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Qualitative Behavior of Solutions of Second Order Differential Equations

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Abstract: In this work, we study the oscillation of second-order delay differential equations, by employing a refinement of the generalized Riccati substitution. We establish a new oscillation criterion. Symmetry ideas are often invisible in these studies, but they help us decide the right way to study them, and to show us the correct direction for future developments. We illustrate the results with some examples.

Keywords: second-order; nonoscillatory solutions; oscillatory solutions; delay differential equations

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1. Introduction

This paper is concerned with oscillation of a second-order differential equation

$$\left[a\left(z\right)w'\left(z\right)\right]' + q\left(z\right)f\left(w\left(\tau\left(z\right)\right)\right) = 0, \quad z \geq z_0, \tag{1}$$

where $a,\ \tau \in C^1\left([z_0,\infty),\ \mathbb{R}^+\right),\ \tau\left(z\right) \leq z, \lim_{z\to\infty}\tau\left(z\right) = \infty,\ \tau'\left(z\right) \geq 0,\ q \in C\left([z_0,\infty),\ [0,\infty)\right),$ the function f is nondecreasing and satisfies the following conditions

$$f \in C\left(\mathbb{R},\mathbb{R}\right),\ wf\left(w\right) > 0,\ f\left(w\right)/w \geq k > 0, \text{ for } w \neq 0, \tag{2}$$

and

$$\lim_{w\to\infty}\frac{w}{f\left(w\right)} = M < \infty, \tag{3}$$

where M is constant.

By a solution of Equation (1) we mean a function $w \in C\left([z_0,\infty),\ \mathbb{R}\right),\ z_w \geq z_0$, which has the property $a\left(z\right)\left[w'\left(z\right)\right] \in C^1\left([z_0,\infty),\ \mathbb{R}\right)$, and satisfies Equation (1) on $[z_w,\infty)$. We consider only those solutions w of Equation (1) which satisfy $\sup\{|w\left(z\right)| : z \geq z_w\} > 0$. Such a solution is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise.

Differential equations play an important role in many branches of mathematics, and they also often appear in other sciences. This fact leads us to more studying such equations and related boundary value problems in more detail, and a theory of solvability and (numerical) solutions for such equations are needed for distinct scientific groups, (see [1–5]).

Usually, one cannot find an exact solution for such equations, and one then needs to describe its qualitative properties in the appropriate functional spaces as well as to suggest a way of reducing the starting equation to a certain well known studied case, or to suggest some computational algorithm for the numerical solution. These studies are the intermediate points for solving equations.

The study of differential equations with deviating argument was initiated in 1918, appearing in the first quarter of the twentieth century as an area of mathematics that has received a lot of attention, (see [6–10]).

The oscillations of second order differential equations have been studied by authors and several techniques have been proposed for obtaining oscillation for these equations. For treatments on this subject, we refer the reader to the texts (see [11–15]). In what follows, we review some results that have provided the background and the motivation for the present work.

Koplatadze [16] is concerned with the oscillation of equations

$$w''(z) + p(z)\, w(\sigma(z)) = 0, \qquad z \geq z_0,$$

and he proved it is oscillatory if

$$\limsup_{s\to\infty} \int_{\sigma(z)}^{z} \sigma(s)\, p(s)\, ds > 1.$$

Moaaz, et al. [17] discussed the equation

$$\left[a(z)\left(w'(z)\right)^{\beta}\right]' + p(z)\, f(w(\tau(z))) = 0, \qquad z \geq z_0,$$

under the condition

$$\int_{z_0}^{\infty} \frac{1}{a^{\frac{1}{\beta}}(z)} dz < \infty.$$

Trench [18] used the comparison technique for the following

$$\left[a(z)\, w'(z)\right]' + q(z)\, w(\tau(z)) = 0,$$

that was compared with the oscillation of certain first order differential equation and under the condition

$$\int_{z_0}^{\infty} \frac{1}{a(z)} dz = \infty.$$

Wei in 1988 [19] discussed the equation

$$w''(z) + q(z)\, w(\tau(z)) = 0, \qquad z \geq z_0,$$

and used the classical Riccati transformation technique.

The present authors in this paper use the generalized Riccati substitution which differs from those reported in [20–22].

This paper deals with oscillatory behavior of second order delay for Equation (1) under the condition

$$\int_{z_0}^{\infty} \frac{1}{a(s)} ds < \infty, \tag{4}$$

which would generalize and extend of the related results reported in the literature. In addition, we use a generalized Riccati substitution. Some examples are included to illustrate the importance of results obtained.

Here we mention some lemmas.

Lemma 1. *(See [23], Lemma 2.1) Let $\beta \geq 1$ be a ratio of two odd numbers, $G,\ H,\ U,\ V \in \mathbb{R}$. Then*

$$G^{\frac{\beta+1}{\beta}} - (G-H)^{\frac{\beta+1}{\beta}} \leq \frac{H^{\frac{1}{\beta}}}{\beta}\left[(1+\beta)\, G - H\right], \quad GH \geq 0,$$

and

$$Uy - Vy^{\frac{\beta+1}{\beta}} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^{\beta}}, \; V > 0.$$

Lemma 2. *(See [13], Lemma 1.1) Let y satisfy $y^{(i)} > 0$, $i = 0, 1, ..., n$, and $y^{(n+1)} < 0$, then*

$$\frac{y\left(z\right)}{z^{n}/n!} \geq \frac{y'\left(z\right)}{z^{n-1}/\left(n-1\right)!}.$$

Lemma 3. *(See [24], Lemma 1) Assume that $w\left(z\right)$ is an eventually positive solution of Equation (1). Then we have two cases*

$$\begin{array}{ll} (C_1) & a\left(z\right)w'\left(z\right) > 0, \left(a\left(z\right)w'\left(z\right)\right)' < 0, \\ (C_2) & a\left(z\right)w'\left(z\right) < 0, \left(a\left(z\right)w'\left(z\right)\right)' < 0, \; for\; z \geq z_1 \geq z_0. \end{array}$$

2. Main Results

In this section, we shall establish some oscillation criteria for Equation (1). For convenience, we denote

$$B\left(z\right) := \int_{z}^{\infty} \frac{1}{a\left(s\right)} ds, \quad A\left(z\right) := kq\left(z\right)\frac{\tau^{2}\left(z\right)}{z^{2}}$$

$$\Phi\left(z\right) := \delta\left(z\right)\left[A\left(z\right) + \frac{1}{a\left(z\right)B^{2}(z)}\right].$$

$$\theta\left(z\right) := \frac{\delta'_{+}\left(z\right)}{\delta\left(z\right)} + \frac{2}{a\left(z\right)B(z)} \text{ and } \delta'_{+}\left(z\right) := \max\left\{0, \; \delta'\left(z\right)\right\}.$$

In what follows, all occurring functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough. As usual and without loss of generality, we can deal only with eventually positive solutions of Equation (1).

Theorem 1. *Assume that Equation (3) holds and $\tau(z) \leq z$. If*

$$\limsup_{s\to\infty} \int_{\tau(z)}^{z} B\left(s\right)q\left(s\right)ds > M, \tag{5}$$

then all solutions of Equation (1) are oscillatory.

Proof. Assume, on the contrary, that $w\left(z\right)$ is an eventually positive solution of Equation (1). Since Equation (5) implies

$$\int_{z_0}^{\infty} B\left(s\right)q\left(s\right)ds < \infty,$$

thereby $w\left(z\right)$ satisfies (C_2) of Lemma 3, which yields

$$\begin{aligned} 0 &\leq w(\tau(z)) + a(\tau(z))w'(\tau(z)) \\ &\leq w(\tau(z)). \end{aligned} \tag{6}$$

Setting

$$\phi\left(z\right) = w(z) + a(z)w'(z)B\left(z\right), \tag{7}$$

we see that

$$f\left(\phi(\tau(z))\right) \leq f\left(w(\tau(z))\right). \tag{8}$$

A simple computation ensures that Equation (1) can be rewritten into the form

$$\left(a(z)w'(z)B\left(z\right)+w(z)\right)'+B\left(z\right)q\left(z\right)f\left(w(\tau(z))\right)=0, \tag{9}$$

which in view of Equation (9) implies that $\phi\left(z\right)$ is a positive decreasing solution of the first order delay differential inequality

$$\phi'\left(z\right)+B\left(z\right)q\left(z\right)f\left(\phi(\tau(z))\right)\leq 0.$$

Integrating from $\tau(z)$ to z, we get

$$\begin{aligned}\phi(\tau(z)) &\geq \int_{\tau(z)}^{z}B\left(s\right)q\left(s\right)f\left(\phi(\tau(s))\right)ds\\ &\geq f\left(\phi(\tau(z))\right)\int_{\tau(z)}^{z}B\left(s\right)q\left(s\right)ds.\end{aligned}$$

Thus, we obtain

$$\frac{\phi(\tau(z))}{f\left(\phi(\tau(z))\right)}\geq\int_{\tau(z)}^{z}B\left(s\right)q\left(s\right)ds.$$

Hence, we obtain

$$\begin{aligned}\limsup_{s\to\infty}\int_{\tau(z)}^{z}B\left(s\right)q\left(s\right)ds &\leq \limsup_{s\to\infty}\left(\frac{\phi(\tau(z))}{f\left(\phi(\tau(z))\right)}\right),\\ &\leq M.\end{aligned}$$

but is a contradiction. Theorem 1 is proved. □

Theorem 2. *Let Equation (4) hold and*

$$w'\left(z\right)<0,\ \left(a\left(z\right)w'\left(z\right)\right)<0.$$

If there exists positive function $\delta\in C^{1}\left([z_0,\infty)\,,\ (0,\infty)\right)$ *such that*

$$\int_{z_0}^{\infty}\left(\Phi\left(s\right)-\frac{\delta\left(s\right)a\left(s\right)\left(\theta\left(s\right)\right)^2}{4}\right)ds=\infty. \tag{10}$$

Then all solutions of Equation (1) are oscillatory.

Proof. Assume that (C_2) holds. By Lemma 2, we find

$$w\left(z\right)\geq\left(\frac{z}{2}\right)w'\left(z\right)$$

and hence

$$\frac{w'\left(z\right)}{w\left(z\right)}\leq\frac{2}{z}.$$

Integrating from $\tau\left(z\right)$ to z, we get

$$\frac{w\left(z\right)}{w\left(\tau\left(z\right)\right)}\leq\frac{z^2}{\tau^2\left(z\right)}$$

and hence

$$\frac{w\left(\tau\left(z\right)\right)}{w\left(z\right)}\geq\frac{\tau^2\left(z\right)}{z^2}. \tag{11}$$

It follows from $(a(z)(w'(z))) \leq 0$, we obtain

$$w'(s) \leq \left(\frac{a(z)}{a(s)}\right) w'(z).$$

Integrating from z to z_1, we get

$$w(z_1) \leq w(z) + a(z) w'(z) \int_z^{z_1} a^{-1}(s)\, ds. \tag{12}$$

Letting $z_1 \to \infty$, we obtain

$$w(z) \geq -B(z) a(z) w'(z)$$

which implies that

$$\left(\frac{w(z)}{B(z)}\right)' \geq 0.$$

Define the function $\psi(z)$ by

$$\psi(z) := \delta(z)\left[\frac{a(z)(w'(z))}{w(z)} + \frac{1}{B(z)}\right], \tag{13}$$

then $\psi(z) > 0$ for $z \geq z_1$ and

$$\begin{aligned}
\psi'(z) &= \delta'(z)\left[\frac{a(z)(w'(z))}{w(z)} + \frac{1}{B(z)}\right] + \delta(z)\frac{(a(z)w'(z))'}{w(z)} \\
&\quad -\delta(z) a(z)\frac{(w')^2(z)}{w^2(z)} + \frac{\delta(z)}{a(z) B^2(z)} \\
&= \frac{\delta'(z)}{\delta(z)}\left(\delta(z)\left[\frac{a(z)(w'(z))}{w(z)} + \frac{1}{B(z)}\right]\right) + \delta(z)\frac{(a(z)w'(z))'}{w(z)} \\
&\quad -\delta(z) a(z)\frac{(w')^2(z)}{w^2(z)} + \frac{\delta(z)}{a(z) B^2(z)}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\psi'(z) &= \frac{\delta'(z)}{\delta(z)}\psi(z) + \delta(z)\frac{(a(z)w'(z))'}{w(z)} \\
&\quad -\delta(z) a(z)\left[\frac{1}{a(z)}\left(\frac{a(z)w'(z)}{w(z)} + \frac{1}{B(z)}\right) - \frac{1}{a(z)B(z)}\right]^2 + \frac{\delta(z)}{a(z)B^2(z)}.
\end{aligned}$$

Using Equation (13) we obtain

$$\begin{aligned}
\psi'(z) &= \frac{\delta'(z)}{\delta(z)}\psi(z) + \delta(z)\frac{(a(z)w'(z))'}{w(z)} \\
&\quad -\delta(z) a(z)\left[\frac{\psi(z)}{\delta(z)a(z)} - \frac{1}{a(z)B(z)}\right]^2 + \frac{\delta(z)}{a(z)B^2(z)}.
\end{aligned} \tag{14}$$

Using Lemma 1 with $G = \frac{\psi(z)}{\delta(z)a(z)}$, $H = \frac{1}{a(z)B(z)}$, $\beta = 1$, we get

$$\begin{aligned}
\left[\frac{\psi(z)}{\delta(z)a(z)} - \frac{1}{a(z)B(z)}\right]^2 &\geq \left(\frac{\psi(z)}{\delta(z)a(z)}\right)^2 \\
&\quad -\frac{1}{a(z)B(z)}\left(\frac{2\psi(z)}{\delta(z)a(z)} - \frac{1}{a(z)B(z)}\right).
\end{aligned} \tag{15}$$

From Equations (14) and (15), we obtain

$$\begin{aligned} \psi'(z) &\leq \frac{\delta'_+(z)}{\delta(z)}\psi(z) - \delta(z)\,kq(z)\,\frac{w(\tau(z))}{w(z)} - \delta(z)\,a(z)\left(\frac{\psi(z)}{\delta(z)\,a(z)}\right)^2 \\ &\quad -\delta(z)\,a(z)\left[\frac{-1}{a(z)\,B(z)}\left(\frac{2\psi(z)}{\delta(z)\,a(z)} - \frac{1}{a(z)\,B(z)}\right)\right]. \end{aligned} \tag{16}$$

From Equations (11) and (16), we get

$$\begin{aligned} \psi'(z) &\leq \frac{\delta'_+(z)}{\delta(z)}\psi(z) - \delta(z)\,A(z) - \delta(z)\,a(z)\left(\frac{\psi(z)}{\delta(z)\,a(z)}\right)^2 \\ &\quad -\delta(z)\,a(z)\left[\frac{-1}{a(z)\,B(z)}\left(\frac{2\psi(z)}{\delta(z)\,a(z)} - \frac{1}{a(z)\,B(z)}\right)\right]. \end{aligned}$$

This implies that

$$\begin{aligned} \psi'(z) &\leq \left(\frac{\delta'_+(z)}{\delta(z)} + \frac{2}{a(z)\,B(z)}\right)\psi(z) - \frac{1}{(\delta(z)\,a(z))}\psi^2(z) \\ &\quad -\delta(z)\left[A(z) + \frac{1}{a(z)\,B^2(z)}\right]. \end{aligned} \tag{17}$$

Thus, by Equation (14) yield

$$\psi'(z) \leq -\Phi(z) + \theta(z)\,\psi(z) - \frac{1}{(\delta(z)\,a(z))}\psi^2(z). \tag{18}$$

Applying the Lemma 1 with $U = \theta(z)\,, V = \frac{1}{(\delta(z)a(z))}$ and $y = \psi(z)$, we get

$$\psi'(z) \leq -\Phi(z) + \frac{\delta(z)\,a(z)\,(\theta(z))^2}{4}. \tag{19}$$

Integrating from z_1 to z, we get

$$\int_{z_1}^{z}\left(\Phi(s) - \frac{\delta(s)\,a(s)\,(\theta(s))^2}{4}\right)ds \leq \psi(z_1) - \psi(z) \leq \psi(z_1)\,,$$

which contradicts Equation (10). The proof is complete. □

Example 1. *As an illustrative example, we consider the following equation:*

$$\left(z^5 w'(z)\right)' + rzw\left(\frac{z}{2}\right) = 0, \tag{20}$$

where $r > 0$. Let

$$a(z) = z^5,\ q(z) = rz,\ \tau(z) = \frac{z}{2}.$$

It is easy to see that all conditions of Theorem 1 are satisfied.

$$\begin{aligned} \int_z^\infty \frac{1}{a(s)}ds &= \int_z^\infty \frac{1}{s^5}ds \\ &= \frac{1}{4z^4} \\ &< \infty, \end{aligned}$$

and

$$\begin{aligned} & \limsup_{z\to\infty} \int_{\tau(z)}^{z} B\left(s\right) q\left(s\right) ds \\ = & \limsup_{z\to\infty} \int_{\frac{z}{2}}^{z} \frac{1}{4s^3} r ds \\ < & \infty. \end{aligned}$$

Hence, by Theorem 1, all solutions of Equation (20) are oscillatory.

Example 2. *We consider the equation:*

$$\left(z^2 x\left(z\right)'\right)' + x\left(\frac{z}{3}\right) = 0,\ z \geq 1. \tag{21}$$

Let

$$a\left(z\right) = z^2,\ q\left(z\right) = 1,\ \tau\left(z\right) = \frac{z}{3}.$$

If we now set $\delta\left(z\right) = 1$ and $k = 1,$ then all conditions of Theorem 2 are satisfied.

$$B\left(z\right) := \int_{z}^{\infty} \frac{1}{a\left(s\right)} ds = \frac{1}{z} < \infty.$$

$$A\left(z\right) = kq\left(z\right)\left(\frac{\tau\left(z\right)}{z}\right)^2 = \frac{1}{9},$$

and

$$\int_{z_0}^{\infty} \left(\Phi\left(s\right) - \frac{\delta\left(s\right) a\left(s\right)\left(\theta\left(s\right)\right)^{\beta+1}}{\left(\beta+1\right)^{\beta+1}}\right) ds = \infty.$$

Applying Theorem 2, we obtain that all solutions of Equation (21) are oscillatory.

Remark 1. *The results in [18] imply those in Equation (21).*

Remark 2. *The results obtained supplement and improve those in [16].*

3. Conclusions

The results of this paper are presented in a form which is essentially new and of high degree of generality. To the best of our knowledge, there are not many studies known about the oscillation of Equation (1) under the assumption of Equation (4). Our primary goal is to fill this gap by presenting simple criteria for the oscillation of all solutions of Equation (1) by using the generalized Riccati transformations which differs from those reported in [22] and using a comparison technique with first order differential equation. Further, we can consider the case of $\tau\left(z\right) \geq z$ in the future work.

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