as $p \to \infty$ in $\mathcal{P}$, for a fixed self-adjoint free random variable $a \in (A, \psi)$ satisfying

(i) $\psi(a) \in \mathbb{R}^\times$, and
(ii) $\psi(a^{2n}) = \psi(a)^{2n}$, for all $n \in \mathbb{N}$.

As an example, the family

$$\mathcal{X}_0^{1_A} = \{Q_{p,0}^{1_A} : p \in \mathcal{P}\} \tag{105}$$

follows the asymptotic semicircular law in $\mathbb{LS}_A$ over $\mathcal{P}$.

We now enlarge such asymptotic behaviors on $\mathbb{LS}_A$ up to certain $*$-isomorphisms.

Define bijective functions g_+ and g_- on $\mathbb{Z}$ by

$$g_+(j) = j+1, \text{ and } g_-(j) = j-1, \tag{106}$$

for all $j \in \mathbb{Z}$.

By (106), one can define bijective functions $g_\pm^{(n)}$ on $\mathbb{Z}$ by

$$g_\pm^{(n)} \overset{def}{=} \underbrace{g_\pm \circ g_\pm \circ g_\pm \circ \cdots \circ g_\pm}_{n\text{-times}}, \tag{107}$$

satisfying $g_\pm^{(1)} = g_\pm$ on $\mathbb{Z}$, with axiomatization:

$$g_\pm^{(0)} = id_{\mathbb{Z}}, \text{ the identity function on } \mathbb{Z},$$

for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For example,

$$g_\pm^{(n)}(j) = j \pm n, \tag{108}$$

for all $j \in \mathbb{Z}$, for all $n \in \mathbb{N}_0$.

From the bijective functions $g_\pm^{(n)}$ of (107), define the bijective functions $\left(g_\pm^o\right)^{(n)}$ on the generator set $\Omega(\mathcal{U}_A)$ of (72) of the A-tensor sub-filterization $\mathbb{LS}_A$ by

$$\begin{gathered} \left(g_+^o\right)^{(n)}\left(Q_{p,j}^a\right) = Q^a_{p,g_+^{(n)}(j)} = Q^a_{p,j+n}, \\ \\ \left(g_-^o\right)^{(n)}\left(Q_{p,j}^a\right) = Q^a_{p,g_-^{(n)}(j)} = Q^a_{p,j-n}, \end{gathered} \tag{109}$$

with

$$\left(g_\pm^o\right)^{(1)} = g_\pm^o, \text{ and } \left(g_\pm^o\right)^{(0)} = id,$$

by (108), for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, for all $n \in \mathbb{N}_0$, where id is the identity function on $\Omega(\mathcal{U}_A)$.

By the construction (73a) of the generator set $\Omega(\mathcal{U}_A)$ of $\mathbb{LS}_A$ under (73b),

$$\Omega(\mathcal{U}_A) = \bigsqcup_{p \in \mathcal{P}} \{Q_{p,j}^a : a \in A, j \in \mathbb{Z}\},$$

the functions $\left(g^o_{\pm}\right)^{(n)}$ of (109) are indeed well-defined bijections on $\Omega(\mathcal{U}_A)$, by the bijectivity of $g^{(n)}_{\pm}$ of (107).

Now, define bounded $*$-homomorphisms $G_{\pm}$ on $\mathbb{LS}_A$ by the bounded multiplicative linear transformations on $\mathbb{LS}_A$ satisfying that:

$$G_{+}\left(Q^a_{p,j}\right) = g^o_{+}\left(Q^a_{p,j}\right) = Q^a_{p,j+1},$$
$$G_{-}\left(Q^a_{p,j}\right) = g^o_{-}\left(Q^a_{p,j}\right) = Q^a_{p,j-1}, \tag{110}$$

in $\mathbb{LS}_A$, by using the bijections $g^o_{\pm}$ of (109), for all $Q^a_{p,j} \in \Omega(\mathcal{U}_A)$.

More precisely, the morphisms $G_{\pm}$ of (110) satisfy that

$$\begin{aligned} G_{\pm}\left(\prod_{l=1}^{N}\left(Q^{a_l}_{p_l,j_l}\right)^{n_l}\right) &= \prod_{l=1}^{N} g^o_{\pm}\left(\left(Q^{a_l}_{p_l,j_l}\right)^{n_l}\right) \\ &= \prod_{l=1}^{N}\left(Q^{a_l}_{p_l,j_l\pm 1}\right)^{n_l}. \end{aligned} \tag{111a}$$

By (111a), one can get that

$$\begin{aligned} G_{\pm}\left(\left(\prod_{l=1}^{N}\left(Q^{a_l}_{p_l,j_l}\right)^{n_l}\right)^*\right) &= G_{\pm}\left(\prod_{l=1}^{N}\left(Q^{a^*_{N-l+1}}_{p_{N-l+1},j_{N-l+1}}\right)^{n_{N-l+1}}\right) \\ &= \prod_{l=1}^{N}\left(\left(Q^{a_{N-l+1}}_{p_{N-l+1},(j_{N-l+1})\pm 1}\right)^{n_{N-l+1}}\right)^* \\ &= \left(\prod_{l=1}^{N}\left(Q^{a_l}_{p_l,j_l\pm 1}\right)^{n_l}\right)^* \\ &= \left(G_{\pm}\left(\prod_{l=1}^{N} Q^{n_l}_{p_l,j_l}\right)\right)^* \end{aligned} \tag{111b}$$

for all $Q^{a_l}_{p_l,j_l} \in \Omega(\mathcal{U}_A)$, for $l = 1, ..., N$, for $N \in \mathbb{N}$.

The formula (111a) are obtained by (110) and the multiplicativity of $G_{\pm}$. The formulas in (111b), obtained from (111a), show that indeed $G_{\pm}$ are $*$-homomorphisms on $\mathbb{LS}_A$, since

$$G_{\pm}\left(T^*\right) = \left(G_{\pm}(T)\right)^*, \forall T \in \mathbb{LS}_A.$$

By (110) and (111a),

$$G^n_{\pm}\left(\prod_{l=1}^{N}\left(Q^{a_l}_{p_l,j_l}\right)^{n_l}\right) = \prod_{l=1}^{N}\left(Q^{a_l}_{p_l,j_l\pm n}\right)^{n_l},$$
$$G^n_{\pm}\left(\left(\prod_{l=1}^{N}\left(Q^{a_l}_{p_l,j_l}\right)^{n_l}\right)^*\right) = \left(G^n_{\pm}\left(\prod_{l=1}^{N}\left(Q^{a_l}_{p_l,j_l}\right)^{n_l}\right)\right)^*, \tag{112}$$

for all $Q^{a_l}_{p_l,j_l} \in \Omega(\mathcal{U}_A)$, for $l = 1, ..., N$, for $N \in \mathbb{N}$, for all $n \in \mathbb{N}_0$.

Definition 16. *We call the bounded $*$-homomorphisms $G^n_{\pm}$ of (110), the n-($\pm$)-integer-shifts on $\mathbb{LS}_A$, for all $n \in \mathbb{N}_0$.*

Based on the integer-shifting processes on $\mathbb{LS}_A$, one can get the following asymptotic behavior on $\mathbb{LS}_A$ over $\mathcal{P}$.

Theorem 8. *Let $\mathcal{X}^a_j$ be a family (100) of the A-tensor sub-filterization $\mathbb{LS}_A$, for any $j \in \mathbb{Z}$, where a is a fixed self-adjoint free random variable of (A, ψ) satisfying the additional conditions (i) and (ii) above. Then, there exists a $(-j)$-integer-shift G_{-j} on $\mathbb{LS}_A$, such that*

$$G_{-j} = \begin{cases} G_{-}^{|j|} = G_{-}^{j} & \text{if } j \geq 0 \text{ in } \mathbb{Z}, \\ G_{+}^{|j|} = G_{+}^{-j} & \text{if } j < 0 \text{ in } \mathbb{Z}, \end{cases} \tag{113}$$

and

$$\tau_0 \left(G_j(T) \right) = \omega_n c_{\frac{n}{2}}, \forall n \in \mathbb{N}, \tag{114}$$

for all $T \in \mathcal{X}_j^a$, where $G_{\mp}^{\pm j}$ on the right-hand sides of (113) are the $|j|$-($\mp$)-integer shifts (110) on $\mathbb{LS}_A$, and where $\tau_0 = \tau \circ \pi$ is the linear functional (95) on $\mathbb{LS}_A$.

Proof. Let $\mathcal{X}_j^a = \left\{ \frac{1}{\psi(a)} Q_{p,j}^a : p \in \mathcal{P} \right\}$ be a family (100) of $\mathbb{LS}_A$, for a fixed $j \in \mathbb{Z}$, where a fixed self-adjoint free random variable $a \in (A, \psi)$ satisfies the above additional conditions (i) and (ii).

Assume first that $j \geq 0$ in $\mathbb{Z}$. Then, one can take the $(-j)$-$(-)$-integer-shift G_{-}^{j} of (110) on $\mathbb{LS}_A$, satisfying

$$G_{-}^{j} \left(Q_{p,j}^a \right) = Q_{p,j-j}^a = Q_{p,0}^a \text{ in } \mathbb{LS}_A,$$

for all $Q_{p,j}^a \in \Omega\left(\mathcal{U}_A\right)$.

Second, if $j < 0$ in $\mathbb{Z}$, then one can have the $|j|$-$(+)$-integer shift G_{+}^{-j} of (110) on $\mathbb{LS}_A$, satisfying that

$$G_{+}^{-j} \left(Q_{p,j}^a \right) = Q_{p,j+(-j)}^a = Q_{p,0}^a \text{ in } \mathbb{LS}_A,$$

for all $Q_{p,j}^a \in \Omega\left(\mathcal{U}_A\right)$.

For example, for any $Q_{p,j}^a \in \Omega(\mathcal{U}_A)$, we have the corresponding $(-j)$-integer-shift G_{-j},

$$G_{-j} = \begin{cases} G_{-}^{j} & \text{if } j \geq 0, \\ G_{+}^{-j} & \text{if } j < 0, \end{cases}$$

on $\mathbb{LS}_A$ in the sense of (113), such that

$$G_{-j} \left(Q_{p,j}^a \right) = Q_{p,0}^a \text{ in } \mathbb{LS}_A,$$

for all $p \in \mathcal{P}$.

Then, for any $X_{p,j}^a = \frac{1}{\psi(a)} Q_{p,j}^a \in \mathcal{X}_j^a$, we have that

$$\tau_0 \left(G_{-j} \left(\left(X_{p,j}^a \right)^n \right) \right) = \tau_0 \left(\frac{1}{\psi(a)^n} \left(G_{-j}(Q_{p,j}^a) \right)^n \right),$$

since G_{-j} is a $*$-homomorphism (113) on $\mathbb{LS}_A$

$$= \tau_0 \left(\frac{1}{\psi(a^n)} \left(Q_{p,0}^a \right)^n \right) = \omega_n c_{\frac{n}{2}},$$

by (96) and (98), for all $n \in \mathbb{N}$. Therefore, formula (114) holds true. □

By the above theorem, we obtain the following result.

Corollary 4. *Let $\mathcal{X}_j^a$ be a family (100) of the A-tensor sub-filterization $\mathbb{LS}_A$, for $j \in \mathbb{Z}$, where a self-adjoint free random variable $a \in (A, \psi)$ satisfies the conditions (i) and (ii). Then, the corresponding family*

$$\mathcal{G}_j^a = \left\{ G_{-j}\left(X\right) : X \in \mathcal{X}_j^a \right\} \tag{115}$$

has its asymptotic free distribution, the semicircular law, in $\mathbb{LS}_A$ over $\mathcal{P}$, where G_{-j} is the $(-j)$-integer shift (113) on $\mathbb{LS}_A$, for all $j \in \mathbb{Z}$.

Proof. The asymptotic semicircular law induced by the family $\mathcal{G}_j^a$ of (115) in $\mathbb{LS}_A$ is guaranteed by (114) and (45), for all $j \in \mathbb{Z}$. □

By the above corollary, the following result is immediately obtained.

Corollary 5. *Let $\mathcal{X}_j^{1_A}$ be in the sense of (100) in $\mathbb{LS}_A$, where 1_A is the unity of (A, ψ), and let*

$$\mathcal{G}_j^{1_A} = \left\{ G_{-j}(X) : X \in \mathcal{X}_j^{1_A} \right\}$$

be in the sense of (115), for all $j \in \mathbb{Z}$. Then, the asymptotic free distributions of $\mathcal{G}_j^{1_A}$ are the semicircular law in $\mathbb{LS}_A$ over $\mathcal{P}$, for all $j \in \mathbb{Z}$.

Proof. The proof is done by Corollary 4. Indeed, the unity 1_A automatically satisfies the conditions (i) and (ii) in (A, ψ). $\square$

More general to Theorem 8, we obtain the following result too.

Theorem 9. *Let $a \in (A, \psi)$ be a self-adjoint free random variable satisfying the conditions (i) and (ii), and let $p_0 \in \mathcal{P}$ be an arbitrarily fixed prime. Let*

$$\mathcal{G}_j^a[\geq p_0] \overset{def}{=} \left\{ G_{-j}\left(X_{p,j}\right) \,\middle|\, \begin{array}{c} X_{p,j}^a \in \mathcal{X}_j^a \text{ and} \\ p \geq p_0 \text{ in } \mathcal{P} \end{array} \right\},$$

where $\mathcal{X}_j^a$ is the family (100), and $\mathcal{G}_j^a$ is the family (115), for $j \in \mathbb{Z}$. Then, the asymptotic free distribution of the family $\mathcal{G}_j^a[\geq p_0]$ is the semicircular law in $\mathbb{LS}_A$.

Proof. The proof of this theorem is similar to that of Theorem 8. One can simply replace

$$\text{“}p \to \infty\text{”} \equiv \text{“}\lim_{n \to \infty} h^n(2); 2 \in \mathcal{P}\text{,”}$$

in the proof of Theorem 8 to

$$\text{“}p \to \infty\text{”} \equiv \text{“}\lim_{n \to \infty} h^n(p_0); p_0 \in \mathcal{P}\text{,”}$$

where $(\equiv)$ means “being symbolically same”. $\square$

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