Article

Filtering Method Based on Symmetrical Second Order Systems

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Abstract: This study presents a filtering and sampling structure based on symmetrical second order systems working on half-period. It is shown that undamped second order oscillating systems working on half-period could provide: (i) a large attenuation coefficient for an alternating signal (due to the filtering second order system), and (ii) a robust sampling procedure (the slope of the generated output being zero at the sampling time moment). Unlike previous studies on the same topics, these results are achieved without the use of an additional integrator.

Keywords: symmetrical response; second order systems

1. Introduction

Usually filtering and sampling devices for analog signal processing are represented by asymptotically stable systems.

The filtering properties of these systems are determined by their transfer function. Thus, a filtering device represented by a low-pass filter of first order with the transfer function

$$H(s) = 1/(T_0 s + 1) \quad (1)$$

provides attenuation for an alternating signal with angular frequency $\omega > \omega_0 = 1/T_0$ of about ω/ω_0 times. A filtering device represented by a low-pass filter of second order with the transfer function

$$H(s) = 1/(T_0^2 s^2 + 2\xi T_0 s + 1) \quad (2)$$

provides a higher attenuation of about $(\omega/\omega_0)^2$ times for an alternating signal with angular frequency $\omega > \omega_0 = 1/T_0$.

For a symmetrical second order system (generating undamped oscillations symmetrical as related to the time moment corresponding to a maximum) the transfer function is represented by

$$H(s) = 1/(T_0^2 s^2 + 1) \quad (3)$$

This system attenuates an alternating input of angular frequency ω about $(\omega/\omega_0)^2$ times (where $\omega_0 = 1/T_0$). By integrating these undamped symmetrical oscillations on the time interval $(0, 2\pi T_0)$, a supplementary attenuation of about $(2\pi\omega/\omega_0)$ is obtained. Moreover, in case of a unity-step input the quantity to be integrated equals zero at the end of the integration period, (see [1] for details).

Thus, the influence of the random variations of the integration period (generated by the switching phenomena) is practically rejected. Overall, this structure provides an attenuation of about $(2\pi)(\omega/\omega_0)^3$ for an alternating input of angular frequency ω on the time interval $2\pi T_0 = 2\pi/\omega_0$.

More advanced mathematical models for detecting sharp jumps (as wavelets methods presented in [2]) are hard to be implemented for fast analog signal processing.

However, the use of any integrative method requires some supplementary analog devices (as operational amplifiers) to be added. As it is well known, these devices should be calibrated in an accurate manner, so (if possible) their number should be reduced.

To do this, we shall try to design a structure for a robust sampling procedure based on generated symmetrical (undamped) oscillations without any additional integrative stage (similar to aspects of generating asymmetrical pulses presented in [3]).

Such an accurate structure could be useful for detecting short-step pulses (as in [4]), for accurate image processing (as in [5]), or for preliminary signal processing before advanced mathematical techniques as wavelets to be applied for an optimal control [6].

2. Materials and Methods

Let us analyze in detail the response of a second order oscillating system at a unity-step input. For the transfer function represented by Equation (3), the response would be

$$y(t) = 1 - \cos(t/T_0) \quad (4)$$

It can be noticed that the first-time moment (excepting the initial moment of time) when the slope of $y(t)$ is zero corresponds to $t' = \pi T_0$, the value of $y(t)$ being two (double as related to the amplitude of the step input). Therefore, a robust sampling procedure could act at this time moment.

For an alternating input with angular frequency ω , the amplitude of the generated output with this angular frequency ω will be $(\omega/\omega_0)^2$ lower than the amplitude of the alternating input (according to partial fraction decomposition applied to generated output within Laplace transform theory).

Since a non-vanishing term with angular frequency ω_0 is also generated (corresponding to proper oscillation of the undamped second order system), we must take care that this alternating component should be added at the sampling time moment to the attenuated output of angular frequency ω . In fact, two alternating terms $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ of angular frequency ω_0 are generated, but $\sin(\omega_0 t)$ equals zero for $\omega_0 t = \pi$ (the suggested sampling moment of time). For null initial conditions, its amplitude corresponds to the opposite value of the oscillating output of angular frequency ω at zero moment of time. Since this has been attenuated $(\omega/\omega_0)^2$ times (as shown before), the sum of these two alternating components (the attenuated component of angular frequency ω and the cosine term $\cos(\omega_0 t)$ of the proper oscillation required for compensating this component at zero moment of time) cannot be greater than two times the amplitude of the attenuated component of angular frequency ω .

Thus it can be preliminary considered that a second order oscillating system working on half-period from initial null conditions could provide a robust sampling procedure for a step input (doubling its amplitude at the sampling time moment) and attenuation for an alternating component of angular frequency ω of about $(\omega/\omega_0)^2/2$.

3. Results

Let us consider a second order system described by the differential equation

$$T_0^2 f''(t) + f(t) = u(t) \quad (5)$$

where $u(t)$ corresponds to the received signal (the input), and $f(t)$ corresponds to the generated output.

First, we analyze the case when the input is represented by an alternating signal of angular frequency ω , amplitude A and initial phase φ . In this case, the output $f_{alt}(t)$ is represented by an alternating function denoted as $f_{alt}(t)$. This means

$$T_0^2 f_{alt}''(t) + f_{alt}(t) = A \times \cos(\omega t + \varphi) \quad (6)$$

Considering null initial conditions, the output $f_{alt}(t)$ for this standard linear differential equation will be represented by

$$f_{alt}(t) = \{A \times \cos\varphi/[1 - T_0^2 \omega^2]\} \cos(\omega t) - \{A \times \sin\varphi/[1 - T_0^2 \omega^2]\} \sin(\omega t) - \{A \times \cos\varphi/[1 - T_0^2 \omega^2]\} \cos(\omega_0 t) + (\omega/\omega_0) \{A \times \sin\varphi/[1 - T_0^2 \omega^2]\} \sin(\omega_0 t) \quad (7)$$

The sampling moment of time t_S is chosen at half-period. This means

$$\omega_0 t_S = \pi, \cos(\omega_0 t_S) = -1, \sin(\omega_0 t_S) = 0 \quad (8)$$

Thus, the function $f_{alt}(t)$ generated by the second order oscillating system for $t = t_S$ would be

$$f_{alt}(t_S) = \{A \times \cos\varphi/[1 - T_0^2 \omega^2]\} \cos(\omega t_S) - \{A \times \sin\varphi/[1 - T_0^2 \omega^2]\} \sin(\omega t_S) + \{A \times \cos\varphi/[1 - T_0^2 \omega^2]\} \quad (9)$$

($\cos(\omega_0 t_S) = -1, \sin(\omega_0 t_S) = 0$ were substituted in Equation (7) according to Equation (8)).

The first two terms in Equation (9) can be grouped according to

$$\cos\varphi \cos(\omega t_S) - \sin\varphi \sin(\omega t_S) = \cos(\omega t_S + \varphi) \quad (10)$$

and $f_{alt}(t_S)$ can be written as

$$f_{alt}(t_S) = \{A/[1 - T_0^2 \omega^2]\} \{\cos(\omega t_S + \varphi) - \cos\varphi\} \quad (11)$$

Since

$$|\cos(\omega t_S + \varphi)| \leq 1, \cos\varphi \leq 1 \quad (12)$$

it results that

$$|\cos(\omega t_S + \varphi) - \cos\varphi| \leq 2 \quad (13)$$

So

$$|\{A/[1 - T_0^2 \omega^2]\} \{\cos(\omega t_S + \varphi) - \cos\varphi\}| \leq 2|\{A/[1 - T_0^2 \omega^2]\}| \quad (14)$$

Thus, for the modulus of the left-hand side of Equation (11) it can be written the inequality

$$|f_{alt}(t_S)| = |\{A/[1 - T_0^2 \omega^2]\} \{\cos(\omega t_S + \varphi) - \cos\varphi\}| \leq |\{A/[1 - T_0^2 \omega^2]\} \cos(\omega t_S + \varphi)| + |\{A \times \cos\varphi / [1 - T_0^2 \omega^2]\}| \leq 2 |\{A/[1 - T_0^2 \omega^2]\}| \quad (15)$$

The response of a second order oscillating system (with a time period which equals unity) on a half-period interval, for an input represented by an alternating function with unity amplitude and angular frequency five times greater, is represented in Figure 1. The amplitude of the generated output is less than 0.08, which means about $2/(5^2)$ times less than input amplitude (the validity of Equation (15) being confirmed).

Figure 1. Response of second order oscillating system on half-period for an input with five times greater angular frequency.

It should be noticed that there is no symmetry as related to a certain vertical axis corresponding to a local maximum since generated oscillations of higher frequency are overlapped to a slow-varying alternating function of unity period.

Let us now consider the case when the input $u(t)$ of the second order oscillating system described by the differential Equation (5) is represented by a step input of amplitude A_0 . This means

$$T_0^2 f''(t) + f(t) = A_0 \quad (16)$$

There are considered initial null conditions for $t = 0$, as in Equation (6). The function $f_0(t)$ generated by this step signal is represented by

$$f_0(t) = A_0 \{1 - \cos(\omega_0 t)\} \quad (17)$$

where $\omega_0 = 1/T_0$.

The sampling moment of time corresponds to half period. This means

$$t_s = \pi/\omega_0 = \pi T_0 \quad (18)$$

and thus,

$$\cos(\omega_0 t_s) = \cos \pi = -1 \quad (19)$$

It results in the value of $f(t)$ at the sampling moment of time as

$$f_0(t_s) = 2A_0 \quad (20)$$

It must be emphasized the fact that $f_0(t)$ possess a saddle point at this sampling time moment (its derivative presents a null value). Thus, a robust sampling procedure can be performed, the filtering and sampling structure being insensitive at small variations of the sampling moment of time.

The response of a second order oscillating system (with a time period which equals unity) on a half-period interval, for a unity-step input, is represented in Figure 2. The sampling moment of time is chosen at half-period, so the symmetry as related to the vertical axis represented at $t = 0.5$ cannot be noticed.

Figure 2. Response of second order oscillating system on half-period for step-input.

Finally, let us consider the case when the input of the oscillating system is represented by a sum of a step-input of amplitude A_0 and an alternating signal of angular frequency ω and amplitude A (the order of magnitude for A and A_0 being the same). An example could be represented by measurements of angular velocity based on the rectification of an alternating signal generated through electromagnetic induction (a significant alternative component being added to the constant component proportional to the angular velocity).

Thus,

$$u(t) = A_0 + A \times \cos(\omega t + \varphi) \quad (21)$$

Since the oscillating system presented in Equation (5) is linear, the output would be simply represented by the sum

$$f(t) = f_0(t) + f_{alt}(t) \quad (22)$$

So,

$$f(t_s) = f_0(t_s) + f_{alt}(t_s) \quad (23)$$

The term $f_0(t_s)$ corresponds to the useful signal (proportional to the step input of amplitude A_0) necessary to be sampled; the term $f_{alt}(t_s)$ is generated by the alternating signal of angular frequency ω necessary to be filtered (its magnitude should be as less than possible).

According to Equations (15) and (20), it results

$$|f_{alt}(t_s)/f_0(t_s)| \leq 2 |A/[1 - T_0^2 \omega^2]| / \{2A_0\} \quad (24)$$

which can be also written as

$$|f_{alt}(t_s)/f_0(t_s)| \leq \{A/A_0\} \{1/[1 - T_0^2 \omega^2]\} \quad (25)$$

The reverse of this ratio is

$$|f_0(t_s)/f_{alt}(t_s)| \geq |1 - T_0^2 \omega^2| \{A_0/A\} \quad (26)$$

corresponds to the signal-to-noise ratio of this filtering and sampling system. Since the time period T_0 of this oscillating second order system would be at least several times greater than the period T of the alternating input component to be filtered, it results that

$$T_0/T \geq 1, (T_0/T)^2 = T_0^2 \omega^2 \geq 1 \quad (27)$$

Thus,

$$|1 - T_0^2 \omega^2| = |T_0^2 \omega^2 - 1| = (T_0/T)^2 - 1 \quad (28)$$

Finally, the signal-to-noise ratio of this filtering and sampling second order system corresponds to

$$|f_0(t_S)/f_{alt}(t_S)| \geq \{(T_0/T)^2 - 1\} \{A_0/A\} \quad (29)$$

The inequality represented in Equation (29) shows that the input signal-to-noise ratio

$$(S/N)_{input} = A_0/A \quad (30)$$

(the ratio of amplitude of step input to amplitude of alternating input) is improved $\{(T_0/T)^2 - 1\}$ times so as to obtain the output signal-to noise-ratio

$$(S/N)_{output} = |f_0(t_S)/f_{alt}(t_S)| \quad (31)$$

(the ratio of modulus of sampled output generated by step input to the modulus of sampled output generated by alternating input).

The sampling moment of time being established at half-period, it results that a response time T_r of

$$T_r = \pi T_0 \approx 3.14 T_0 \quad (32)$$

can be considered for this oscillating system working on half-period.

After each working time of half-period, the initial null conditions should be restored before a new working time to start. Thus, possible instabilities due to the use of a second order system without damping on extended time intervals can be avoided.

4. Discussion

For an efficient attenuation of an alternating input of period T is necessary that the time constant T_0 of any second order oscillating system used for filtering to be several times greater than T (it should include several periods of this oscillating input). Preliminary we denote this required number by N .

This means $T_0 = N T$, $T_0/T = N$, so

$$\{(T_0/T)^2 - 1\} \approx (T_0/T)^2 = 4 N^2 \quad (33)$$

(the accuracy of approximation above being higher than 96% for $N > 5$). Thus, it can be considered that a filtering device based on an oscillating system working on half period improves the signal-to-noise ratio as

$$|f_0(t_S)/f_{alt}(t_S)| \approx (4N^2) \{A_0/A\} \quad (34)$$

(usually, N equals 4 ... 6 so $(T_0/T)^2$ is much greater than unity in right-hand-side of (29)), with the response time

$$T_r = \pi T_0 \approx 3.14 N T \quad (35)$$

For a method based on the integration of an oscillating second order system on a period (this means on $2\pi T_0$), the working time $T_{r(i)}$ equals

$$T_{r(i)} = 2\pi T_0 = N (2\pi T) \approx 6.28 N T \quad (36)$$

Since the signal-to-noise ratio for this system based on integration is represented by

$$|f_0(t_S)/f_{alt}(t_S)|_{(i)} \approx (2\pi)(\omega/\omega_0)^3 \{A_0/A\} \quad (37)$$

(according to [2]), by substituting $\omega_0 = 1/T_0$, $\omega = 1/T$ in Equation (37) it results

$$|f_0(t_S)/f_{alt}(t_S)|_{(i)} \approx (2\pi)(T_0/T)^3 \{A_0/A\} = (2\pi) N^3 \{A_0/A\} \quad (38)$$

If a damped second order system would have been used, the response time would have been $T_{r(d)} = 4T_0/\xi$. The time constant T_0 is considered to include the same number N of periods of the oscillating input (with amplitude A and angular frequency ω). Thus

$$T_{r(d)} = 4T_0/\xi = 4N T/\xi \approx 4 \dots 6 N T \quad (39)$$

An alternating input signal of angular frequency ω is attenuated about $(\omega/\omega_0)^2$ times (according to the basic theory of second order systems for ω much greater than ω_0), so the signal-to-noise ratio for a damped second order system is

$$|f_0(t_S)/f_{alt}(t_S)|_{(d)} \approx (\omega/\omega_0)^2 \{A_0/A\} = (T_0/T)^2 \{A_0/A\} \quad (40)$$

Using $T_0/T = N$ it results

$$|f_0(t_S)/f_{alt}(t_S)| \approx N^2 \{A_0/A\} \quad (41)$$

By comparing Equations (34) and (41) by one side, and Equations (35) and (39) by the other side, it results that an oscillating second order system working on half-period presents a signal-to-noise ratio about four times greater than a damped second order system, within a shorter response time (for the same time constant). By comparing Equations (34) and (37) it results that its signal-to-noise ratio is much lower than the ratio of a second order oscillating system based on integration. However, the lack of the integrator electronic circuit corresponds to a more robust structure working on a time interval reduced to 50% (according to Equations (34) and (35)). Thus, the filtering performances of an oscillating system working on half-period should be preliminarily checked before a supplementary integrator circuit (usually based on operational amplifiers) to be added. For $T_0/T = 5$ and $A = A_0 = 1$ (a significant alternating input) it results an alternating output (the noise) less than 1% (the same order of magnitude as for the error of the step output considered on the transient time interval of first or second order damped systems), according to Equations (33) and (34).

5. Conclusions

This paper has presented a filtering and sampling structure based on an oscillating system working on half-period from initial null conditions. The time constant of this second order system is chosen several times greater than the time constant of alternating signals to be filtered. By choosing the sampling time moment at half-period (instead of a full-time period as for oscillating systems connected to an integrator or after several time periods as for damped second order systems) the response time is significantly decreased. For an input represented by a useful continuous signal (to be sampled) the slope of the generated output equals zero at half-period (the sampling time moment). Thus, a robust sampling procedure is obtained without the use of additional integrators (as in previous studies on the same topics).

The structure is recommended for industrial applications as measuring velocities through electromagnetic induction in electrical drives, where significant alternating components are added to the continuous component.

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