

## Article

# Multi-Criteria Group Decision-Making Using an $m$ -Polar Hesitant Fuzzy TOPSIS Approach

Muhammad Akram <sup>1,\*</sup> , Arooj Adeel <sup>1</sup> and José Carlos R. Alcantud <sup>2</sup> 

<sup>1</sup> Department of Mathematics, University of the Punjab, New Campus, Lahore 54590, Pakistan; arooj\_adeel@ymail.com

<sup>2</sup> BORDA Research Unit and IME, Universidad de Salamanca, Salamanca 37008, Spain; jcr@usal.es

\* Correspondence: m.akram@pucit.edu.pk

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**Abstract:** The  $m$ -polar fuzzy sets ( $mF$  sets) have a representative and fundamental role in several fields of science and decision-making. The fusion of  $mF$  sets with several other theories of mathematics has become a favorable practice for depicting numerous types of uncertainties under multi-polar information. In this article, we introduce an innovative hybrid model, called  $m$ -polar hesitant fuzzy sets ( $mHF$ -sets), a hybridization of hesitancy and  $mF$  sets, which enables us to tackle multi-polar information with hesitancy. Hesitancy incorporates symmetry into the treatment of the data, whereas the  $m$ -polar fuzzy format allows for differentiated or asymmetric sources of information. We highlight and explore basic key properties of  $mHF$ -sets and formulate intrinsic operations. Moreover, we develop an  $m$ -polar hesitant fuzzy TOPSIS ( $mHF$ -TOPSIS) approach for multi-criteria group decision-making (MCGDM), which is a natural extension of the TOPSIS method to this framework. We describe applications of  $mHF$ -sets in group decision-making. Further, we show the efficiency of our proposed approach by applying it to the industrial field. Finally, we generate a computer programming code that implements our decision-making procedure for ease of lengthy calculations.

**Keywords:**  $m$ -Polar fuzzy set; hesitant fuzzy set;  $m$ -polar hesitant fuzzy set; decision-making; TOPSIS

## 1. Introduction

Most of the classical tools for conventional modeling, computing, and reasoning are absolute, deterministic, and classic in character. In classical set theory, an element can either be a member of a set or not; consequently, a solution is either achievable or not. However, we cannot take advantage of these conventional mathematical tools in order to overcome the uncertainties which arise in real-life problems. Therefore, a number of theories have been developed to account for vague forms of knowledge, including probability, rough set theory, and fuzzy set theory and its extensions. Our contribution is in line with the latter theory and allows us to avail ourselves of information fusion with a novel structure. To be precise, we first succeed in combining the advantages of two characteristics that have been added to the fuzzy set model of Zadeh [1]; namely, multi-polarity and hesitancy.

In 1965, Zadeh introduced a mathematical framework to discuss the phenomena of uncertainty and vagueness in existing problems. Zhang [2] enhanced the concept of fuzzy sets and proposed the idea of bipolar fuzzy sets, whose range of membership degrees is described in the interval  $[-1, 1]$ . As a generalization of bipolar fuzzy sets, Chen et al. [3] introduced the study of  $m$ -polar fuzzy (or  $mF$ ) sets, and concluded that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical approaches. The ethos supporting their method is that multi-polar knowledge (particularly, bipolar knowledge—which coincides with two-valued logic) occurs, as facts and knowledge in real world problems often originate from  $n \geq 2$  sources; for example, the correct degree of telecommunication, provided that human species are points in  $[0, 1]^n$  ( $n \approx 7 \times 10^9$ ), because distinct characters have been

supervised at distinct times. Other examples include the truth degree of two-logic formulae (which are based on  $n \geq 2$  logical implication operators), regulating results of universities or magazines, and inclusion degrees (rough measures, accuracy measures, fuzziness measures, approximation qualities, and decision pre-formation evaluations) of rough sets. Akram [4] introduced several novel concepts, including  $m$ -polar fuzzy graphs,  $m$ -polar fuzzy labeling graphs, and certain metrics in  $m$ -polar fuzzy graphs. Akram et al. [5–7] proposed multi-attribute decision-making methods based on  $mF$  rough and  $mF$  soft rough information.

Motivated by a different concern, Torra and Narukawa [8,9] introduced the notion of hesitant fuzzy sets (HFSs). This concept is reasonable for the modeling of situations where decision-makers hesitate to submit their estimations and judgments of objects, and also when we combine the assumptions of distinct experts into an individual input. Certainly, experts are hesitant or doubtful in many decision-making cases, a circumstance that prevents them from producing unequivocal assessments [10,11]. The concept of hesitancy is responsive to hybridization with other theories of uncertainty and vagueness [12–16]. The ideas of uncertainty or hesitancy in MCGDM have also been dealt with in the evidential reasoning framework [17,18]. Alcantud and Torra [19] discussed extension principles and decomposition theorems for HFSs. On information fusion in decision-making, Rodríguez et al. [20] provided a perspective analysis and position of HFSs. Chen et al. [21] generalized the concept of HFSs by proposing the idea of interval-valued hesitant fuzzy sets (IVHFSs). Xia and Xu [22] presented some aggregation operators and applications to tackle the hesitancy of multi-criteria decision-making (MCDM) problems. Zhang et al. [23] introduced the operations and integration of probabilistic hesitant fuzzy data in decision-making. Xia et al. [24] introduced some further hesitant fuzzy aggregation approaches and showed their feasibility in group decision-making. For the exploration of specialized literature and other hybrid models related to HFSs, readers are referred to [25–29].

Decision-making is considered to be a rational action depending upon specific reasoning, which leads to the election of the most suitable alternative from a set of required options in a decision situation [30]. The TOPSIS method, proposed in [31], highlighted the effective, favorable, and widely used decision results to handle MCGDM problems. The method of TOPSIS derives from the belief that the preferred alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. Since its introduction, several extended TOPSIS methods have been applied to different MCDM problems [32–37]. Xu and Zhang [38] established a new approach, based on TOPSIS and the maximizing deviation method, for the interpretation of MCDM problems. Ashtiani [39] developed an extension of the fuzzy TOPSIS method, based on interval-valued fuzzy sets. Wang and Lee [40] introduced the generalized TOPSIS method for fuzzy MCGDM. Chen [41] introduced an extended TOPSIS method for MCDM by considering triangular fuzzy numbers and defining the crisp Euclidean distance between two fuzzy numbers. Roszkowska [42] proposed to approach MCDM models by application of the TOPSIS method to crisp and interval data. Further, Roszkowska and Wachowicz [43] applied the fuzzy TOPSIS method to rate negotiation actions in poorly formatted negotiation problems. Ren et al. [44] developed a novel hesitant fuzzy linguistic TOPSIS method for group multi-criteria linguistic decision-making. Akram and Adeel [45] extended the TOPSIS approach for MAGDM, based on an interval-valued hesitant fuzzy  $N$ -soft environment. Adeel et al. [46] extended the TOPSIS approach to introduce an  $m$ -polar fuzzy linguistic TOPSIS approach for group decision-making. Further, Akram et al. proposed the novel concepts related to MCDM methods to enhance and support the theory of decision-making including [47,48].

The scope of our article is absolute. The methods proposed in the above-cited articles are unable to provide information about preferences, when data appear in multi-polar form and hesitancy is allowed in relation with them. Traditional methods are ineffectual to study this type of imprecise behavior of multiple computations and assessments having hesitant surroundings. In order to handle this kind of decision-making problems, in this article, we introduce the concept of  $mHF$ -sets with an associated novel approach of TOPSIS for MCGDM problems. The  $mHF$ -TOPSIS method is, therefore,

capable of dealing with problems when they incorporate multi-polar information, in terms of hesitancy. It is able to deal with pessimistic and optimistic decisions, in which decision-makers are free from any external conditions and requirements. In this method, all the aspects related to alternatives, according to the preferences of the decision-makers, are discussed. Our novel approach increases the affluence of multiple information incited by hesitancy and to choose the best alternative without a ranking relation of the remaining alternatives.

The organization of this research article is as follows: In Section 2, we introduce our hybrid model (namely,  $m$ HF-sets), we construct its basic operations and we also investigate some of its fundamental properties. In Section 3, we propose a TOPSIS approach based on  $m$ HF-sets and describe some potential applications of the proposed model. We also present our proposed method as an algorithm. In Section 4, we study theoretical comparison analysis of proposed approach. In Section 5, we present some conclusions and future directions for research.

## 2. $m$ -Polar Hesitant Fuzzy Set Model

In this section, we introduce our novel hybrid model, which is the combination of  $m$ F sets and HFSs. For detailed basic concepts of  $m$ F sets and HFSs, the readers are referred to [3,9]. Our proposed concept is designed to deal with a hesitant situation separately for each degree of membership in an  $m$ F set:

**Definition 1.** Let  $Z$  be a reference set, an  $m$ HF-set on  $Z$  is a function  $\hbar_m$  that returns a subset of values in  $[0, 1]^m$ :

$$\hbar_m : Z \rightarrow (\mathcal{P}\{[0, 1]^m\}).$$

Mathematically, an  $m$ HF-set is represented as follows:

$$H = \{\langle z, \hbar_m(z) \rangle \mid \forall z \in Z\},$$

where  $\hbar_m(z) = \left( \{\zeta_h \mid \zeta_h \in p_1 \circ \hbar_m(z)\}, \{\zeta_h \mid \zeta_h \in p_2 \circ \hbar_m(z)\}, \dots, \{\zeta_h \mid \zeta_h \in p_m \circ \hbar_m(z)\} \right)$ . This notation shows that  $\hbar_m(z)$  is an  $m$ -tuple of sets, having possible membership degrees of each element  $z \in Z$  in set  $H$ , where  $\hbar_m = \hbar_m(z)$  is called an  $m$ -polar hesitant fuzzy element ( $m$ HFE).

It is apparent that, when  $m = 1$ ,  $m$ HFEs are hesitant fuzzy elements (HFEs), as defined in [22], and  $m$ HFSs are standard HFSs. The following example illustrates the concepts above:

**Example 1.** Let  $Z = \{z_1, z_2, z_3\}$  be a reference set and

$$\hbar_m(z_1) = \left( \{0.3, 0.4\}, \{0.3, 0.5\}, \{0.4, 0.5, 0.65\} \right),$$

$$\hbar_m(z_2) = \left( \{0.1, 0.3, 0.5\}, \{0.2, 0.3, 0.7\}, \{0.1, 0.4\} \right),$$

$$\hbar_m(z_3) = \left( \{0.4, 0.55\}, \{0.5, 0.6\}, \{0.3, 0.4, 0.7, 0.75\} \right),$$

be respective 3-polar hesitant fuzzy elements (3HFEs). Then, a 3HF-set  $H$  is given as

$$H = \left\{ \left\langle z_1, \left( \{0.3, 0.4\}, \{0.3, 0.5\}, \{0.4, 0.5, 0.65\} \right) \right\rangle, \right. \\ \left. \left\langle z_2, \left( \{0.1, 0.3, 0.5\}, \{0.2, 0.3, 0.7\}, \{0.1, 0.4\} \right) \right\rangle, \right. \\ \left. \left\langle z_3, \left( \{0.4, 0.55\}, \{0.5, 0.6\}, \{0.3, 0.4, 0.7, 0.75\} \right) \right\rangle \right\}.$$

Some special  $m$ HFEs for  $z \in Z$  are given as follows:

1. Empty set:  $\hbar_m^e = (\{0\}_m)$ .
2. Full set:  $\hbar_m^f = (\{1\}_m)$ .
3. Complete ignorance: (All values are possible)  $\hbar_m = [\mathbf{0}, \mathbf{1}]$ , where  $\mathbf{0} = (0, 0, \dots, 0)$  and  $\mathbf{1} = (1, 1, \dots, 1)$ .
4. Nonsense set:  $\Phi$ .

The next subsection reports on the basic operations in the framework that we have defined. Then, Section 2.2 refers to the problem of comparing mHFEs.

### 2.1. Basic Operations

In this subsection, we define the basic operations of mHF-sets and describe them with an example.

1. Lower bound:

$$\hbar_m^-(z) = \left( \inf\{\zeta_h | \zeta_h \in p_1 \circ \hbar_m(z)\}, \inf\{\zeta_h | \zeta_h \in p_2 \circ \hbar_m(z)\}, \dots, \inf\{\zeta_h | \zeta_h \in p_m \circ \hbar_m(z)\} \right),$$

for all  $z \in Z$ .

2. Upper bound:

$$\hbar_m^+(z) = \left( \sup\{\zeta_h | \zeta_h \in p_1 \circ \hbar_m(z)\}, \sup\{\zeta_h | \zeta_h \in p_2 \circ \hbar_m(z)\}, \dots, \sup\{\zeta_h | \zeta_h \in p_m \circ \hbar_m(z)\} \right),$$

for all  $z \in Z$ .

3. Complement:

$$\hbar_m^c(z) = \left( \{1 - \zeta_h | \zeta_h \in p_1 \circ \hbar_m(z)\}, \{1 - \zeta_h | \zeta_h \in p_2 \circ \hbar_m(z)\}, \dots, \{1 - \zeta_h | \zeta_h \in p_m \circ \hbar_m(z)\} \right),$$

for all  $z \in Z$ .

4. Union:

$$(\hbar_m^{(H_1)} \cup \hbar_m^{(H_2)})(z) = \left( \{\zeta_h \in p_i \circ \hbar_m^{(H_1)}(z) \cup p_i \circ \hbar_m^{(H_2)}(z) | \zeta_h \geq \sup\{\hbar_m^{(H_1)-}(z), \hbar_m^{(H_2)-}(z)\}\} \right),$$

where  $p_i \circ \hbar_m^{(H_1)}(z) \in \hbar_m^{(H_1)}(z)$  and  $p_i \circ \hbar_m^{(H_2)}(z) \in \hbar_m^{(H_2)}(z)$ , for all  $z \in Z$  and  $i \in m$ .

5. Intersection:

$$(\hbar_m^{(H_1)} \cap \hbar_m^{(H_2)})(z) = \left( \{\zeta_h \in p_i \circ \hbar_m^{(H_1)}(z) \cap p_i \circ \hbar_m^{(H_2)}(z) | \zeta_h \leq \inf\{\hbar_m^{(H_1)+}(z), \hbar_m^{(H_2)+}(z)\}\} \right),$$

where  $p_i \circ \hbar_m^{(H_1)}(z) \in \hbar_m^{(H_1)}(z)$  and  $p_i \circ \hbar_m^{(H_2)}(z) \in \hbar_m^{(H_2)}(z)$ , for all  $z \in Z$  and  $i \in m$ .

6. Direct sum:

$$(\hbar_m^{(H_1)} \oplus \hbar_m^{(H_2)})(z) = \left( \{\zeta_{h_1} + \zeta_{h_2} - \zeta_{h_1}\zeta_{h_2} | \zeta_{h_1} \in p_i \circ \hbar_m^{(H_1)}(z), \zeta_{h_2} \in p_i \circ \hbar_m^{(H_2)}(z)\} \right),$$

where  $p_i \circ \hbar_m^{(H_1)}(z) \in \hbar_m^{(H_1)}(z)$  and  $p_i \circ \hbar_m^{(H_2)}(z) \in \hbar_m^{(H_2)}(z)$ , for all  $z \in Z$  and  $i \in m$ .

7. Direct product:

$$(\hbar_m^{(H_1)} \otimes \hbar_m^{(H_2)})(z) = \left( \{\zeta_{h_1}\zeta_{h_2} | \zeta_{h_1} \in p_i \circ \hbar_m^{(H_1)}(z), \zeta_{h_2} \in p_i \circ \hbar_m^{(H_2)}(z)\} \right),$$

where  $p_i \circ \hbar_m^{(H_1)}(z) \in \hbar_m^{(H_1)}(z)$  and  $p_i \circ \hbar_m^{(H_2)}(z) \in \hbar_m^{(H_2)}(z)$ , for all  $z \in Z$  and  $i \in m$ .

The following example illustrates the operations defined above:

**Example 2.** Let  $Z = \{z_1, z_2, z_3\}$  be the reference set. Then, two 3HF-sets  $H_1$  and  $H_2$  on  $Z$  are, respectively, given as

$$H_1 = \left\{ \left\langle z_1, \left( \{0.2, 0.3\}, \{0.4, 0.5, 0.6\}, \{0.4, 0.6\} \right) \right\rangle, \right. \\ \left. \left\langle z_2, \left( \{0.3, 0.5\}, \{0.4, 0.6\}, \{0.7, 0.8\} \right) \right\rangle, \right. \\ \left. \left\langle z_3, \left( \{0.1, 0.2\}, \{0.5, 0.6, 0.7\}, \{0.7, 0.8\} \right) \right\rangle \right\}, \text{ and}$$

$$H_2 = \left\{ \left\langle z_1, \left( \{0.4, 0.6, 0.7\}, \{0.6, 0.7\}, \{0.7, 0.8\} \right) \right\rangle, \right. \\ \left. \left\langle z_2, \left( \{0.5, 0.6\}, \{0.2, 0.3, 0.4\}, \{0.3, 0.5, 0.8, 0.9\} \right) \right\rangle, \right. \\ \left. \left\langle z_3, \left( \{0.3, 0.4\}, \{0.2, 0.4, 0.6\}, \{0.5, 0.7\} \right) \right\rangle \right\}.$$

The aforementioned operations on these two 3HF-sets  $H_1$  and  $H_2$  are calculated as follows:

1. Lower bound:

$$\begin{aligned} \hbar_m^{(H_1)^-}(z_1) &= \left( \inf\{0.2, 0.3\}, \inf\{0.4, 0.5, 0.6\}, \inf\{0.4, 0.6\} \right) \\ &= (0.2, 0.4, 0.4), \\ \hbar_m^{(H_2)^-}(z_3) &= \left( \inf\{0.3, 0.4\}, \inf\{0.2, 0.4, 0.6\}, \inf\{0.5, 0.7\} \right) \\ &= (0.3, 0.2, 0.5). \end{aligned}$$

2. Upper bound:

$$\begin{aligned} \hbar_m^{(H_1)^+}(z_2) &= \left( \sup\{0.3, 0.5\}, \sup\{0.4, 0.6\}, \sup\{0.7, 0.8\} \right) \\ &= (0.5, 0.6, 0.8), \\ \hbar_m^{(H_2)^+}(z_3) &= \left( \sup\{0.3, 0.4\}, \sup\{0.2, 0.4, 0.6\}, \sup\{0.5, 0.7\} \right) \\ &= (0.4, 0.6, 0.7). \end{aligned}$$

3. Complement:

$$\begin{aligned} \hbar_m^{(H_1)^c}(z_1) &= \left( \{1 - 0.2, 1 - 0.3\}, \{1 - 0.4, 1 - 0.5, 1 - 0.6\}, \{1 - 0.4, 1 - 0.6\} \right) \\ &= \left( \{0.8, 0.7\}, \{0.5, 0.6, 0.4\}, \{0.6, 0.4\} \right), \end{aligned}$$

$$\begin{aligned} \hbar_m^{(H_2)^c}(z_2) &= \left( \{1 - 0.5, 1 - 0.6\}, \{1 - 0.2, 1 - 0.3, 1 - 0.4\}, \{1 - 0.3, 1 - 0.5, 1 - 0.8, 1 - 0.9\} \right) \\ &= \left( \{0.5, 0.4\}, \{0.8, 0.7, 0.6\}, \{0.7, 0.5, 0.2, 0.1\} \right). \end{aligned}$$

## 4. Union:

$$\begin{aligned}
 (\hbar_m^{(H_1)} \cup \hbar_m^{(H_2)})(z_1) &= \max \left\{ (0.2, 0.4, 0.4), (0.4, 0.6, 0.7) \right\} = (0.4, 0.6, 0.7) \\
 &= \left( \{0.4, 0.6, 0.7\}, \{0.6, 0.7\}, \{0.7, 0.8\} \right), \\
 (\hbar_m^{(H_1)} \cup \hbar_m^{(H_2)})(z_3) &= \max \left\{ (0.1, 0.5, 0.7), (0.3, 0.2, 0.5) \right\} = (0.3, 0.5, 0.7) \\
 &= \left( \{0.3, 0.4\}, \{0.5, 0.6, 0.7\}, \{0.7, 0.8\} \right).
 \end{aligned}$$

## 5. Intersection:

$$\begin{aligned}
 (\hbar_m^{(H_1)} \cap \hbar_m^{(H_2)})(z_2) &= \max \left\{ (0.5, 0.6, 0.8), (0.6, 0.4, 0.9) \right\} = (0.5, 0.4, 0.8) \\
 &= \left( \{0.3, 0.5\}, \{0.2, 0.3, 0.4\}, \{0.3, 0.5, 0.7, 0.8\} \right), \\
 (\hbar_m^{(H_1)} \cap \hbar_m^{(H_2)})(z_3) &= \max \left\{ (0.2, 0.7, 0.8), (0.4, 0.6, 0.7) \right\} = (0.2, 0.6, 0.7) \\
 &= \left( \{0.1, 0.2\}, \{0.2, 0.4, 0.5, 0.6\}, \{0.5, 0.7\} \right).
 \end{aligned}$$

## 6. Direct sum:

$$\begin{aligned}
 (\hbar_m^{(H_1)} \oplus \hbar_m^{(H_2)})(z_1) &= \left( \{0.52, 0.68, 0.76, 0.79, 0.58, 0.72\}, \{0.76, 0.82, 0.8, 0.85, 0.84, 0.88\}, \right. \\
 &\quad \left. \{0.88, 0.82, 0.92\} \right).
 \end{aligned}$$

## 7. Direct product:

$$\begin{aligned}
 (\hbar_m^{(H_1)} \otimes \hbar_m^{(H_2)})(z_3) &= \left( \{0.03, 0.04, 0.06, 0.08\}, \{0.1, 0.2, 0.3, 0.12, 0.24, 0.36, 0.14, 0.28, 0.42\}, \right. \\
 &\quad \left. \{0.35, 0.49, 0.40, 0.56\} \right).
 \end{aligned}$$

The following propositions show that the union and intersection of  $m$ HFSs satisfy the commutativity, associativity, and idempotency properties, under limited conditions.

**Proposition 1.** For any  $m$ HFEs  $\hbar_m^{(1)}$ ,  $\hbar_m^{(2)}$ , and  $\hbar_m^{(3)}$  in  $H(Z)$  and  $z \in Z$ , we have

## 1. Commutativity:

- (i)  $(\hbar_m^{(1)} \cup \hbar_m^{(2)})(z) = (\hbar_m^{(2)} \cup \hbar_m^{(1)})(z),$
- (ii)  $(\hbar_m^{(1)} \cap \hbar_m^{(2)})(z) = (\hbar_m^{(2)} \cap \hbar_m^{(1)})(z).$

## 2. Associativity:

- (i)  $((\hbar_m^{(1)} \cup \hbar_m^{(2)}) \cup \hbar_m^{(3)})(z) = (\hbar_m^{(1)} \cup (\hbar_m^{(2)} \cup \hbar_m^{(3)}))(z),$
- (ii)  $((\hbar_m^{(1)} \cap \hbar_m^{(2)}) \cap \hbar_m^{(3)})(z) = (\hbar_m^{(1)} \cap (\hbar_m^{(2)} \cap \hbar_m^{(3)}))(z).$

## 3. Idempotency:

- (i)  $(\hbar_m^{(1)} \cup \hbar_m^{(1)})(z) = \hbar_m^{(1)}(z),$
- (ii)  $(\hbar_m^{(1)} \cap \hbar_m^{(1)})(z) = \hbar_m^{(1)}(z).$

**Proof.** All three described properties are trivial to prove.  $\square$

Next, we state some operational rules, in the form of Propositions:

**Proposition 2.** For any  $\hbar_m \in H(Z)$  and  $z \in Z$ , we have

1.  $(\hbar_m^c)^-(z) = \mathbf{1} - \hbar_m^+(z)$ ,
2.  $(\hbar_m^c)^+(z) = \mathbf{1} - \hbar_m^-(z)$ .

**Proof.**

$$\begin{aligned}
 1. \quad (\hbar_m^c)^-(z) &= \inf \hbar_m^c(z) \\
 &= \left( \inf \{1 - \zeta_h | \zeta_h \in p_1 \circ \hbar_m(z)\}, \inf \{1 - \zeta_h | \zeta_h \in p_2 \circ \hbar_m(z)\}, \right. \\
 &\quad \left. \dots, \inf \{\zeta_h | 1 - \zeta_h \in p_m \circ \hbar_m(z)\} \right), \forall z \in Z \\
 &= \mathbf{1} - \left( \sup \{\zeta_h | \zeta_h \in p_1 \circ \hbar_m(z)\}, \sup \{\zeta_h | \zeta_h \in p_2 \circ \hbar_m(z)\}, \right. \\
 &\quad \left. \dots, \sup \{\zeta_h | \zeta_h \in p_m \circ \hbar_m(z)\} \right), \forall z \in Z \\
 &= \mathbf{1} - \hbar_m^+(z). \\
 \\
 2. \quad (\hbar_m^c)^+(z) &= \sup \hbar_m^c(z) \\
 &= \left( \sup \{1 - \zeta_h | \zeta_h \in p_1 \circ \hbar_m(z)\}, \sup \{1 - \zeta_h | \zeta_h \in p_2 \circ \hbar_m(z)\}, \right. \\
 &\quad \left. \dots, \sup \{\zeta_h | 1 - \zeta_h \in p_m \circ \hbar_m(z)\} \right), \forall z \in Z \\
 &= \mathbf{1} - \left( \inf \{\zeta_h | \zeta_h \in p_1 \circ \hbar_m(z)\}, \inf \{\zeta_h | \zeta_h \in p_2 \circ \hbar_m(z)\}, \right. \\
 &\quad \left. \dots, \inf \{\zeta_h | \zeta_h \in p_m \circ \hbar_m(z)\} \right), \forall z \in Z \\
 &= \mathbf{1} - \hbar_m^-(z).
 \end{aligned}$$

$\square$

**Proposition 3.** For any  $\hbar_m \in H(Z)$  and  $z \in Z$ , we have

1.  $(\hbar_m \cup \hbar_m^f)(z) = \hbar_m^f(z)$  and  $(\hbar_m \cap \hbar_m^f)(z) = \hbar_m(z)$ ,
2.  $(\hbar_m \cup \hbar_m^e)(z) = \hbar_m(z)$  and  $(\hbar_m \cap \hbar_m^e)(z) = \hbar_m^e(z)$ .

**Proof.**

$$\begin{aligned}
 1. (\hbar_m \cup \hbar_m^f)(z) &= \left( \{\zeta_h \in p_i \circ \hbar_m(z) \cup \{1\}_{i \in m} \mid \zeta_h \geq \sup\{\hbar_m^-(z), (\{1\}_m)\}\} \right), \\
 &\quad \forall z \in Z \text{ and } i \in m \\
 &= \left( \{\zeta_h \in p_i \circ \hbar_m(z) \cup \{1\}_{i \in m} \mid \zeta_h \geq (\{1\}_m)\} \right), \forall z \in Z \text{ and } i \in m \\
 &= (\{1\}_m) = \hbar_m^f(z), \\
 (\hbar_m \cap \hbar_m^f)(z) &= \left( \{\zeta_h \in p_i \circ \hbar_m(z) \cap \{1\}_{i \in m} \mid \zeta_h \leq \inf\{\hbar_m^+(z), (\{1\}_m)\}\} \right), \\
 &\quad \forall z \in Z \text{ and } i \in m \\
 &= \left( \{\zeta_h \in p_i \circ \hbar_m(z) \cap \{1\}_{i \in m} \mid \zeta_h \leq \hbar_m^+(z)\} \right), \forall z \in Z \text{ and } i \in m \\
 &= \hbar_m(z).
 \end{aligned}$$

$$\begin{aligned}
 2. (\hbar_m \cup \hbar_m^e)(z) &= \left( \{\zeta_h \in p_i \circ \hbar_m(z) \cup \{0\}_{i \in m} \mid \zeta_h \geq \sup\{\hbar_m^-(z), (\{0\}_m)\}\} \right), \\
 &\quad \forall z \in Z \text{ and } i \in m \\
 &= \left( \{\zeta_h \in p_i \circ \hbar_m(z) \cup \{0\}_{i \in m} \mid \zeta_h \geq \hbar_m^-(z)\} \right), \forall z \in Z \text{ and } i \in m \\
 &= \hbar_m(z), \\
 (\hbar_m \cap \hbar_m^e)(z) &= \left( \{\zeta_h \in p_i \circ \hbar_m(z) \cap \{0\}_{i \in m} \mid \zeta_h \leq \inf\{\hbar_m^+(z), (\{0\}_m)\}\} \right), \\
 &\quad \forall z \in Z \text{ and } i \in m \\
 &= \left( \{\zeta_h \in p_i \circ \hbar_m(z) \cap \{0\}_{i \in m} \mid \zeta_h \leq (\{0\}_m)\} \right), \forall z \in Z \text{ and } i \in m \\
 &= (\{0\}_m) = \hbar_m^e(z).
 \end{aligned}$$

□

## 2.2. Comparison Laws of mHFEs

A score function is a standard tool for comparing hesitant fuzzy elements and generalized concepts such as mHFEs. The notion of score for a hesitant fuzzy element was first defined in [12] (Section 2.4), although there were some examples of typical scores in the previous literature. The properties of the scores of a hesitant fuzzy element were explored in [12] (Appendix A). Finally, [12] (Section 4) explained how scores can be applied to individual and group decision-making in NaP-HFSs, an extension of HFSs that uses necessary and possible information.

These ideas can be incorporated, in our case, by recourse to the following concept:

**Definition 2.** The score function  $s(\hbar_m)$  of the mHFEs of an mHF-set is defined as

$$s(\hbar_m) = \left( \frac{1}{\gamma^{p_i \circ \hbar_m(z)}} \sum_{\zeta_h \in p_i \circ \hbar_m(z)} \zeta_h \right), i \in m,$$

where  $\gamma^{p_i \circ \hbar_m(z)}$  is the number of elements in  $p_i \circ \hbar_m(z)$ .

The score function can help us compare mHFEs, according to the following rules:



**Remark 1.** For any two mHFEs  $\mathfrak{h}_m^{(1)}$  and  $\mathfrak{h}_m^{(2)}$  of an mHF-set:

- If  $s(\mathfrak{h}_m^{(1)}) > s(\mathfrak{h}_m^{(2)})$ , then  $\mathfrak{h}_m^{(1)}$  is superior to (or finer than)  $\mathfrak{h}_m^{(2)}$ .
- If  $s(\mathfrak{h}_m^{(1)}) < s(\mathfrak{h}_m^{(2)})$ , then  $\mathfrak{h}_m^{(1)}$  is inferior to (or weaker than)  $\mathfrak{h}_m^{(2)}$ .
- If  $s(\mathfrak{h}_m^{(1)}) = s(\mathfrak{h}_m^{(2)})$ , then  $\mathfrak{h}_m^{(1)}$  is indifferent to  $\mathfrak{h}_m^{(2)}$ .
- If none of the above are true, then  $\mathfrak{h}_m^{(1)}$  is totally different from  $\mathfrak{h}_m^{(2)}$ .

**Example 3.** Consider the two 3HFEs  $\mathfrak{h}_m^{(1)} = \left( \{0.4, 0.5\}, \{0.6, 0.8, 0.85\}, \{0.5, 0.7, 0.9\} \right)$  and  $\mathfrak{h}_m^{(2)} = \left( \{0.5, 0.6\}, \{0.7, 0.8, 0.9\}, \{0.5, 0.65, 0.7, 0.75\} \right)$ . Then, by Definition 2, the score functions of  $\mathfrak{h}_m^{(1)}$  and  $\mathfrak{h}_m^{(2)}$  are calculated as follows:

$$\begin{aligned} s(\mathfrak{h}_m^{(1)}) &= \left( \frac{0.4 + 0.5}{2}, \frac{0.6 + 0.8 + 0.85}{3}, \frac{0.5 + 0.7 + 0.9}{3} \right) \\ &= (0.45, 0.75, 0.7), \text{ and} \\ s(\mathfrak{h}_m^{(2)}) &= \left( \frac{0.5 + 0.6}{2}, \frac{0.7 + 0.8 + 0.9}{3}, \frac{0.5 + 0.65 + 0.7 + 0.75}{4} \right) \\ &= (0.55, 0.8, 0.65). \end{aligned}$$

From these calculations, one readily concludes that  $\mathfrak{h}_m^{(1)}$  is totally different to  $\mathfrak{h}_m^{(2)}$ .

However, the score function is not fully discriminative (see [12] (Section 2.4) for a formal discussion of the cause for this). Let us exemplify this feature:

**Example 4.** Consider the two 4HFE  $\mathfrak{h}_m^{(1)} = \left( \{0.2, 0.4, 0.6\}, \{0.1, 0.2\}, \{0.2, 0.3, 0.5, 0.6\}, \{0.7, 0.8, 0.9\} \right)$  and  $\mathfrak{h}_m^{(2)} = \left( \{0.1, 0.7\}, \{0.1, 0.15, 0.2\}, \{0.1, 0.5, 0.6\}, \{0.7, 0.8, 0.8, 0.9\} \right)$ . Then, by Definition 2, the score functions of  $\mathfrak{h}_m^{(1)}$  and  $\mathfrak{h}_m^{(2)}$  are calculated as follows:

$$\begin{aligned} s(\mathfrak{h}_m^{(1)}) &= (0.4, 0.15, 0.4, 0.8), \text{ and} \\ s(\mathfrak{h}_m^{(2)}) &= (0.4, 0.15, 0.4, 0.8). \end{aligned}$$

From these calculations, one observes that  $\mathfrak{h}_m^{(1)}$  is deemed indifferent to  $\mathfrak{h}_m^{(2)}$ , and we are unable to give formal support to the difference between  $\mathfrak{h}_m^{(1)}$  and  $\mathfrak{h}_m^{(2)}$  by the application of the score function alone.

In other words, the example above shows that, sometimes, we cannot perform a comparison when two mHFEs have coincident score functions, as computed by Definition 2. In order to break ties in such a situation, we define the deviation degree of an mHFE. In case of indifference between two mHFEs, this figure may tell us which one is superior.

**Definition 3.** The deviation degree  $\Delta(\mathfrak{h}_m)$  of the mHFEs of an mHF-set is defined as

$$\Delta(\mathfrak{h}_m) = \left( \left[ \frac{1}{\gamma^{p_i \circ \mathfrak{h}_m(z)}} \sum_{\zeta_h \in p_i \circ \mathfrak{h}_m(z)} (\zeta_h - s(\mathfrak{h}_m))^2 \right]^{\frac{1}{2}} \right), \quad i \in m,$$

where  $\gamma^{p_i \circ \mathfrak{h}_m(z)}$  is the number of elements in  $p_i \circ \mathfrak{h}_m(z)$ .

We are ready to refine the criterion in Remark 1, using the following terms:

**Remark 2.** For any two indifferent mHFEs  $\tilde{h}_m^{(1)}$  and  $\tilde{h}_m^{(2)}$  of an mHF-set:

- If  $\Delta(\tilde{h}_m^{(1)}) > \Delta(\tilde{h}_m^{(2)})$ , then  $\tilde{h}_m^{(1)}$  is superior to (or finer than)  $\tilde{h}_m^{(2)}$ .
- If  $\Delta(\tilde{h}_m^{(1)}) < \Delta(\tilde{h}_m^{(2)})$ , then  $\tilde{h}_m^{(1)}$  is inferior to (or weaker than)  $\tilde{h}_m^{(2)}$ .
- If  $\Delta(\tilde{h}_m^{(1)}) = \Delta(\tilde{h}_m^{(2)})$ , then  $\tilde{h}_m^{(1)}$  is indifferent to  $\tilde{h}_m^{(2)}$ .
- If none of the above are true, then  $\tilde{h}_m^{(1)}$  is completely different from  $\tilde{h}_m^{(2)}$ .

**Example 5.** By reconsidering Example 4 above, where  $s(\tilde{h}_m^{(1)}) = s(\tilde{h}_m^{(2)})$ , we calculate the deviation degrees of these mHFEs as follows:

$$\Delta(\tilde{h}_m^{(1)}) = (0.163, 0.05, 0.158, 0.082), \text{ and}$$

$$\Delta(\tilde{h}_m^{(2)}) = (0.3, 0.041, 0.216, 0.071).$$

From these calculations, we observe that none of  $\Delta(\tilde{h}_m^{(1)}) > \Delta(\tilde{h}_m^{(2)})$ ,  $\Delta(\tilde{h}_m^{(1)}) < \Delta(\tilde{h}_m^{(2)})$ , or  $\Delta(\tilde{h}_m^{(1)}) = \Delta(\tilde{h}_m^{(2)})$  are true. Thus,  $\tilde{h}_m^{(1)}$  is completely different from  $\tilde{h}_m^{(2)}$ .

### 3. The $m$ -Polar Hesitant Fuzzy TOPSIS Approach

In this section, we propose an  $m$ HF-TOPSIS approach for MCGDM, which is flexible and compatible with multi-polar data under hesitancy. Our proposed TOPSIS approach based on  $m$ HF-sets deals with MCGDM problems, in which we choose among a set  $A = \{a_1, a_2, \dots, a_p\}$  of different alternatives and  $C = \{c_1, c_2, \dots, c_q\}$  is the set of criteria which are distinguished by  $m$  different characteristics under a hesitant situation. The framework of the problem is as follows: The decision-makers are subject to evaluating the  $p$  different alternatives having  $q$  criteria, and the convenient valuations of the alternatives are determined having  $m$  different characteristics under  $r$  different membership values, due to hesitancy. We proceed to describe the steps for the proposed approach:

**Step 1:** The degree of each alternative ( $a_j \in A$ ,  $j = 1, 2, \dots, p$ ) over all criteria ( $c_k \in C$ ,  $k = 1, 2, \dots, q$ ) is given by mHFEs as

$$\tilde{h}_m^{jk}(z) = \tilde{h}_m^{jk} = \left( \{ \zeta_h | \zeta_h \in p_1 \circ \tilde{h}_m^{jk}(z) \}, \{ \zeta_h | \zeta_h \in p_2 \circ \tilde{h}_m^{jk}(z) \}, \dots, \{ \zeta_h | \zeta_h \in p_m \circ \tilde{h}_m^{jk}(z) \} \right),$$

where  $(p_i \circ \tilde{h}_m^{jk}(z) | i = 1, 2, \dots, m)$  classify the several other characteristics of each criterion. The tabular representation of the  $m$ HF decision matrix  $H$  is given by Table 1, which describes the ratings of alternatives.

**Table 1.** Tabular representation of a generic  $m$ -polar hesitant fuzzy set ( $m$ HF) decision matrix.

Alternatives	Criteria's			
	$c_1$	$c_2$	$\dots$	$c_q$
$a_1$	$\tilde{h}_m^{11}$	$\tilde{h}_m^{12}$	$\dots$	$\tilde{h}_m^{1q}$
$a_2$	$\tilde{h}_m^{21}$	$\tilde{h}_m^{22}$	$\dots$	$\tilde{h}_m^{2q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_p$	$\tilde{h}_m^{p1}$	$\tilde{h}_m^{p2}$	$\dots$	$\tilde{h}_m^{pq}$

For each possible  $j, k$  in Table 1,

$$\tilde{h}_m^{jk}(z) = \tilde{h}_m^{jk} = \left( \{ \zeta_h | \zeta_h \in p_1 \circ \tilde{h}_m^{jk}(z) \}, \{ \zeta_h | \zeta_h \in p_2 \circ \tilde{h}_m^{jk}(z) \}, \dots, \{ \zeta_h | \zeta_h \in p_m \circ \tilde{h}_m^{jk}(z) \} \right).$$

Note that, in general, the number of  $m$ HFEs is not comparable in all  $m$ HF-sets. In order to increase efficiency, we can prolong the largest or smallest membership values, until the lengths of all  $m$ HFEs become equal, as the decision-makers want to choose the best alternative in an optimistic or pessimistic spirit. For this reason, the information fusion shows an optimistic or pessimistic response and improves the  $m$ HF data by adding the maximal or minimal values.

**Step 2:** The decision-makers have the ability to attach weights to each criteria of alternatives, according to their experience and the priority of each criteria. The desired weights assigned by the decision-makers are

$$W = (w_1, w_2, \dots, w_q) \in (0, 1].$$

Weights assigned by the decision-makers satisfy a normalization condition; that is,

$$\sum_{k=1}^q w_k = 1.$$

Note that the only condition for the weights assigned by decision-makers is that the weights should be normalized. Readers are free to take the weights according to their own method and choice. It is not necessary to take two, three, or four digits after the decimal point, as we chose in examples in Sections 3.1 and 3.2. We take two or four values after the decimal point, for our convenience, which satisfy the normalized condition. These weights totally depend upon the choice of the decision-maker and the importance of the required criteria. In the case of a lack of information about these figures, we divide the weights equally.

**Step 3:** The weighted  $m$ HF decision matrix  $H'$  is calculated in Table 2.

**Table 2.** Tabular representation of a weighted  $m$ HF decision matrix.

Alternatives	Criteria's			
	$c_1$	$c_2$	$\dots$	$c_q$
	Weights			
	$w_1$	$w_2$	$\dots$	$w_q$
$a_1$	$\tilde{h}_m^{11'}$	$\tilde{h}_m^{12'}$	$\dots$	$\tilde{h}_m^{1q'}$
$a_2$	$\tilde{h}_m^{21'}$	$\tilde{h}_m^{22'}$	$\dots$	$\tilde{h}_m^{2q'}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_p$	$\tilde{h}_m^{p1'}$	$\tilde{h}_m^{p2'}$	$\dots$	$\tilde{h}_m^{pq'}$

For each possible  $j, k$  in Table 2,

$$\begin{aligned} \tilde{h}_m^{jk'} &= w_k \tilde{h}_m^{jk} \\ &= \left( w_k \{ \zeta_h | \zeta_h \in p_1 \circ \tilde{h}_m^{jk}(z) \}, w_k \{ \zeta_h | \zeta_h \in p_2 \circ \tilde{h}_m^{jk}(z) \}, \dots, w_k \{ \zeta_h | \zeta_h \in p_m \circ \tilde{h}_m^{jk}(z) \} \right) \\ &= \left( \{ \zeta'_h | \zeta'_h \in p_1 \circ \tilde{h}_m^{jk}(z) \}, \{ \zeta'_h | \zeta'_h \in p_2 \circ \tilde{h}_m^{jk}(z) \}, \dots, \{ \zeta'_h | \zeta'_h \in p_m \circ \tilde{h}_m^{jk}(z) \} \right). \end{aligned}$$

**Step 4:** The  $m$ HF positive ideal solution  $mHP_{IS}$  and  $m$ HF negative ideal solution  $mHN_{IS}$  of the alternatives, under our  $m$ HF environment, can be calculated by the following Equations (1) and (2):

$$mHP_{IS} = \{ (\tilde{h}_m^{1'})^+, (\tilde{h}_m^{2'})^+, \dots, (\tilde{h}_m^{q'})^+ \} \text{ and} \quad (1)$$

$$mHN_{IS} = \{(\hbar_m^{1'})^-, (\hbar_m^{2'})^-, \dots, (\hbar_m^{q'})^-\}, \quad (2)$$

where

$$\begin{aligned} (\hbar_m^{k'})^+ &= \max_j (\hbar_m^{k'}) \\ &= \left( \max_j \{ \zeta'_h | \zeta'_h \in p_1 \circ \hbar_m^{jk}(z) \}, \max_j \{ \zeta'_h | \zeta'_h \in p_2 \circ \hbar_m^{jk}(z) \}, \dots, \max_j \{ \zeta'_h | \zeta'_h \in p_m \circ \hbar_m^{jk}(z) \} \right) \\ &= \left( \{ (\zeta'_h)^+ | (\zeta'_h)^+ \in p_1 \circ \hbar_m^{jk}(z) \}, \{ (\zeta'_h)^+ | (\zeta'_h)^+ \in p_2 \circ \hbar_m^{jk}(z) \}, \dots, \{ (\zeta'_h)^+ | (\zeta'_h)^+ \in p_m \circ \hbar_m^{jk}(z) \} \right), \text{ and} \\ (\hbar_m^{k'})^- &= \min_j (\hbar_m^{k'}) \\ &= \left( \min_j \{ \zeta'_h | \zeta'_h \in p_1 \circ \hbar_m^{jk}(z) \}, \min_j \{ \zeta'_h | \zeta'_h \in p_2 \circ \hbar_m^{jk}(z) \}, \dots, \min_j \{ \zeta'_h | \zeta'_h \in p_m \circ \hbar_m^{jk}(z) \} \right) \\ &= \left( \{ (\zeta'_h)^- | (\zeta'_h)^- \in p_1 \circ \hbar_m^{jk}(z) \}, \{ (\zeta'_h)^- | (\zeta'_h)^- \in p_2 \circ \hbar_m^{jk}(z) \}, \dots, \{ (\zeta'_h)^- | (\zeta'_h)^- \in p_m \circ \hbar_m^{jk}(z) \} \right). \end{aligned}$$

**Step 5:** The *m*HF Euclidean distance of each alternative  $a_j$  from  $mHP_{IS}$  and  $mHN_{IS}$ , respectively, can be calculated by Equations (3) and (4):

$$D'_e(a_j, mHP_{IS}) = \sqrt{\frac{1}{rm} \sum_{k=1}^q \left[ \sum_{i=1}^m \left\{ (\zeta_{h1}^{jk'} - (\zeta_{h1}^{k'})^+)^2 + (\zeta_{h2}^{jk'} - (\zeta_{h2}^{k'})^+)^2 + \dots + (\zeta_{hr}^{jk'} - (\zeta_{hr}^{k'})^+)^2 \right\} \right]}, \quad (3)$$

where  $\zeta_{hl}^{jk'}$  and  $(\zeta_{hl}^{k'})^+ \in p_i \circ \hbar_m^{jk}(z)$ .

$$D'_e(a_j, mHN_{IS}) = \sqrt{\frac{1}{rm} \sum_{k=1}^q \left[ \sum_{i=1}^m \left\{ (\zeta_{h1}^{jk'} - (\zeta_{h1}^{k'})^-)^2 + (\zeta_{h2}^{jk'} - (\zeta_{h2}^{k'})^-)^2 + \dots + (\zeta_{hr}^{jk'} - (\zeta_{hr}^{k'})^-)^2 \right\} \right]}, \quad (4)$$

where,  $\zeta_{hl}^{jk'}$  and  $(\zeta_{hl}^{k'})^- \in p_i \circ \hbar_m^{jk}(z)$ .

**Step 6:** The relative *m*HF closeness coefficient of each alternative  $a_j$  can be computed by using following formula, as described in Equation (5):

$$E'_j = \frac{D'_e(a_j, mHN_{IS})}{D'_e(a_j, mHP_{IS}) + D'_e(a_j, mHN_{IS})}, \quad j = 1, 2, \dots, p. \quad (5)$$

The alternative with highest *m*HF closeness coefficient is best one, and we can determine the ranking order of each alternative.

We present our proposed method of decision-making in Algorithm 1:

**Algorithm 1** Algorithm of the proposed approach for multi-criteria group decision-making (MCGDM).**Step 1. Input**

$p$ : number of alternatives against  $m$ HF information.  
 $q$ : number of criteria.  
 $m$ : number of poles, according to characteristics.  
 $r$ : number of membership values due to hesitancy.  
 $w$ : weights, according to decision-makers.

**Step 2.** Compute an  $m$ HF decision matrix  $H$ .**Step 3.** Compute the weighted  $m$ HF decision matrix  $H'$ .**Step 4.** Compute the  $m$ HF positive ideal solution  $mHP_{IS}$ .**Step 5.** Compute the  $m$ HF negative ideal solution  $mHN_{IS}$ .**Step 6.** Compute the  $m$ HF distances of alternatives from  $mHP_{IS}$  and  $mHN_{IS}$ .**Step 7.** Compute the relative  $m$ HF closeness coefficients.**Step 8. Output**

Rank the alternatives for final decision and select the best one.

In Sections 3.1 and 3.2 below, we discuss the practical use of our proposed model; in particular, we show how  $m$ HF-TOPSIS is useful in the selection of a brand name and a product design for a company, respectively.

*3.1. Selection of a Perfect Brand Name*

In this subsection, we apply our decision model to a problem in strategic marketing; namely, the choice of a perfect brand name. This is one of the fundamental decisions when launching a new product into the market (especially if it is introduced under the umbrella of a new brand). A perfect brand name is not something that looks fine on a business card or a web banner, or is cool to say, or somebody likes it. It is perfect, when it conveys the right feelings to customers, from whom the demand for good brand names emanates. A good name can be the most prized property of a company. There exist many theories and have been many studies about what makes a good brand name, and common principles, which make a brand name simpler for the owner to use and easier for customers to remember, have also been established.

Ideally, one must take advantage of a fusion of information that derives from multi-polar advice under hesitant directions. An  $m$ HF-set handles all the characteristics and tools to select a brand name, in terms of  $m$  different numeric values under hesitant situations, due to the guidelines of different decision-makers or experts. For this purpose, we consider  $B_N = \{B_{n1}, B_{n2}, B_{n3}, B_{n4}, B_{n5}\}$  to be a set of five different brand names and  $C = \{c_1, c_2, c_3\}$  to be a set of three different evaluating criteria (or characteristics) of a brand name. According to the literature [27–29], the criteria can be described and characterized by respective features, as follows:

1. “Articulate core identity”, which may include the following features:
  - The “Vision”, or why your company exists;
  - the “Mission”, or what your company does;
  - the “Value”, or how you do what you do; and
  - the “Direction”, or where it goes on.
2. “Brainstorm”, which may include the following features:
  - The “Founder”, a name based on a real or fictional person;
  - the “Description”, a name that describes what you do or make;
  - the “Magic spell”, a name that is a portmanteau (two words together) or a real word with a made-up spelling; and
  - the “Fabricated”, a totally made-up name or word.
3. “Test”, which may include the following features:

- “Sounds good”, it is good to hear;
- “Not confusing”, it is not linked with other brand names;
- “Not mispronounced”, it is easy to pronounce; and
- “Related publicity”, it focuses on a targeted group of customers.

All these characteristics are assessed by a group of three different decision-makers, who are responsible for evaluating the evaluating the brand names. Due to their mutual decision, each criteria has a specified condition for further classification of described values into three hesitant values for any given candidate brand name. The decision-makers have the authority to choose further membership values from the interval  $[0, 1]$ ; their assigned values are described in Table 3. As described in Section 3, the desired computations of the 4HFEs are not proportionate with the 4HF-sets. In order to attain efficiency, we prolong the largest membership values until the lengths of all 4HFEs become equal, as the company wants to base the perfect brand name on an optimistic spirit. For this reason, the information fusion shows an optimistic response and improves the 4HF data by adding the maximal values, as mentioned in Table 4.

The weights that satisfy the normalized condition are given in Equation (6) below:

$$w = (0.23, 0.34, 0.43). \quad (6)$$

The weighted optimistic 4HF decision matrix is calculated in Table 5.

**Table 3.** Tabular representation of the 4HF decision matrix detailed in Section 3.1.

Brand Names	Articulate Core Identity			
	Vision	Mission	Value	Direction
$B_{n1}$	{0.40, 0.50}	{0.30, 0.60, 0.70}	{0.20, 0.70}	{0.30, 0.70, 0.80}
$B_{n2}$	{0.60, 0.70}	{0.30, 0.70}	{0.50, 0.60}	{0.40, 0.60, 0.80}
$B_{n3}$	{0.40}	{0.40, 0.50, 0.80}	{0.20, 0.30, 0.50}	{0.60, 0.80}
$B_{n4}$	{0.70, 0.80}	{0.60, 0.80}	{0.50}	{0.70, 0.80, 0.90}
$B_{n5}$	{0.40, 0.60}	{0.55, 0.70}	{0.40, 0.50, 0.70}	{0.75, 0.80}
Brand Names	Brainstorm			
	Founder	Descriptive	Magic spell	Fabricated
$B_{n1}$	{0.30, 0.70}	{0.40, 0.50, 0.80}	{0.60, 0.80}	{0.70, 0.80}
$B_{n2}$	{0.10, 0.20, 0.30}	{0.50, 0.60}	{0.10, 0.50}	{0.60, 0.80}
$B_{n3}$	{0.10, 0.15}	{0.20, 0.50}	{0.40}	{0.70, 0.80, 0.90}
$B_{n4}$	{0.40, 0.50}	{0.65, 0.70}	{0.40, 0.70}	{0.70, 0.80}
$B_{n5}$	{0.45, 0.50}	{0.50, 0.70}	{0.10, 0.20}	{0.50, 0.60, 0.70}
Brand Names	Test			
	Sounds good	Not confusing	Not mispronounced	Related publicity
$B_{n1}$	{0.40, 0.60}	{0.70, 0.80}	{0.30, 0.50}	{0.60, 0.70, 0.90}
$B_{n2}$	{0.30, 0.40, 0.60}	{0.20}	{0.40, 0.70}	{0.20, 0.30, 0.50}
$B_{n3}$	{0.30, 0.50, 0.70}	{0.50, 0.80}	{0.60, 0.90}	{0.50, 0.70}
$B_{n4}$	{0.20, 0.50}	{0.10, 0.25}	{0.60, 0.80}	{0.50, 0.60, 0.80}
$B_{n5}$	{0.60, 0.80, 0.90}	{0.50, 0.60}	{0.70}	{0.20, 0.30, 0.35}

**Table 4.** Tabular representation of the optimistic 4HF decision matrix by adding maximal values.

Brand Names	Articulate Core Identity			
	Vision	Mission	Value	Direction
$B_{n1}$	{0.40, 0.50, 0.50}	{0.30, 0.60, 0.70}	{0.20, 0.70, 0.70}	{0.30, 0.70, 0.80}
$B_{n2}$	{0.60, 0.70, 0.70}	{0.30, 0.70, 0.70}	{0.50, 0.60, 0.60}	{0.40, 0.60, 0.80}
$B_{n3}$	{0.40, 0.40, 0.40}	{0.40, 0.50, 0.80}	{0.20, 0.30, 0.50}	{0.60, 0.80, 0.80}
$B_{n4}$	{0.70, 0.80, 0.80}	{0.60, 0.80, 0.80}	{0.50, 0.50, 0.50}	{0.70, 0.80, 0.90}
$B_{n5}$	{0.40, 0.60, 0.60}	{0.55, 0.70, 0.70}	{0.40, 0.50, 0.70}	{0.75, 0.80, 0.80}
Brand Names	Brainstorm			
	Founder	Descriptive	Magic spell	Fabricated
$B_{n1}$	{0.30, 0.70, 0.70}	{0.40, 0.50, 0.80}	{0.60, 0.80, 0.80}	{0.70, 0.80, 0.80}
$B_{n2}$	{0.10, 0.20, 0.30}	{0.50, 0.60, 0.60}	{0.10, 0.50, 0.50}	{0.60, 0.80, 0.80}
$B_{n3}$	{0.10, 0.15, 0.15}	{0.20, 0.50, 0.50}	{0.40, 0.40, 0.40}	{0.70, 0.80, 0.90}
$B_{n4}$	{0.40, 0.500.50}	{0.65, 0.70, 0.70}	{0.40, 0.70, 0.70}	{0.70, 0.80, 0.80}
$B_{n5}$	{0.45, 0.50, 0.50}	{0.50, 0.70, 0.70}	{0.10, 0.20, 0.20}	{0.50, 0.60, 0.70}
Brand Names	Test			
	Sounds good	Not confusing	Not mispronounced	Related publicity
$B_{n1}$	{0.40, 0.60, 0.60}	{0.70, 0.80, 0.80}	{0.30, 0.50, 0.50}	{0.60, 0.70, 0.90}
$B_{n2}$	{0.30, 0.40, 0.60}	{0.20, 0.20, 0.20}	{0.40, 0.70, 0.70}	{0.20, 0.30, 0.50}
$B_{n3}$	{0.30, 0.50, 0.70}	{0.50, 0.80, 0.80}	{0.60, 0.90, 0.90}	{0.50, 0.70, 0.70}
$B_{n4}$	{0.20, 0.50, 0.50}	{0.10, 0.25, 0.25}	{0.60, 0.80, 0.80}	{0.50, 0.60, 0.80}
$B_{n5}$	{0.60, 0.80, 0.90}	{0.50, 0.60, 0.60}	{0.70, 0.70, 0.70}	{0.20, 0.30, 0.35}

**Table 5.** Tabular representation of the weighted optimistic 4HF decision matrix.

Brand Names	Articulate Core Identity			
	Vision	Mission	Value	Direction
$B_{n1}$	{0.0920, 0.1150, 0.1150}	{0.0690, 0.1380, 0.1610}	{0.0460, 0.1610, 0.1610}	{0.0690, 0.1610, 0.1840}
$B_{n2}$	{0.1380, 0.1610, 0.1610}	{0.0690, 0.1610, 0.1610}	{0.1150, 0.1380, 0.1380}	{0.0920, 0.1380, 0.1840}
$B_{n3}$	{0.0920, 0.0920, 0.0920}	{0.0920, 0.1150, 0.1840}	{0.0460, 0.0690, 0.1150}	{0.1380, 0.1840, 0.1840}
$B_{n4}$	{0.1610, 0.1840, 0.1840}	{0.1380, 0.1840, 0.1840}	{0.1150, 0.1150, 0.1150}	{0.1610, 0.1840, 0.2070}
$B_{n5}$	{0.0920, 0.1380, 0.1380}	{0.1265, 0.1610, 0.1610}	{0.0920, 0.1150, 0.1610}	{0.1725, 0.1840, 0.1840}
Brand Names	Brainstorm			
	Founder	Descriptive	Magic spell	Fabricated
$B_{n1}$	{0.1020, 0.2380, 0.2380}	{0.1360, 0.1700, 0.2720}	{0.2040, 0.2720, 0.2720}	{0.2380, 0.2720, 0.2720}
$B_{n2}$	{0.0340, 0.0680, 0.1020}	{0.1700, 0.2040, 0.2040}	{0.0340, 0.1700, 0.1700}	{0.2040, 0.2720, 0.2720}
$B_{n3}$	{0.0340, 0.0510, 0.0510}	{0.0680, 0.1700, 0.1700}	{0.1360, 0.1360, 0.1360}	{0.2380, 0.2720, 0.3060}
$B_{n4}$	{0.1360, 0.1700, 0.1700}	{0.2210, 0.2380, 0.2380}	{0.1360, 0.2380, 0.2380}	{0.2380, 0.2720, 0.2720}
$B_{n5}$	{0.1530, 0.1700, 0.1700}	{0.1700, 0.2380, 0.2380}	{0.0340, 0.0680, 0.0680}	{0.1700, 0.2040, 0.2380}
Brand Names	Test			
	Sounds good	Not confusing	Not mispronounced	Related publicity
$B_{n1}$	{0.1720, 0.2580, 0.2580}	{0.3010, 0.3440, 0.3440}	{0.1290, 0.2150, 0.2150}	{0.2580, 0.3010, 0.3870}
$B_{n2}$	{0.1290, 0.1720, 0.2580}	{0.0860, 0.0860, 0.0860}	{0.1720, 0.3010, 0.3010}	{0.0860, 0.1290, 0.2150}
$B_{n3}$	{0.1290, 0.2150, 0.3010}	{0.2150, 0.3440, 0.3440}	{0.2580, 0.3870, 0.3870}	{0.2150, 0.3010, 0.301}
$B_{n4}$	{0.0860, 0.2150, 0.2150}	{0.0430, 0.1075, 0.1075}	{0.2580, 0.3440, 0.3440}	{0.2150, 0.2580, 0.3440}
$B_{n5}$	{0.2580, 0.3440, 0.3870}	{0.2150, 0.2580, 0.2580}	{0.3010, 0.3010, 0.3010}	{0.0860, 0.1290, 0.1505}

We use Equations (1) and (2) to determine the  $4HP_{IS}$  and  $4HN_{IS}$ , respectively:

$$4HP_{IS} = \left\{ \left( \{0.1610, 0.1840, 0.1840\}, \{0.1380, 0.1840, 0.1840\}, \{0.1150, 0.1610, 0.1610\}, \{0.1725, 0.1840, 0.2070\} \right), \right. \\ \left( \{0.1530, 0.2380, 0.2380\}, \{0.2210, 0.2380, 0.2720\}, \{0.2040, 0.2720, 0.2720\}, \{0.2380, 0.2720, 0.3060\} \right), \\ \left( \{0.2580, 0.3440, 0.3870\}, \{0.3010, 0.3440, 0.3440\}, \{0.3010, 0.3870, 0.3870\}, \{0.2580, 0.3010, 0.3870\} \right) \Big\}. \\ 4HN_{IS} = \left\{ \left( \{0.0920, 0.0920, 0.0920\}, \{0.0690, 0.1150, 0.1610\}, \{0.0460, 0.0690, 0.1150\}, \{0.0690, 0.1380, 0.1840\} \right), \right. \\ \left( \{0.0340, 0.0510, 0.0510\}, \{0.0680, 0.1700, 0.1700\}, \{0.0340, 0.0680, 0.0680\}, \{0.1700, 0.2040, 0.2380\} \right), \\ \left( \{0.0860, 0.1720, 0.2150\}, \{0.0430, 0.0860, 0.0860\}, \{0.1290, 0.2150, 0.2150\}, \{0.0860, 0.1290, 0.1505\} \right) \Big\}.$$

We use Equations (3) and (4) to calculate the 4HF Euclidean distances of the brand names from  $4HP_{IS}$  and  $4HN_{IS}$ , producing the following figures:

$$\begin{aligned} D'_e(B_{n1}, 4HP_{IS}) &= 0.1204, & D'_e(B_{n1}, 4HN_{IS}) &= 0.2153, \\ D'_e(B_{n2}, 4HP_{IS}) &= 0.2045, & D'_e(B_{n2}, 4HN_{IS}) &= 0.0885, \\ D'_e(B_{n3}, 4HP_{IS}) &= 0.1497, & D'_e(B_{n3}, 4HN_{IS}) &= 0.1732, \\ D'_e(B_{n4}, 4HP_{IS}) &= 0.1550, & D'_e(B_{n4}, 4HN_{IS}) &= 0.1641, \\ D'_e(B_{n5}, 4HP_{IS}) &= 0.1593, & D'_e(B_{n5}, 4HN_{IS}) &= 0.1615. \end{aligned}$$

Using Equation (5), we calculate the relative 4HF closeness coefficients  $E_j$  of the brand names:

$$\begin{aligned} E'_1 &= 0.6413, & E'_2 &= 0.3021, \\ E'_3 &= 0.5364, & E'_4 &= 0.5142, \\ E'_5 &= 0.5034. \end{aligned}$$

For the comparison, we arrange the brand names  $\{B_{nj} | j = 1, 2, \dots, 5\}$  according to the ranking in the 4HF closeness coefficients; that is,

$$B_{n1} > B_{n3} > B_{n4} > B_{n5} > B_{n2}.$$

Hence,  $B_{n1}$  is the perfect brand name, according to this ranking.

### 3.2. Selection of Suitable Product Design for a Company

In this subsection, we focus on product design, which is an action to determine the unique aspect and attributes of a product. We also discuss the process of selection, which is the development of techniques to manufacture the designed product, because these two processes are usually designed together. Product design and its selection process are induced by the quality of the product, its cost, and customer satisfaction. If the product design is not suitable and its manufacturing process is not appropriate, then the quality of the product may suffer. Further, products are composed and synthesized by using materials, machinery, and labor expertise, which should be valuable, productive, and profitable. We call this the product composition, according to which the product can be manufactured. Finally, if the product accomplishes customer satisfaction, it should have the associated features of good design, the capacity to fulfill the needs of a market, and competitive prices.



An  $m$ HF-set deals with all of the characteristics of a product, in terms of  $m$  different numeric or fuzzy values under hesitant situations by different decision-makers or experts. For this purpose, we consider  $P_D = \{P_{d1}, P_{d2}, P_{d3}, P_{d4}\}$  to be a set of four different product designs and  $C = \{c_1, c_2, c_3, c_4\}$  to be a set of four different evaluation criteria or characteristics of product design. Product design defines the aspects of a product, according to the specialized literature described in ([www.google.com/search?q=selection+of+product+design](http://www.google.com/search?q=selection+of+product+design)) having characteristics such as appearance, the materials it is made of, its dimensions and tolerances, and its performance standards. These characteristics or criteria are further classified, as follows:

1. The "Appearance" of a product design may include the following features:
  - "Contrast and symmetry";
  - "Color and shade"; and
  - "Body texture and surface".
2. The "Material" of a product design may include the following features:
  - "Fine quality";
  - "Low cost"; and
  - "Reversibility".
3. The "Dimensions and Tolerances" of a product design may include the following features:
  - "Size and functions";
  - "Flexibility"; and
  - "Nominal geometry".
4. The "Performance Standards" of a product design may include the following features:
  - "Market value";
  - "Customer satisfaction"; and
  - "Availability and evaluating report".

In our example, all these characteristics are assessed by a group of four different experts or decision-makers, who are responsible for evaluating the product designs. Due to their mutual decision, each criteria has a specified condition for further classification of described values into four hesitant values for any single product design. The decision-makers have the authority to choose further membership values in the interval  $[0, 1]$ ; their assigned values are described in Table 6. As described in Section 3, the desired computations of the 3HFEs are not proportionate with the 3HF-sets. In order to attain efficiency, we prolong the smallest membership values until the lengths of all 3HFEs become equal as, in this case, the company wants to base the best product design on a pessimistic decision. Now, the information fusion is pessimistically responsive and, so, we reform the 3HF data by adding the minimal values, as given in Table 7.

The weights that satisfy the normalized condition are given by Equation (7), below:

$$w = (0.2012, 0.2259, 0.2631, 0.3098). \quad (7)$$

The weighted pessimistic 3HF decision matrix is calculated in Table 8.

Table 6. Tabular representation of the 3HF decision matrix detailed in Section 3.2.

Product Design	Appearance		
	Contrast and Symmetry	Color and Shade	Body Texture and Surface
$P_{d1}$	{0.25, 0.45, 0.47}	{0.30, 0.31, 0.36}	{0.20, 0.25, 0.26}
$P_{d2}$	{0.46, 0.48, 0.49}	{0.47, 0.49}	{0.55, 0.60, 0.61, 0.63}
$P_{d3}$	{0.51, 0.53, 0.57, 0.60}	{0.46, 0.52, 0.70}	{0.29, 0.30, 0.51, 0.52}
$P_{d4}$	{0.39, 0.41, 0.43}	{0.60, 0.68, 0.71, 0.73}	{0.50, 0.67, 0.69}
Product Design	Material		
	Fine Quality	Low Cost	Reversibility
$P_{d1}$	{0.45, 0.49, 0.51, 0.59}	{0.67, 0.68, 0.71}	{0.50, 0.56, 0.63, 0.64}
$P_{d2}$	{0.49, 0.50}	{0.71, 0.74, 0.79}	{0.35, 0.59, 0.61, 0.65}
$P_{d3}$	{0.71, 0.73, 0.77}	{0.46, 0.52, 0.70}	{0.29, 0.30, 0.51, 0.52}
$P_{d4}$	{0.53, 0.54, 0.56, 0.58}	{0.60, 0.63, 0.73, 0.79}	{0.40, 0.47, 0.49}
Product Design	Dimension and Tolerance		
	Size and Functions	Flexibility	Nominal Geometry
$P_{d1}$	{0.85, 0.86, 0.87}	{0.53, 0.59, 0.66}	{0.72, 0.75, 0.76, 0.78}
$P_{d2}$	{0.66, 0.68, 0.69}	{0.47, 0.50, 0.51, 0.64}	{0.65, 0.66, 0.81}
$P_{d3}$	{0.51, 0.55}	{0.66, 0.68, 0.75, 0.76}	{0.39, 0.40, 0.58, 0.62}
$P_{d4}$	{0.59, 0.61, 0.73, 0.74}	{0.26, 0.38, 0.41, 0.43}	{0.51, 0.77}
Product Design	Performance Standards		
	Market Value	Customer Satisfaction	Availability/Evaluating Report
$P_{d1}$	{0.55, 0.65}	{0.40, 0.48, 0.60, 0.61}	{0.80, 0.85, 0.86}
$P_{d2}$	{0.54, 0.58, 0.59, 0.61}	{0.77, 0.79, 0.84}	{0.55, 0.60, 0.68}
$P_{d3}$	{0.81, 0.83, 0.87}	{0.56, 0.62, 0.70}	{0.69, 0.70, 0.76, 0.82}
$P_{d4}$	{0.37, 0.48, 0.49, 0.59}	{0.26, 0.38, 0.41, 0.43}	{0.60, 0.67}

Table 7. Tabular representation of the pessimistic 3HF decision matrix by adding minimal values.

Product Design	Appearance		
	Contrast and Symmetry	Color and Shade	Body Texture and Surface
$P_{d1}$	{0.25, 0.25, 0.45, 0.47}	{0.30, 0.30, 0.31, 0.36}	{0.20, 0.20, 0.25, 0.26}
$P_{d2}$	{0.46, 0.46, 0.48, 0.49}	{0.47, 0.47, 0.47, 0.49}	{0.55, 0.60, 0.61, 0.63}
$P_{d3}$	{0.51, 0.53, 0.57, 0.60}	{0.46, 0.46, 0.52, 0.70}	{0.29, 0.30, 0.51, 0.52}
$P_{d4}$	{0.39, 0.39, 0.41, 0.43}	{0.60, 0.68, 0.71, 0.73}	{0.50, 0.50, 0.67, 0.69}
Product Design	Material		
	Fine Quality	Low Cost	Reversibility
$P_{d1}$	{0.45, 0.49, 0.51, 0.59}	{0.67, 0.67, 0.68, 0.71}	{0.50, 0.56, 0.63, 0.64}
$P_{d2}$	{0.49, 0.49, 0.49, 0.50}	{0.71, 0.71, 0.74, 0.79}	{0.35, 0.59, 0.61, 0.65}
$P_{d3}$	{0.71, 0.71, 0.73, 0.77}	{0.46, 0.46, 0.52, 0.70}	{0.29, 0.30, 0.51, 0.52}
$P_{d4}$	{0.53, 0.54, 0.56, 0.58}	{0.60, 0.63, 0.73, 0.79}	{0.40, 0.40, 0.47, 0.49}
Product Design	Dimension and Tolerance		
	Size and Functions	Flexibility	Nominal Geometry
$P_{d1}$	{0.85, 0.85, 0.86, 0.87}	{0.53, 0.53, 0.59, 0.66}	{0.72, 0.75, 0.76, 0.78}
$P_{d2}$	{0.66, 0.66, 0.68, 0.69}	{0.47, 0.50, 0.51, 0.64}	{0.65, 0.65, 0.66, 0.81}
$P_{d3}$	{0.51, 0.51, 0.51, 0.55}	{0.66, 0.68, 0.75, 0.76}	{0.39, 0.40, 0.58, 0.62}
$P_{d4}$	{0.59, 0.61, 0.73, 0.74}	{0.26, 0.38, 0.41, 0.43}	{0.51, 0.51, 0.51, 0.77}
Product Design	Performance Standards		
	Market Value	Customer Satisfaction	Availability/Evaluating Report
$P_{d1}$	{0.55, 0.55, 0.55, 0.65}	{0.40, 0.48, 0.60, 0.61}	{0.80, 0.80, 0.85, 0.86}
$P_{d2}$	{0.54, 0.58, 0.59, 0.61}	{0.77, 0.77, 0.79, 0.84}	{0.55, 0.55, 0.60, 0.68}
$P_{d3}$	{0.81, 0.81, 0.83, 0.87}	{0.56, 0.56, 0.62, 0.70}	{0.69, 0.70, 0.76, 0.82}
$P_{d4}$	{0.37, 0.48, 0.49, 0.59}	{0.26, 0.38, 0.41, 0.43}	{0.60, 0.60, 0.60, 0.67}

**Table 8.** Tabular representation of the weighted pessimistic 3HF decision matrix.

Product Design	Appearance		
	Contrast and Symmetry	Color and Shade	Body Texture and Surface
$P_{d1}$	{0.0503, 0.0503, 0.0905, 0.0946}	{0.0604, 0.0604, 0.0624, 0.0724}	{0.0402, 0.0402, 0.0503, 0.0523}
$P_{d2}$	{0.0926, 0.0926, 0.0966, 0.0986}	{0.0946, 0.0946, 0.0946, 0.0986}	{0.1107, 0.1207, 0.1227, 0.1268}
$P_{d3}$	{0.1026, 0.1066, 0.1147, 0.1207}	{0.0926, 0.0926, 0.1046, 0.1408}	{0.0583, 0.0604, 0.1026, 0.1046}
$P_{d4}$	{0.0785, 0.0785, 0.0825, 0.0865}	{0.1207, 0.1368, 0.1429, 0.1469}	{0.1006, 0.1006, 0.1348, 0.1388}
Product Design	Material		
	Fine Quality	Low Cost	Reversibility
$P_{d1}$	{0.1017, 0.1107, 0.1152, 0.1333}	{0.1514, 0.1514, 0.1536, 0.1604}	{0.1129, 0.1265, 0.1423, 0.1446}
$P_{d2}$	{0.1107, 0.1107, 0.1107, 0.1129}	{0.1604, 0.1604, 0.1672, 0.1785}	{0.0791, 0.1333, 0.1378, 0.1468}
$P_{d3}$	{0.1604, 0.1604, 0.1649, 0.1739}	{0.1039, 0.1039, 0.1175, 0.1581}	{0.0655, 0.0678, 0.1152, 0.1175}
$P_{d4}$	{0.1197, 0.1220, 0.1265, 0.1310}	{0.1355, 0.1423, 0.1649, 0.1785}	{0.0904, 0.0904, 0.1062, 0.1107}
Product Design	Dimension and Tolerance		
	Size and Functions	Flexibility	Nominal Geometry
$P_{d1}$	{0.2236, 0.2236, 0.2263, 0.2289}	{0.1394, 0.1394, 0.1552, 0.1736}	{0.1894, 0.1973, 0.2000, 0.2052}
$P_{d2}$	{0.1736, 0.1736, 0.1789, 0.1815}	{0.1237, 0.1316, 0.1342, 0.1684}	{0.1710, 0.1710, 0.1736, 0.2131}
$P_{d3}$	{0.1342, 0.1342, 0.1342, 0.1447}	{0.1736, 0.1789, 0.1973, 0.2000}	{0.1026, 0.1052, 0.1526, 0.1631}
$P_{d4}$	{0.1552, 0.1605, 0.1921, 0.1947}	{0.0684, 0.1000, 0.1079, 0.1131}	{0.1342, 0.1342, 0.1342, 0.2026}
Product Design	Performance Standards		
	Market Value	Customer Satisfaction	Availability/Evaluating Report
$P_{d1}$	{0.1704, 0.1704, 0.1704, 0.2014}	{0.1239, 0.1487, 0.1859, 0.1890}	{0.2478, 0.2478, 0.2633, 0.2664}
$P_{d2}$	{0.1673, 0.1797, 0.1828, 0.1890}	{0.2385, 0.2385, 0.2447, 0.2602}	{0.1704, 0.1704, 0.1859, 0.2107}
$P_{d3}$	{0.2509, 0.2509, 0.2571, 0.2695}	{0.1735, 0.1735, 0.1921, 0.2169}	{0.2138, 0.2169, 0.2354, 0.2540}
$P_{d4}$	{0.1146, 0.1487, 0.1518, 0.1828}	{0.0805, 0.1177, 0.1270, 0.1332}	{0.1859, 0.1859, 0.1859, 0.2076}

We use Equations (1) and (2) to determine the  $3HP_{IS}$  and  $3HN_{IS}$ , respectively.

$$\begin{aligned}
 3HP_{IS} = & \left\{ \left( \{0.1026, 0.1066, 0.1147, 0.1207\}, \{0.1207, 0.1368, 0.1429, 0.1469\}, \{0.1107, 0.1207, 0.1348, 0.1388\} \right), \right. \\
 & \left( \{0.1604, 0.1604, 0.1649, 0.1739\}, \{0.1604, 0.1604, 0.1672, 0.1785\}, \{0.1129, 0.1333, 0.1423, 0.1468\} \right), \\
 & \left( \{0.2236, 0.2236, 0.2263, 0.2289\}, \{0.1736, 0.1789, 0.1973, 0.2000\}, \{0.1894, 0.1973, 0.2000, 0.2131\} \right), \\
 & \left. \left( \{0.2509, 0.2509, 0.2571, 0.2695\}, \{0.2385, 0.2385, 0.2447, 0.2602\}, \{0.2478, 0.2478, 0.2633, 0.2664\} \right) \right\}. \\
 3HN_{IS} = & \left\{ \left( \{0.0503, 0.0503, 0.0825, 0.0865\}, \{0.0604, 0.0604, 0.0624, 0.0724\}, \{0.0402, 0.0402, 0.0503, 0.0523\} \right), \right. \\
 & \left( \{0.1017, 0.1107, 0.1107, 0.1129\}, \{0.1039, 0.1039, 0.1175, 0.1581\}, \{0.0655, 0.0678, 0.1062, 0.1107\} \right), \\
 & \left( \{0.1342, 0.1342, 0.1342, 0.1447\}, \{0.0684, 0.1000, 0.1079, 0.1131\}, \{0.1026, 0.1052, 0.1342, 0.1631\} \right), \\
 & \left. \left( \{0.1146, 0.1487, 0.1518, 0.1828\}, \{0.0805, 0.1177, 0.1270, 0.1332\}, \{0.1704, 0.1704, 0.1859, 0.2076\} \right) \right\}.
 \end{aligned}$$

We use Equations (3) and (4) to calculate the 3HF Euclidean distances of the product designs from  $3HP_{IS}$  and  $3HN_{IS}$ . They produce the following figures:

$$\begin{aligned}
 D'_e(P_{d1}, 3HP_{IS}) &= 0.1023, & D'_e(P_{d1}, 3HN_{IS}) &= 0.0983, \\
 D'_e(P_{d2}, 3HP_{IS}) &= 0.0854, & D'_e(P_{d2}, 3HN_{IS}) &= 0.1110,
 \end{aligned}$$

$$\begin{aligned} D'_e(P_{d3}, 3HP_{IS}) &= 0.0907, & D'_e(P_{d3}, 3HN_{IS}) &= 0.1111, \\ D'_e(P_{d4}, 3HP_{IS}) &= 0.1311, & D'_e(P_{d4}, 3HN_{IS}) &= 0.0725. \end{aligned}$$

Using Equation (5), we calculate the relative 3HF closeness coefficients  $E_j$  of the product designs:

$$\begin{aligned} E'_1 &= 0.4899, & E'_2 &= 0.5654, \\ E'_3 &= 0.5504, & E'_4 &= 0.3563. \end{aligned}$$

For the comparison, we arrange the product designs  $\{P_{dj} | j = 1, 2, \dots, 4\}$  according to the ranking of their 3HF closeness coefficients; that is,

$$P_{d2} > P_{d3} > P_{d1} > P_{d4}.$$

Hence, the product design  $P_{d2}$  is selected for manufacture.

#### 4. Comparison Analysis of Proposed Approach

In this section, we theoretically describe the comparison analysis of our proposed approach with the  $m$ -polar fuzzy ( $mF$ ) linguistic TOPSIS method proposed in [46].

1. All previously proposed TOPSIS methods for decision-making were not suitable for such situations, where the alternatives are assessed depending on hesitant situations of decision-makers, under the conditions of huge data with multi-polar information. An  $mHF$ -TOPSIS method is able to deal with these situations, having such kinds of multi-polar data under hesitant situations. This method is also preferable, because it is able to deal with both pessimistic and optimistic decisions, in which the decision-makers are free from any external conditions and requirements. In this method, all aspects related to alternatives, according to the preferences of the decision-makers, are discussed. The proposed approach is able to provide more flexible and precise results, in order to choose the best alternative considering multi-polar information under hesitancy. Although its calculations are complex and difficult to handle, we have generated a computer programming code to make these complex calculations easier.
2. An  $mF$  linguistic TOPSIS method is also considered as a flexible approach, as compared to various other extensions of TOPSIS, but this approach is limited, as a linguistic variable and its values are considered as fixed criteria for the evaluation and ranking of alternatives. This approach is valid only when the alternatives have linguistic variables and corresponding values. In this method, the alternatives are assessed depending on the linguistic values of a variable, which are further classified by  $m$  different characteristics. This approach is only able to observe and recognize expertise about the linguistic variable and the values of alternatives, in the form of words and sentences having multi-polar information. It is unable to provide any information about the hesitant situation of a decision. This approach is unable to discuss general cases, other than those with linguistic values and variables.

#### 5. Conclusions

In a short period of time, the powerful simplicity of HFSs has fascinated many researchers, as a number of cases, in different real-world problems, depend upon a hesitant framework and are not smooth and uncomplicated situations. However, in the present case of interest, this approach is incapable of dealing with problems having multi-polar data. In order to facilitate the professional and to tackle the problem of multi-polar information under hesitancy, we have proposed a model, called  $mHF$ -set, which is able to facilitate decisions in problems and situations with knowledge of  $m$  different numeric or fuzzy values under hesitant surroundings. We have also defined the basic operations and investigated some of its properties. From a practical perspective and to handle MCGDM problems, we have developed an  $mHF$ -TOPSIS approach, which is a natural extension of the TOPSIS method

and is able to assess alternatives depending on the hesitant situations of decision-makers, under the conditions of huge data with multi-polar information. Finally, we have generated a computer programming code that implements our decision-making procedure for ease of lengthy calculations. Our future work will be targeted at the exploration of real-life applications related to the concept based on (1)  $m$ -polar hesitant fuzzy rough sets, and (2) the TOPSIS method for MCDM with  $m$ -polar hesitant fuzzy rough sets.

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## Appendix A

We show the computer programming code of our proposed approach in Table A1, using MATLAB R2014a.

**Table A1.** MATLAB computer programming code of proposed approach for MCGDM.

MATLAB Computer Programming Code of Proposed Approach	
1.	clc
2.	m=input('enter the total number of poles');
3.	H=input('enter the $r \times m$ decision matrix in each entry');
4.	w=input('enter the weights as dimension $1 \times q$ ');
5.	[u,q]=size(w);
6.	[p,v]=size(H);
7.	$r = v / (q * m)$ ;
8.	if sum(w,2)==1
9.	W=zeros(p,v);
10.	for j=1:p
11.	for k=1:q
12.	for v1=k*r*m-(r*m-1):k*r*m
13.	W(j,v1)=w(1,k).*H(j,v1);
14.	end
15.	end
16.	end
17.	W
18.	mHPIS=zeros(1,v); mHNIS=ones(1,v);
19.	for j=1:p
20.	for v1=1:v
21.	mHPIS(1,v1)=max(mHPIS(1,v1),W(j,v1));
22.	mHNIS(1,v1)=min(mHNIS(1,v1),W(j,v1));
23.	end
24.	end
25.	mHPIS
26.	mHNIS
27.	Y=zeros(p,v); Z=zeros(p,v);
28.	for j=1:p
29.	for v1=1:v
30.	Y(j,v1)=(W(j,v1)-mHPIS(1,v1)).^ 2;
31.	Z(j,v1)=(W(j,v1)-mHNIS(1,v1)).^ 2;
32.	end
33.	end
34.	D_p=zeros(p,q);D_n=zeros(p,q);
35.	for j=1:p
36.	for k=1:q
37.	for v1=k*r*m-(r*m-1):k*r*m
38.	D_p(j,k)=D_p(j,k)+Y(j,v1);
39.	D_n(j,k)=D_n(j,k)+Z(j,v1);
40.	end
41.	end
42.	end
43.	D=[sqrt(sum(D_p,2)./(r*m)) sqrt(sum(D_n,2)./(r*m))]
44.	E=D(:,2)./sum(D,2)
45.	end

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