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Simpson's Type Inequalities for Co-Ordinated Convex Functions on Quantum Calculus

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Abstract: In the present paper, we aim to prove a new lemma and quantum Simpson's type inequalities for functions of two variables having convexity on co-ordinates over $[\alpha, \beta] \times [\psi, \phi]$. Moreover, our deduction introduce new direction as well as validate the previous results.

Keywords: quantum calculus; simpson's type inequality; quantum simpson's type inequality on co-ordinates; q_1q_2-partial derivatives; q_1q_2–Hölder's inequality; co-ordinated convexity

1. Introduction

In mathematics, q–calculus, also known as Quantum calculus, is the study of calculus with no limits. In quantum calculus, we obtain q-analogues of mathematical objects that can be recaptured as $q \to 1^-$. This concept was given by Euler who introduced q in infinite series and further defined in detail by Newton. Later on, Jackson [1] proposed the notation of q–definite integrals and extended the concept of q–calculus. Diverse fields of q–calculus have plentiful applications in orthogonal polynomials, number theory, information technology, quantum mechanics and relativity theory. Profound work of quantum calculus and theory of inequalities is addressed in [2–4] and the references cited therein. The idea of q–derivatives over the finite interval $[\alpha, \beta] \subset \mathbb{R}$ is introduced by Tariboon et al. [5,6] and discussed numerous problems on quantum analogues like q–Hölder inequality, q–Ostrowski inequality, q–Cauchy–Schwarz inequality, q–Grüss–Čhebyšev inequality, q–Grüss inequality and other integral inequalities.

Noor et al. [7,8], Sudsutad et al. [9] and Zhuang et al. [10] used q-differentiable convex functions as well as quasi-convex functions to investigate integral inequalities in different ways and their results are helpful in estimation of the right-hand side of quantum analogue of Hermite–Hadamard inequality.

Inequalities and theory of convex functions have a great dependency on each other. This relationship is the main sanity behind the vast literature published using convex functions. The Hermite–Hadamard inequalities have been studied extensively over the past three decades. The Hermite–Hadamard inequalities provide a necessary and sufficient conditions for a continuous function $h : V \subset \mathbb{R} \to \mathbb{R}$ to be convex on $[\alpha, \beta]$, where $\alpha, \beta \in V$ with $\alpha < \beta$. These inequalities are stated as follows:

$$h\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} h(z)dz \leq \frac{h(\alpha)+h(\beta)}{2}. \tag{1}$$

The following inequality is recognized as Simpson's inequality:

Let $h : [\alpha, \beta] \to \mathbb{R}$ be a four times continuously differentiable mapping on the interval $[\alpha, \beta]$ and $||h^{(4)}||_\infty = \sup_{z \in (\alpha,\beta)} \left| h^{(4)}(z) \right| < \infty$. Then, the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{h(\alpha) + h(\beta)}{2} + 2h\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha} \int_\alpha^\beta h(z) dz \right] \right| \leq \frac{(\beta-\alpha)^4}{2880} \left|\left| h^{(4)} \right|\right|_\infty. \tag{2}$$

A number of results on Simpson's type inequalities have been proved by many researchers. For more details—see [11–14].

Tunç et al. first proposed Simpson type quantum integral inequalities for the function of one variable based on convexity—see [15].

Lemma 1. *Letting $h : V \to \mathbb{R}$ be a continuous function and $q \in (0, 1)$. If ${}_\alpha D_q h$ is an integrable function on V^o (the interior of V), then the following inequality holds:*

$$\frac{1}{6}\left[h(\alpha) + 4h\left(\frac{\alpha+\beta}{2}\right) + h(\beta) \right] - \frac{1}{\beta-\alpha} \int_\alpha^\beta h(x)_\alpha d_q x = (\beta-\alpha) \int_0^1 p(z)_\alpha D_q h\left((1-z)\alpha + z\beta\right) {}_0 d_q z,$$

where

$$p(z) = \begin{cases} qz - \frac{1}{6}, & z \in [0, \frac{1}{2}), \\ qz - \frac{5}{6}, & z \in [\frac{1}{2}, 1). \end{cases}$$

Preliminaries of Hermite–Hadamard type inequality for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$ are addressed by Dragomir—see [16].

The foundation of Simpson's type inequality for co-ordinated convex functions is laid by Özdemir et al.—see [17].

Lemma 2. *Let $h : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [\alpha, \beta] \times [\psi, \phi]$. If $\frac{\partial^2 h}{\partial z \partial w} \in L(\Delta)$, then the following equality holds:*

$$\begin{aligned} &\frac{h\left(\alpha, \frac{\psi+\phi}{2}\right) + h\left(\beta, \frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2}, \psi\right) + h\left(\frac{\alpha+\beta}{2}, \phi\right)}{9} \\ &+ \frac{h(\alpha, \psi) + h(\beta, \psi) + h(\alpha, \phi) + h(\beta, \phi)}{36} \\ &- \frac{1}{6(\beta-\alpha)} \int_\alpha^\beta \left[h(x, \psi) + 4h\left(x, \frac{\psi+\phi}{2}\right) + h(x, \phi) \right] dx \\ &- \frac{1}{6(\phi-\psi)} \int_\psi^\phi \left[h(\alpha, y) + 4h\left(\frac{\alpha+\beta}{2}, y\right) + h(\beta, y) \right] dy \\ &+ \frac{1}{(\beta-\alpha)(\phi-\psi)} \int_\alpha^\beta \int_\psi^\phi h(x, y) dy dx \end{aligned} \tag{3}$$

$$= (\beta-\alpha)(\phi-\psi) \int_0^1 \int_0^1 p(x, z) q(y, w) \frac{\partial^2 h((1-z)\alpha + z\beta, (1-w)\psi + w\phi)}{\partial z \partial w} dz dw,$$

where

$$p(x, z) = \begin{cases} z - \frac{1}{6}, & z \in \left[0, \frac{1}{2}\right] \\ z - \frac{5}{6}, & z \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and

$$q(y, w) = \begin{cases} w - \frac{1}{6}, & w \in [0, \frac{1}{2}] \\ w - \frac{5}{6}, & w \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

The main result from [17] is stated in the theorem below.

Theorem 1. *Let $h : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a second order partially differentiable function on Δ. If $\frac{\partial^2 h}{\partial z \partial w}$ is a convex function on the co-ordinates on Δ, then the given inequality holds:*

$$\Bigg| \frac{h\left(\alpha, \frac{\psi+\phi}{2}\right) + h\left(\beta, \frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2}, \psi\right) + h\left(\frac{\alpha+\beta}{2}, \phi\right)}{9}$$
$$+ \frac{h(\alpha,\psi) + h(\beta,\psi) + h(\alpha,\phi) + h(\beta,\phi)}{36} + \frac{1}{(\beta-\alpha)(\phi-\psi)} \int_\alpha^\beta \int_\psi^\phi h(x,y)dydx - A \Bigg|$$
$$\leq \frac{25(\beta-\alpha)(\phi-\psi)}{144} \left[\left| \frac{\partial^2 h(\alpha,\psi)}{\partial t \partial s} \right| + \left| \frac{\partial^2 h(\alpha,\phi)}{\partial t \partial s} \right| + \left| \frac{\partial^2 h(\beta,\psi)}{\partial t \partial s} \right| + \left| \frac{\partial^2 h(\beta,\phi)}{\partial t \partial s} \right| \right],$$

where

$$A = \frac{1}{6(\beta-\alpha)} \int_\alpha^\beta \left[h(x,\psi) + 4h\left(x, \frac{\psi+\phi}{2}\right) + h(x,\phi) \right] dx$$
$$+ \frac{1}{6(\phi-\psi)} \int_\psi^\phi \left[h(\alpha,y) + 4h\left(\frac{\alpha+\beta}{2}, y\right) + h(\beta,y) \right] dy.$$

2. Preliminaries

For attaining our main aim, we recall some previously known concepts and basic results on q–calculus. The concept of q-calculus in single variable was given by Tariboon et al. [5,6]

Definition 1. *Let $h : V = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function and let $s \in V$. Then, the q–derivative of h on V at s with $q \in (0,1)$ is defined as*

$$ {}_\alpha D_q h(s) = \frac{h(s) - h(qs + (1-q)\alpha)}{(1-q)(s-\alpha)}, \quad s \neq \alpha. \tag{4}$$

It is obvious that

$$\lim_{s \to \alpha} {}_\alpha D_q h(s) = {}_\alpha D_q h(\alpha).$$

A function h is q-differentiable on V; then, ${}_\alpha D_q h(s)$ exists for all $s \in V$. Moreover, if we take $\alpha = 0$ in (4), then ${}_0 D_q h = D_q h$, where $D_q h$ is well-known q–derivative of $h(s)$, which is defined by

$$D_q h(s) = \frac{h(s) - h(qs)}{(1-q)(s)}.$$

In addition, we shall define higher-order q-derivatives of functions on V .

Definition 2. *Let $h : V = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function and $q \in (0,1)$ be a constant, denoted by ${}_\alpha D_q^2 h$ (provided that ${}_\alpha D_q h$ is q-differentiable on V), which is the function from $V \to \mathbb{R}$ defined by*

$${}_\alpha D_q^2 h = {}_\alpha D_q \left({}_\alpha D_q h \right).$$

Similarly, provided that ${}_\alpha D_q^{n-1} h$ is q-differentiable on V for some integer $n > 2$, the n^{th} -order q-derivative of h on $V$is the function from $V \to \mathbb{R}$ defined by

$${}_\alpha D_q^n h = {}_\alpha D_q \left({}_\alpha D_q^{n-1} h \right).$$

Definition 3. *Let $h : V = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function and $q \in (0,1)$ be a constant. Then, the q–integral on V is defined by*

$$\int_{\alpha}^{s} h(z)_{\alpha} d_q z = (1-q)(s-\alpha) \sum_{n=0}^{\infty} q^n h(q^n s + (1-q^n)\alpha) \tag{5}$$

for $s \in V$.

Note that, if we take $\alpha = 0$ in (5), then we obtain the concept of classical q–integral of function $h(z)$ as

$$\int_{0}^{s} h(z)_{0} d_q z = (1-q)s \sum_{n=0}^{\infty} q^n (q^n s).$$

Theorem 2. *Let $h : V = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function and $q \in (0,1)$ be a constant; then, we have the following*

$$\begin{aligned}
&(i) \quad {}_{\alpha}D_q \int_{\alpha}^{s} h(z)_{\alpha} d_q z = h(s); \\
&(ii) \quad \int_{\alpha}^{s} {}_{\alpha}D_q h(z)_{\alpha} d_q z = h(s); \\
&(iii) \quad \int_{\psi}^{s} {}_{\alpha}D_q h(z)_{\alpha} d_q z = h(s) - h(\psi), \quad \psi \in (\alpha, s).
\end{aligned}$$

Theorem 3. *Letting $h_1, h_2 : V = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function, and $a \in \mathbb{R}$ and $q \in (0,1)$ be a constant, then we have the following:*

$$\begin{aligned}
&(i) \quad \int_{\alpha}^{s} [h_1(z) + h_2(z)]\, {}_{\alpha}d_q z = \int_{\alpha}^{s} h_1(z)_{\alpha} d_q z + \int_{\alpha}^{s} h_2(z)_{\alpha} d_q z, \\
&(ii) \quad \int_{\alpha}^{s} (a h_1(z))_{\alpha} d_q z = a \int_{\alpha}^{s} h_1(z)_{\alpha} d_q z.
\end{aligned}$$

Latif et al. [18] evolve quantum integral inequalities theory for functions of two variables and introduced q–Hermite–Hadamard type inequality of functions of two variables over finite rectangles. It is easy to discern that the preliminaries in Latif et al. [18] contain the preliminaries in Tariboon et al. [5] as a special case when h is a function of a single variable.

Definition 4. *Let $h : [\alpha, \beta] \times [\psi, \phi] \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables and $q_1, q_2 \in (0,1)$ be constants. Then, partial q_1-derivative, q_2-derivative and q_1q_2-derivative at $(s,t) \in [\alpha,\beta] \times [\psi,\phi]$ are, respectively, characterized by expressions as:*

$$\begin{aligned}
&\frac{{}_{\alpha}\partial_{q_1} h(s,t)}{{}_{\alpha}\partial_{q_1} s} = \frac{h(q_1 s + (1-q_1)\alpha, t) - h(s,t)}{(1-q_1)(s-\alpha)}, \quad s \neq \alpha, \\
&\frac{{}_{\psi}\partial_{q_2} h(s,t)}{{}_{\psi}\partial_{q_2} t} = \frac{h(s, q_2 t + (1-q_2)\psi) - h(s,t)}{(1-q_2)(t-\psi)}, \quad t \neq \psi, \\
&\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(s,t)}{{}_{\alpha}\partial_{q_1} s\, {}_{\psi}\partial_{q_2} t} = \frac{1}{(1-q_1)(1-q_2)(s-\alpha)(t-\psi)} \\
&\times \Big[h(q_1 s + (1-q_1)\alpha, q_2 t + (1-q_2)\psi) - h(q_1 s + (1-q_1)\alpha, t) \\
&- h(s, q_2 t + (1-q_2)\psi) + h(s,t) \Big], \quad s \neq \alpha, \quad t \neq \psi.
\end{aligned}$$

The function $h : [\alpha,\beta] \times [\psi,\phi] \subset \mathbb{R}^2 \to \mathbb{R}$ is called partially q_1- q_2- and q_1q_2-differentiable on $[\alpha,\beta] \times [\psi,\phi]$ if $\frac{{}_{\alpha}\partial_{q_1} h(s,t)}{{}_{\alpha}\partial_{q_1} s}$, $\frac{{}_{\psi}\partial_{q_2} h(s,t)}{{}_{\psi}\partial_{q_2} t}$ and $\frac{{}_{\alpha,\psi}\partial^2_{q_1 q_2} h(s,t)}{{}_{\alpha}\partial_{q_1} s\, {}_{\psi}\partial_{q_2} t}$ exist for all $(s,t) \in [\alpha,\beta] \times [\psi,\phi]$.

Similarly, we can define partial derivatives of higher order.

Definition 5. *Let $h : [\alpha, \beta] \times [\psi, \phi] \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables and $q_1, q_2 \in (0, 1)$ be constants. Then, the definite q_1q_2-integral on $[\alpha, \beta] \times [\psi, \phi]$ are delineated as:*

$$\int_{\psi}^{t} \int_{\alpha}^{s} h(z, w)_{\alpha}d_{q_1}z_{\psi}d_{q_2}w$$
$$= (1 - q_1)(1 - q_2)(s - \alpha)(t - \psi) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m h\left(q_1^n s + (1 - q_1^n)\,\alpha, q_2^m t + (1 - q_2^m)\,\psi\right)$$

for $(s, t) \in [\alpha, \beta] \times [\psi, \phi]$.

Theorem 4. *Let $h : [\alpha, \beta] \times [\psi, \phi] \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then,*

$$(i) \quad \frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}}{{}_{\alpha}\partial_{q_1}s_{\psi}\partial_{q_2}t} \int_{\psi}^{t} \int_{\alpha}^{s} h(z, w)_{\alpha}d_{q_1}z_{\psi}d_{q_2}w = h(s, t)$$

$$(ii) \quad \int_{\psi}^{t} \int_{\alpha}^{s} \frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(z, w)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}{}_{\alpha}d_{q_1}z_{\psi}d_{q_2}w = h(s, t)$$

$$(iii) \quad \int_{t_1}^{t} \int_{s_1}^{s} \frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(z, w)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}{}_{\alpha}d_{q_1}z_{\psi}d_{q_2}w$$
$$= h(s, t) - h(s, t_1) - h(s_1, t) + h(s_1, t_1), \quad (s_1, t_1) \in (\alpha, s) \times (\psi, t).$$

Theorem 5. *Let $h_1, h_2 : [\alpha, \beta] \times [\psi, \phi] \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous functions and $a \in \mathbb{R}$. Then, for $(s, t) \in [\alpha, \beta] \times [\psi, \phi]$,*

$$(i) \quad \int_{\psi}^{t} \int_{\alpha}^{s} [h_1(z, w) + h_2(z, w)]\, {}_{\alpha}d_{q_1}z_{\psi}d_{q_2}w$$
$$= \int_{\psi}^{t} \int_{\alpha}^{s} h_1(z, w)_{\alpha}d_{q_1}z_{\psi}d_{q_2}w + \int_{\psi}^{t} \int_{\alpha}^{s} h_2(z, w)_{\alpha}d_{q_1}z_{\psi}d_{q_2}w.$$

$$(ii) \quad \int_{\psi}^{t} \int_{\alpha}^{s} ah(z, w)_{\alpha}d_{q_1}z_{\psi}d_{q_2}w = a \int_{\psi}^{t} \int_{\alpha}^{s} h(z, w)_{\alpha}d_{q_1}z_{\psi}d_{q_2}w.$$

Theorem 6. *(Hölder inequality for double sums). Suppose $(x_{mn})_{m,n\in\mathbb{N}}$ and $(y_{mn})_{m,n\in\mathbb{N}}$ be sequences of real (or complex) numbers and $r_1^{-1} + r_2^{-1} = 1$, $r_1, r_2 > 1$, the following inequality for double sums holds:*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |x_{mn}y_{mn}| \leq \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |x_{mn}|^{r_1} \right)^{\frac{1}{r_1}} \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |y_{mn}|^{r_2} \right)^{\frac{1}{r_2}},$$

where all the sums are assumed to be finite.

Theorem 7. *(q_1q_2–Hölder inequality for functions of two variables). Let g and h be functions defined on $[\alpha, \beta] \times [\psi, \phi]]$ and $q_1, q_2 \in (0, 1)$ be constants. If $r_1^{-1} + r_2^{-1} = 1$ with $r_1r_2 > 1$, the following inequality for q_1q_2–Hölder inequality for functions of two variables holds:*

$$\int_{\alpha}^{\beta} \int_{\psi}^{\phi} |g(z, w)h(z, w)|\, {}_{\psi}d_{q_2}w_{\alpha}d_{q_1}z \tag{6}$$
$$\leq \left(\int_{\alpha}^{\beta} \int_{\psi}^{\phi} |g(z, w)|^{r_1}\, {}_{\psi}d_{q_2}w_{\alpha}d_{q_1}z \right)^{\frac{1}{r_1}} \left(\int_{\alpha}^{\beta} \int_{\psi}^{\phi} |h(z, w)|^{r_2}\, {}_{\psi}d_{q_2}w_{\alpha}d_{q_1}z \right)^{\frac{1}{r_2}}. \tag{7}$$

For more detail see [15].

Lemma 3. *Let $0 < q < 1$ be a constant. Then, hold*

$$\begin{aligned}
A_q &:= \int_0^{\frac{1}{2}} (1-t)\left|qt - \frac{1}{6}\right| {}_0d_qt = \frac{36q^3 + 12q^2 + 12q + 1}{216(q^3 + 2q^2 + 2q + 1)}. \\
B_q &:= \int_0^{\frac{1}{2}} t\left|qt - \frac{1}{6}\right| {}_0d_qt = \frac{18q^2 + 18q - 7}{216(q^3 + 2q^2 + 2q + 1)}. \\
C_q &:= \int_{\frac{1}{2}}^{1} (1-t)\left|qt - \frac{5}{6}\right| {}_0d_qt = \frac{12q^2 + 12q + 5}{216(q^3 + 2q^2 + 2q + 1)}. \\
D_q &:= \int_{\frac{1}{2}}^{1} t\left|qt - \frac{5}{6}\right| {}_0d_qt = \frac{18q^2 + 18q + 25}{216(q^3 + 2q^2 + 2q + 1)}. \\
E_q &:= \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| {}_0d_qt = \frac{6q - 1}{36(q+1)}. \\
F_q &:= \int_{\frac{1}{2}}^{1} \left|qt - \frac{5}{6}\right| {}_0d_qt = \frac{5}{36(q+1)}. \\
G_q &:= \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right|^p {}_0d_qt = \frac{\left(1 + (3q-1)^{p+1}\right)(1-q)}{6^{p+1}q(1-q^{p+1})}. \\
H_q &:= \int_{\frac{1}{2}}^{1} \left|qt - \frac{5}{6}\right|^p {}_0d_qt = \frac{\left[(5-3q)^{p+1} + (6q-5)^{p+1}\right](1-q)}{6^{p+1}q(1-q^{p+1})}.
\end{aligned}$$

The main objective of this paper is to formulate lemmas and derive some quantum analogues of Simpson's type inequalities of functions of two variables over finite rectangles by taking under consideration the theory quantum calculus of functions of two variables.

Moreover, we also provide some quantum estimates for Simpson's type inequalities of functions of two variables using convexity on co-ordinates of the absolute value of the q_1q_2-partial derivatives.

3. Main Results

Lemma 4. *Let $h : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a mixed partial q_1q_2-differentiable function over Δ° (the interior of Δ). Moreover, if mixed partial q_1q_2-derivative $\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(z,w)}{{}_{\alpha}\partial_{q_1} z_{\psi}\partial_{q_2} w}$ is continuous and integrable over $[\alpha,\beta] \times [\psi,\phi] \subset \Delta^{\circ}$ for $q_1, q_2 \in (0,1)$, then the following equality holds:*

$$\begin{aligned}
&\frac{h\left(\alpha, \frac{\psi+\phi}{2}\right) + h\left(\beta, \frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2}, \psi\right) + h\left(\frac{\alpha+\beta}{2}, \phi\right)}{9} \\
&+ \frac{h(\alpha,\psi) + h(\beta,\psi) + h(\alpha,\phi) + h(\beta,\phi)}{36} \\
&- \frac{1}{6(\beta-\alpha)} \int_{\alpha}^{\beta} \left[h(x,\psi) + 4h\left(x, \frac{\psi+\phi}{2}\right) + h(x,\phi)\right] {}_0d_{q_1}x \\
&- \frac{1}{6(\phi-\psi)} \int_{\psi}^{\phi} \left[h(\alpha,y) + 4h\left(\frac{\alpha+\beta}{2}, y\right) + h(\beta,y)\right] {}_0d_{q_2}y \\
&+ \frac{1}{(\beta-\alpha)(\phi-\psi)} \int_{\alpha}^{\beta}\int_{\psi}^{\phi} h(x,y)\,{}_0d_{q_2}y\,{}_0d_{q_1}x
\end{aligned} \tag{8}$$

$$= (\beta-\alpha)(\phi-\psi) \int_0^1\int_0^1 P(x,z,q_1)T(y,w,q_2) \frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h((1-z)\alpha + z\beta, (1-w)\psi + w\phi)}{{}_{\alpha}\partial_{q_1} z_{\psi}\partial_{q_2} w} {}_0d_{q_1}z\,{}_0d_{q_2}w,$$

where

$$P(x,z,q_1) = \begin{cases} q_1 z - \frac{1}{6}, & z \in \left[0, \frac{1}{2}\right), \\ q_1 z - \frac{5}{6}, & z \in [\frac{1}{2}, 1), \end{cases}$$

and

$$T(y,w,q_2) = \begin{cases} q_2 w - \frac{1}{6}, & w \in [0, \frac{1}{2}), \\ q_2 w - \frac{5}{6}, & w \in [\frac{1}{2}, 1). \end{cases}$$

Proof. Now, we consider

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(q_1 z-\frac{1}{6}\right)\left(q_2 w-\frac{1}{6}\right)\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h((1-z)\alpha+z\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1} z\,{}_{\psi}\partial_{q_2} w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left(q_1 z-\frac{1}{6}\right)\left(q_2 w-\frac{5}{6}\right)\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h((1-z)\alpha+z\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1} z\,{}_{\psi}\partial_{q_2} w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&+\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\left(q_1 z-\frac{5}{6}\right)\left(q_2 w-\frac{1}{6}\right)\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h((1-z)\alpha+z\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1} z\,{}_{\psi}\partial_{q_2} w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left(q_1 z-\frac{5}{6}\right)\left(q_2 w-\frac{5}{6}\right)\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h((1-z)\alpha+z\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1} z\,{}_{\psi}\partial_{q_2} w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w.
\end{aligned} \tag{9}$$

By the definition of partial q_1q_2-derivatives and definite q_1q_2-integrals, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(q_1 z-\frac{1}{6}\right)\left(q_2 w-\frac{1}{6}\right)\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h((1-z)\alpha+z\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1} z\,{}_{\psi}\partial_{q_2} w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&=\frac{1}{(1-q_1)(1-q_2)(\beta-\alpha)(\phi-\psi)}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{\left(q_1 z-\frac{1}{6}\right)\left(q_2 w-\frac{1}{6}\right)}{zw} \\
&\times\Big[h(zq_1\beta+(1-zq_1)\alpha, wq_2\phi+(1-wq_2)\psi)-h(zq_1\beta+(1-zq_1)\alpha, w) \\
&-h(z, wq_2\phi+(1-wq_2)\psi)+h(z,w)\Big]\,{}_0d_{q_1}z\,{}_0d_{q_2}w
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right) \\
&=-\frac{h\left(\frac{\alpha+\beta}{2},\frac{\psi+\phi}{2}\right)}{(\beta-\alpha)(\phi-\psi)}-\frac{1}{(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}q^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha,\frac{\psi+\phi}{2}\right) \\
&-\frac{1}{(\beta-\alpha)(\phi-\psi)}\sum_{m=0}^{\infty}q_2^m h\left(\frac{\alpha+\beta}{2},\frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right) \\
&+\frac{1}{(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right), \\
&-\frac{q_2}{(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right) \\
&=\frac{q_2}{(\beta-\alpha)(\phi-\psi)}\sum_{m=0}^{\infty}q_2^m h\left(\frac{a+b}{2},\frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right) \\
&-\frac{q_2}{(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right), \\
&-\frac{q_1}{(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right) \\
&=\frac{q_1}{(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha,\frac{\psi+\phi}{2}\right) \\
&-\frac{q_1}{(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right),
\end{aligned}$$

$$\frac{q_1 q_2}{(\phi-\psi)(\beta-\alpha)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$-\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$=\frac{h\left(\frac{\alpha+\beta}{2},\psi\right)}{6(\beta-\alpha)(\phi-\psi)}-\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha,\psi\right)$$

$$-\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$=-\frac{h\left(\frac{\alpha+\beta}{2},\frac{\psi+\phi}{2}\right)}{6(\beta-\alpha)(\phi-\psi)}+\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha,\frac{\psi+\phi}{2}\right)$$

$$+\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$\frac{q_1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$=\frac{q_1}{6(\beta-\alpha)(\phi-\psi)}\left[-\sum_{n=0}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha,\frac{\psi+\phi}{2}\right)+\sum_{n=0}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha,\psi\right)\right]$$

$$+\frac{q_1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$-\frac{q_1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$-\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$=\frac{h\left(\alpha,\frac{\psi+\phi}{2}\right)}{6(\beta-\alpha)(\phi-\psi)}-\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{m=0}^{\infty} q_2^m h\left(\alpha,\frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$-\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$\frac{q_2}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$=-\frac{q_2}{6(\beta-\alpha)(\phi-\psi)}\left[-\sum_{m=0}^{\infty} q_2^m h\left(\frac{\alpha+\beta}{2},\frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)+\sum_{m=0}^{\infty} q_2^m h\left(\alpha,\frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)\right]$$

$$+\frac{q_2}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$=-\frac{h\left(\frac{\alpha+\beta}{2},\frac{\psi+\phi}{2}\right)}{6(\beta-\alpha)(\phi-\psi)}+\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{m=0}^{\infty} q_2^m h\left(\frac{\alpha+\beta}{2},\frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$+\frac{1}{6(\beta-\alpha)(\phi-\psi)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_2^m h\left(\frac{q_1^n}{2}\beta+\left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi+\left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$- \frac{q_2}{6(\beta-\alpha)(\phi-\psi)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_2^m h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right)$$

$$\frac{1}{36(\beta-\alpha)(\phi-\psi)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right)$$
$$= \frac{h(\alpha,\psi)}{36(\beta-\alpha)(\phi-\psi)} + \frac{1}{36(\beta-\alpha)(\phi-\psi)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$- \frac{1}{36(\beta-\alpha)(\phi-\psi)} \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right)$$
$$= -\frac{h\left(\alpha, \frac{\psi+\phi}{2}\right)}{36(\beta-\alpha)(\phi-\psi)} - \frac{1}{36(\beta-\alpha)(\phi-\psi)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$- \frac{1}{36(\beta-\alpha)(\phi-\psi)} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right)$$
$$= -\frac{h\left(\frac{\alpha+\beta}{2}, \psi\right)}{36(\beta-\alpha)(\phi-\psi)} - \frac{1}{36(\beta-\alpha)(\phi-\psi)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right),$$

$$\frac{1}{36(\beta-\alpha)(\phi-\psi)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right)$$
$$= \frac{h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right)}{36(\beta-\alpha)(\phi-\psi)} + \frac{1}{36(\beta-\alpha)(\phi-\psi)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} h\left(\frac{q_1^n}{2}\beta + \left(1-\frac{q_1^n}{2}\right)\alpha, \frac{q_2^m}{2}\phi + \left(1-\frac{q_2^m}{2}\right)\psi\right).$$

We can calculate the value of the remaining three integrals in the same way as shown above, respectively, and adding them together to get the following result:

$$\int_0^1\int_0^1 P(x,z,q_1)T(y,w,q_2)\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h((1-z)\alpha+z\beta,(1-s)\psi+s\phi)}{{}_{\alpha}\partial_{q_1} z_{\psi}\partial_{q_2} w}{}_0d_{q_1}z_0d_{q_2}w$$
$$= \frac{h\left(\alpha,\frac{\psi+\phi}{2}\right) + h\left(\beta,\frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2},\frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2},\psi\right) + h\left(\frac{\alpha+\beta}{2},\phi\right)}{9}$$
$$+ \frac{h(\alpha,\psi)+h(\beta,\psi)+h(\alpha,\phi)+h(\beta,\phi)}{36}$$
$$- \frac{1-q_1}{6(\beta-\alpha)(\psi-\phi)}\left[\sum_{n=0}^{\infty} q_1^n h\left(q_1^n\beta + (1-q_1^n)\alpha,\psi\right) + 4\sum_{n=0}^{\infty} q_1^n h\left(q_1^n\beta + (1-q_1^n)\alpha, \frac{\psi+\phi}{2}\right)\right.$$
$$\left. + \sum_{n=0}^{\infty} q_1^n h\left(q_1^n\beta + (1-q_1^n)\alpha,\phi\right)\right]$$

$$- \frac{1-q_2}{6(\beta-\alpha)(\psi-\phi)}\left[\sum_{m=0}^{\infty} q_2^m h\left(\alpha, q_2^m\phi + (1-q_2^m)\psi\right) + 4\sum_{m=0}^{\infty} q_2^m h\left(\frac{\alpha+\beta}{2}, q_2^m\phi + (1-q_2^m)\psi\right)\right.$$
$$\left. + \sum_{m=0}^{\infty} q_2^m h\left(\beta, q_2^m\phi + (1-q_2^m)\psi\right)\right]$$
$$+ \frac{(1-q_1)(1-q_2)}{(\beta-\alpha)(\psi-\phi)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[h\left(q_1^n\beta + (1-q_1^n)\alpha, q_2^m\phi + (1-q_2^m)\psi\right)\right]$$

$$
\begin{aligned}
&= \frac{h\left(\alpha, \frac{\psi+\phi}{2}\right) + h\left(\beta, \frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2}, \psi\right) + h\left(\frac{\alpha+\beta}{2}, \phi\right)}{9} \\
&+ \frac{h(\alpha,\psi) + h(\beta,\psi) + h(\alpha,\phi) + h(\beta,\phi)}{36} \\
&- \frac{1}{6(\beta-\alpha)^2(\psi-\phi)} \int_\alpha^\beta \left[h(x,\psi) + 4h\left(x, \frac{\psi+\phi}{2}\right) + h(x,\phi)\right] {}_0d_{q_1}x \qquad (10) \\
&- \frac{1}{6(\beta-\alpha)(\phi-\psi)^2} \int_\psi^\phi \left[h(\alpha,y) + 4h\left(\frac{\alpha+\beta}{2}, y\right) + h(\beta,y)\right] {}_0d_{q_1}y \\
&+ \frac{1}{(\beta-\alpha)^2(\phi-\psi)^2} \int_\alpha^\beta \int_\psi^\phi h(x,y) {}_0d_{q_1}x {}_0d_{q_1}y.
\end{aligned}
$$

Multiplying both sides of (10) by $(\beta-\alpha)(\psi-\phi)$, we get the desired result. □

Theorem 8. *Let $h : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a mixed partially q_1q_2-differentiable mapping over Δ^o(the interior of Δ). Moreover, if mixed partially q_1q_2–derivative $\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(z,w)}{{}_\alpha\partial_{q_1} z {}_\psi\partial_{q_2} w}$ is continuous and integrable over $[\alpha,\beta]\times[\psi,\phi] \subset \Delta^o$ for $q_1, q_2 \in (0,1)$ and $\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(z,w)}{{}_\alpha\partial_{q_1} z {}_\psi\partial_{q_2} w}\right|$ is convex on co-ordinates over $[\alpha,\beta]\times[\psi,\phi]$, then the given inequality holds:*

$$
\begin{aligned}
&\left| \frac{h\left(\alpha, \frac{\psi+\phi}{2}\right) + h\left(\beta, \frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2}, \psi\right) + h\left(\frac{\alpha+\beta}{2}, \phi\right)}{9} \right. \\
&\left. + \frac{h(\alpha,\psi) + h(\beta,\psi) + h(\alpha,\phi) + h(\beta,\phi)}{36} + \frac{1}{(\beta-\alpha)(\phi-\psi)} \int_\alpha^\beta \int_\psi^\phi h(x,y)dydx - A \right| \\
&\leq (\beta-\alpha)(\phi-\psi)\left[M_{q_1}(A_{q_2}+C_{q_2}) \left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,\psi)}{{}_\alpha\partial_{q_1} z {}_\psi\partial_{q_2} w}\right| + M_{q_1}(B_{q_2}+D_{q_2}) \left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,\phi)}{{}_\alpha\partial_{q_1} z {}_\psi\partial_{q_2} w}\right| \right. \\
&\left. + N_{q_1}(A_{q_2}+C_{q_2}) \left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,\psi)}{{}_\alpha\partial_{q_1} z {}_\psi\partial_{q_2} w}\right| + N_{q_1}(B_{q_2}+D_{q_2}) \left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,\phi)}{{}_\alpha\partial_{q_1} z {}_\psi\partial_{q_2} w}\right| \right],
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{1}{6(\beta-\alpha)} \int_\alpha^\beta \left[h(x,\psi) + 4h\Big(x, \frac{\psi+\phi}{2}\Big) + h(x,\phi)\right] {}_0d_{q_1}x \\
&+ \frac{1}{6(\phi-\psi)} \int_\psi^\phi \left[h(\alpha,y) + 4h\Big(\frac{\alpha+\beta}{2}, y\Big) + h(\beta,y)\right] {}_0d_{q_2}y.
\end{aligned}
$$

Proof. Taking the absolute value on both sides of the equality of Lemma 4 and convexity of $\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(z,w)}{{}_\alpha\partial_{q_1} z {}_\psi\partial_{q_2} w}\right|$ on co-ordinates over $[\alpha,\beta]\times[\psi,\phi]$, then we have

$$
\begin{aligned}
&\left| \frac{h\left(\alpha, \frac{\psi+\phi}{2}\right) + h\left(\beta, \frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2}, \psi\right) + h\left(\frac{\alpha+\beta}{2}, \phi\right)}{9} \right. \\
&\left. + \frac{h(\alpha,\psi) + h(\beta,\psi) + h(\alpha,\phi) + h(\beta,\phi)}{36} + \frac{1}{(\beta-\alpha)(\phi-\psi)} \int_\alpha^\beta \int_\psi^\phi h(x,y)dydx - A \right|
\end{aligned}
$$

$$\leq (\beta-\alpha)(\phi-\psi)\int_0^1\int_0^1 |P(x,z,q_1)T(y,w,q_2)| \left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h((1-z)\alpha+z\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right| {}_0d_{q_1}z\,{}_0d_{q_2}w$$
$$\leq (\beta-\alpha)(\phi-\psi)\int_0^1 |T(y,w,q_2)|\left[\int_0^1 |P(x,z,q_1)|\left\{(1-z)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\right.\right.$$
$$\left.\left.+z\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\right\}{}_0d_{q_1}z\right]{}_0d_{q_2}w.$$

Computing the integral on the right-hand side of the above inequality, we have

$$\int_0^1 |P(x,z,q_1)|\left\{(1-z)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|+z\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\right\}{}_0d_{q_1}z$$
$$=\int_0^{\frac{1}{2}}\left|q_1z-\frac{1}{6}\right|\left\{(1-z)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|+z\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\right\}{}_0d_{q_1}z$$
$$+\int_{\frac{1}{2}}^1\left|q_1z-\frac{5}{6}\right|\left\{(1-z)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|+z\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\right\}{}_0d_{q_1}z$$
$$=\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\int_0^{\frac{1}{2}}(1-z)\left|q_1z-\frac{1}{6}\right|{}_0d_{q_1}z$$
$$+\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\int_0^{\frac{1}{2}}z\left|q_1z-\frac{1}{6}\right|{}_0d_{q_1}z$$
$$+\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\int_{\frac{1}{2}}^1(1-z)\left|q_1z-\frac{5}{6}\right|{}_0d_{q_1}z$$
$$+\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|\int_{\frac{1}{2}}^1 z\left|q_1z-\frac{5}{6}\right|{}_0d_{q_1}z.$$

Utilizing Lemma 3,

$$=A_{q_1}\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|+B_{q_1}\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|$$
$$+C_{q_1}\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|+D_{q_1}\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|.$$

After simplification, we get

$$=\frac{6q_1^3+4q_1^2+4q_1+1}{36(q_1^3+2q_1^2+2q_1+1)}\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|$$
$$+\frac{2q_1^2+2q_1+1}{12(q_1^3+2q_1^2+2q_1+1)}\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|$$
$$=M_{q_1}\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|+N_{q_1}\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,(1-w)\psi+w\phi)}{{}_{\alpha}\partial_{q_1}z\,{}_{\psi}\partial_{q_2}w}\right|.$$

We obtain

$$\left| \frac{h\left(\alpha, \frac{\psi+\phi}{2}\right) + h\left(\beta, \frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2}, \psi\right) + h\left(\frac{\alpha+\beta}{2}, \phi\right)}{9} \right.$$

$$\left. + \frac{h(\alpha,\psi) + h(\beta,\psi) + h(\alpha,\phi) + h(\beta,\phi)}{36} + \frac{1}{(\beta-\alpha)(\phi-\psi)} \int_\alpha^\beta \int_\psi^\phi h(x,y)dydx - A \right|$$

$$\leq (\beta-\alpha)(\phi-\psi) \int_0^1 |T(y,w,q_2)| \left[M_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2}}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} h(\alpha,(1-w)\psi + w\phi) \right| \right.$$

$$\left. + N_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2}}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} h(\beta,(1-w)\psi + w\phi) \right| \right] {}_0d_{q_2}w.$$

By a similar argument for the above integral, we have

$$\leq (\beta-\alpha)(\phi-\psi) \left[\int_0^{\frac{1}{2}} \left| q_2 w - \frac{1}{6} \right| \left\{ (1-w)M_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,\psi)}{{}_\alpha\partial_{q_1}\psi\partial_{q_2} w} \right| + wM_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,\phi)}{{}_\alpha\partial_{q_1} w_\psi \partial_{q_2} w} \right| \right. \right.$$

$$\left. + (1-w)N_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,\psi)}{{}_\alpha\partial_{q_1}\psi\partial_{q_2} w} \right| + wN_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,\phi)}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} \right| \right\}$$

$$+ \int_{\frac{1}{2}}^1 \left| q_2 w - \frac{5}{6} \right| \left\{ (1-w)M_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,\psi)}{{}_\alpha\partial_{q_1}\psi\partial_{q_2} w} \right| + wM_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,\phi)}{{}_\alpha\partial_{q_1} w_\psi \partial_{q_2} w} \right| \right.$$

$$\left. \left. + (1-w)N_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,\psi)}{{}_\alpha\partial_{q_1}\psi\partial_{q_2} w} \right| + wN_{q_1} \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,\phi)}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} \right| \right\} \right] {}_0d_{q_2}w.$$

Again utilizing Lemma 3, we get

$$\leq (\beta-\alpha)(\phi-\psi) \left[M_{q_1}(A_{q_2} + C_{q_2}) \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,\psi)}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} \right| + M_{q_1}(B_{q_2} + D_{q_2}) \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\alpha,\phi)}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} \right| \right.$$

$$\left. + N_{q_1}(A_{q_2} + C_{q_2}) \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,\psi)}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} \right| + N_{q_1}(B_{q_2} + D_{q_2}) \left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(\beta,\phi)}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} \right| \right].$$

We obtain our desired result. □

Remark 1. *Letting $q_1 \to 1^-$ and $q_2 \to 1^-$ in Theorem 8, then Theorem 8 converts into Theorem 3 proved in [17].*

Theorem 9. *Let $h : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a mixed partially q_1q_2-differentiable mapping over Δ^o (the interior of Δ). Moreover, if mixed partially q_1q_2-derivative $\frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(z,w)}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w}$ is continuous and integrable over $[\alpha,\beta] \times [\psi,\phi] \subset \Delta^o$ for $q_1, q_2 \in (0,1)$ and $\left| \frac{{}^{\alpha,\psi}\partial^2_{q_1,q_2} h(z,w)}{{}_\alpha\partial_{q_1} z_\psi \partial_{q_2} w} \right|^{r_1}$ is convex on co-ordinates over $[\alpha,\beta] \times [\psi,\phi]$for $r_1 > 1$ with $p_1^{-1} + r_1^{-1} = 1$, then the given inequality holds:*

$$\left| \frac{h\left(\alpha, \frac{\psi+\phi}{2}\right) + h\left(\beta, \frac{\psi+\phi}{2}\right) + 4h\left(\frac{\alpha+\beta}{2}, \frac{\psi+\phi}{2}\right) + h\left(\frac{\alpha+\beta}{2}, \psi\right) + h\left(\frac{\alpha+\beta}{2}, \phi\right)}{9} \right.$$

$$\left. + \frac{h(\alpha,\psi) + h(\beta,\psi) + h(\alpha,\phi) + h(\beta,\phi)}{36} + \frac{1}{(\beta-\alpha)(\phi-\psi)} \int_\alpha^\beta \int_\psi^\phi h(x,y)dydx - A \right|$$

$$\leq \frac{(\beta-\alpha)(\phi-\psi)\left(\left(G_{q_1}+H_{q_1}\right)\left(G_{q_2}+H_{q_2}\right)\right)^{\frac{1}{p_1}}}{[(1+q_1)(1+q_2)]^{\frac{1}{r_1}}}$$

$$\times\left(q_1q_2\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}+q_1\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}+q_2\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}+\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}\right)^{\frac{1}{r_1}},$$

where A is defined in Theorem 8.

Proof. Taking the absolute value on both sides of the equality of Lemma 4, using the (q_1,q_2)–Hölder inequality for functions of two variables inequality and convexity of $\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(z,w)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}$ on co-ordinates over $[\alpha,\beta]\times[\psi,\phi]$, then we have

$$\Bigg|\frac{h\left(\alpha,\frac{\psi+\phi}{2}\right)+h\left(\beta,\frac{\psi+\phi}{2}\right)+4h\left(\frac{\alpha+\beta}{2},\frac{\psi+\phi}{2}\right)+h\left(\frac{\alpha+\beta}{2},\psi\right)+h\left(\frac{\alpha+\beta}{2},\phi\right)}{9}$$

$$+\frac{h(\alpha,\psi)+h(\beta,\psi)+h(\alpha,\phi)+h(\beta,\phi)}{36}+\frac{1}{(\beta-\alpha)(\phi-\psi)}\int_\alpha^\beta\int_\psi^\phi h(x,y)dydx-A\Bigg|$$

$$\leq(\beta-\alpha)(\phi-\psi)\left(\int_0^1\int_0^1|(P(x,z,q_1)T(y,w,q_2)|^{p_2}\,{}_0d_{q_1}z_0d_{q_2}w\right)^{\frac{1}{p_2}}$$

$$\times\left(\int_0^1\int_0^1\left[(1-z)(1-w)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}+(1-z)w\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}\right.\right.$$

$$\left.\left.+(1-w)z\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}+wz\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}\right]{}_0d_{q_1}z_0d_{q_2}w\right)^{\frac{1}{r_1}}.$$

Utilizing Lemma 3, we obtain

$$\begin{aligned}&\int_0^1\int_0^1|P(x,z,q_1)T(y,w,q_2)|^{p_1}\,{}_0d_{q_1}z_0d_{q_2}w\\&=\left(\int_0^1|P(x,z,q_1)|^{p_1}\,{}_0d_{q_1}\right)\left(\int_0^1|T(y,w,q_2)|^{p_1}\,{}_0d_{q_2}w\right)\\&=\left(G_{q_1}+H_{q_1}\right)\left(G_{q_2}+H_{q_2}\right).\end{aligned}$$

Utilizing the q_1-integral and q_2-integral, we get

$$\begin{aligned}&\int_0^1(1-z)_0d_{q_1}z=\frac{q_1}{1+q_1},\\&\int_0^1(1-w)_0d_{q_2}w=\frac{q_2}{1+q_2},\\&\int_0^1 z_0d_{q_1}z=\frac{1}{1+q_1},\\&\int_0^1 w_0d_{q_2}w=\frac{1}{1+q_2}.\end{aligned}$$

Finally,

$$\leq \frac{(\beta-\alpha)(\phi-\psi)\left(\left(G_{q_1}+H_{q_1}\right)\left(G_{q_2}+H_{q_2}\right)\right)^{\frac{1}{p_1}}}{\left[(1+q_1)(1+q_2)\right]^{\frac{1}{r_1}}}$$

$$\times\left(q_1q_2\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}+q_1\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}+q_2\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}+\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r_1}\right)^{\frac{1}{r_1}},$$

which is our expected result. □

Theorem 10. *Let $h:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a mixed partially q_1q_2-differentiable mapping over Δ° (the interior of Δ). Moreover, if mixed partially q_1q_2-derivative $\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(z,w)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}$ is continuous and integrable over $[\alpha,\beta]\times[\psi,\phi]\subset\Delta^{\circ}$ for $q_1,q_2\in(0,1)$ and $\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(z,w)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}$ is convex on co-ordinates over $[\alpha,\beta]\times[\psi,\phi]$for $r\geq 1$, then the given inequality holds:*

$$\Bigg|\frac{h\left(\alpha,\frac{\psi+\phi}{2}\right)+h\left(\beta,\frac{\psi+\phi}{2}\right)+4h\left(\frac{\alpha+\beta}{2},\frac{\psi+\phi}{2}\right)+h\left(\frac{\alpha+\beta}{2},\psi\right)+h\left(\frac{\alpha+\beta}{2},\phi\right)}{9}$$
$$+\frac{h(\alpha,\psi)+h(\beta,\psi)+h(\alpha,\phi)+h(\beta,\phi)}{36}+\frac{1}{(\beta-\alpha)(\phi-\psi)}\int_{\alpha}^{\beta}\int_{\psi}^{\phi}h(x,y)dydx-A\Bigg|$$

$$\leq(\beta-\alpha)(\phi-\psi)\left[\left(E_{q_1}+F_{q_1}\right)\left(E_{q_2}+F_{q_2}\right)\right]^{1-\frac{1}{r}}$$
$$\times\Bigg(\left(A_{q_1}+C_{q_1}\right)\left(A_{q_2}+C_{q_2}\right)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}+\left(A_{q_1}+C_{q_1}\right)\left(B_{q_2}+D_{q_2}\right)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}$$
$$+\left(B_{q_1}+D_{q_1}\right)\left(A_{q_2}+C_{q_2}\right)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}+\left(B_{q_1}+D_{q_1}\right)\left(B_{q_2}+D_{q_2}\right)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}\Bigg)^{\frac{1}{r}},$$

where A is defined in Theorem 8.

Proof. Taking the absolute value on both sides of the equality of Lemma 4, using the (q_1,q_2)–Hölder inequality for functions of two variables inequality and convexity of $\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(z,w)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}$ on co-ordinates over $[\alpha,\beta]\times[\psi,\phi]$, then we have

$$\leq(\beta-\alpha)(\phi-\psi)\left(\int_0^1\int_0^1|P(x,z,q_1)T(y,w,q_2)|\,{}_0d_{q_1}z_0d_{q_2}w\right)^{1-\frac{1}{r}}\left(\int_0^1\int_0^1|P(x,z,q_1)T(y,w,q_2)|\right)$$
$$\times\Bigg[\Bigg((1-z)(1-w)\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}+(1-z)w\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\alpha,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}$$
$$+(1-w)z\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\psi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}+wz\left|\frac{{}_{\alpha,\psi}\partial^2_{q_1,q_2}h(\beta,\phi)}{{}_{\alpha}\partial_{q_1}z_{\psi}\partial_{q_2}w}\right|^{r}\Bigg]{}_0d_{q_1}z_0d_{q_2}w\Bigg)^{\frac{1}{r}}.$$

Utilizing Lemma 3, we observe that

$$\begin{aligned}
&\int_0^1 \int_0^1 |P(x,z,q_1)T(y,w,q_2)|\, {}_0d_{q_1}z\, {}_0d_{q_2}w \\
&= \left(\int_0^1 |P(x,z,q_1)|\, {}_0d_{q_1}z\right) \left(\int_0^1 |T(y,w,q_2)|\, {}_0d_{q_2}w\right) \\
&= \left(E_{q_1} + F_{q_1}\right) \left(E_{q_2} + F_{q_2}\right), \\
&\int_0^1 \int_0^1 (1-z)(1-w)\, |P(x,z,q_1)T(y,w,q_2)|\, {}_0d_{q_1}z\, {}_0d_{q_2}w \\
&= \left(\int_0^1 (1-z)\, |P(x,z,q_1)|\, {}_0d_{q_1}z\right) \left(\int_0^1 (1-w)\, |T(y,w,q_2)|\, {}_0d_{q_2}w\right) \\
&= \left(A_{q_1} + C_{q_1}\right) \left(A_{q_2} + C_{q_2}\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 (1-z)w\, |P(x,z,q_1)T(y,w,q_2)|\, {}_0d_{q_1}z\, {}_0d_{q_2}w \\
&= \left(\int_0^1 (1-z)\, |P(x,z,q_1)|\, {}_0d_{q_1}z\right) \left(\int_0^1 w\, |T(y,w,q_2)|\, {}_0d_{q_2}w\right) \\
&= \left(A_{q_1} + C_{q_1}\right) \left(B_{q_2} + D_{q_2}\right), \\
&\int_0^1 \int_0^1 z(1-w)\, |P(x,z,q_1)T(y,w,q_2)|\, {}_0d_{q_1}z\, {}_0d_{q_2}w \\
&= \left(\int_0^1 z\, |P(x,z,q_1)|\, {}_0d_{q_1}z\right) \left(\int_0^1 (1-w)\, |T(y,w,q_2)|\, {}_0d_{q_2}w\right) \\
&= \left(B_{q_1} + D_{q_1}\right) \left(A_{q_2} + C_{q_2}\right), \\
&\int_0^1 \int_0^1 zw\, |P(x,z,q_1)T(y,w,q_2)|\, {}_0d_{q_1}z\, {}_0d_{q_2}w \\
&= \left(\int_0^1 z\, |P(x,z,q_1)|\, {}_0d_{q_1}z\right) \left(\int_0^1 w\, |T(y,w,q_2)|\, {}_0d_{q_2}w\right) \\
&= \left(B_{q_1} + D_{q_1}\right) \left(B_{q_2} + D_{q_2}\right).
\end{aligned}$$

Using the values of the above q_1q_2-integrals, we get the desired inequality. □

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