symmetry

MDPI

Article

# Sufficiency Criterion for A Subfamily of Meromorphic Multivalent Functions of Reciprocal Order with Respect to Symmetric Points

**Shahid Mahmood [1,*], Gautam Srivastava [2,3], H.M. Srivastava [4,5], Eman S.A. Abujarad [6], Muhammad Arif [7] and Fazal Ghani [7]**

1 Department of Mechanical Engineering, Sarhad University of Science & I. T Landi Akhun Ahmad, Hayatabad Link. Ring Road, Peshawar 25000, Pakistan
2 Department of Mathematics and Computer Science, Brandon University, 270 18th Street, Brandon, MB R7A 6A9, Canada; srivastavag@brandonu.ca
3 Research Center for Interneural Computing, China Medical University, Taichung 40402, Taiwan
4 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
5 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
6 Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India; emanjarad2@gmail.com
7 Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan; marifmaths@awkum.edu.pk (M.A.); fazalghanimaths@gmail.com (F.G.)
* Correspondence: shahidmahmood757@gmail.com

Received: 5 May 2019; Accepted: 30 May 2019; Published: 5 June 2019

check for updates

**Abstract:** In the present research paper, our aim is to introduce a new subfamily of meromorphic $p$-valent (multivalent) functions. Moreover, we investigate sufficiency criterion for such defined family.

**Keywords:** meromorphic multivalent starlike functions; subordination

## 1. Introduction

Let the notation $\Omega_p$ be the family of meromorphic $p$-valent functions $f$ that are holomorphic (analytic) in the region of punctured disk $\mathbb{E} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and obeying the following normalization

$$f(z) = \frac{1}{z^p} + \sum_{j=1}^{\infty} a_{j+p}\, z^{j+p} \quad (z \in \mathbb{E}). \tag{1}$$

In particular $\Omega_1 = \Omega$, the familiar set of meromorphic functions. Further, the symbol $\mathcal{MS}^*$ represents the set of meromorphic starlike functions which is a subfamily of $\Omega$ and is given by

$$\mathcal{MS}^* = \left\{ f : f(z) \in \Omega \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) < 0 \ \ (z \in \mathbb{E}) \right\}.$$

Two points $p$ and $p'$ are said to be symmetrical with respect to $o$ if $o'$ is the midpoint of the line segment $pp'$. This idea was further nourished in [1,2] by introducing the family $\mathcal{MS}_s^*$ which is defined in set builder form as;

$$\mathcal{MS}_s^* = \left\{ f : f(z) \in \Omega \text{ and } \Re\left(\frac{-2zf'(z)}{f(-z) - f(z)}\right) < 0 \ \ (z \in \mathbb{E}) \right\}.$$

Now, for $-1 \leq t < s \leq 1$ with $s \neq 0 \neq t$, $0 < \xi < 1$, $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$ and $p \in \mathbb{N}$, we introduce a subfamily of $\Omega_p$ consisting of all meromorphic $p$-valent functions of reciprocal order $\xi$, denoted by $\mathcal{NS}_p^{\lambda}(s,t,\xi)$, and is defined by

$$\mathcal{NS}_p^{\lambda}(s,t,\xi) = \left\{ f : f(z) \in \Omega_p \text{ and } \Re\left(e^{-i\lambda}\frac{ps^pt^p}{s^p - t^p}\frac{f(sz) - f(tz)}{zf'(z)}\right) > \xi\cos\lambda \ \ (z \in \mathbb{E}) \right\}.$$

We note that for $p = s = 1$ and $t = -1$, the class $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ reduces to the class $\mathcal{NS}_1^{\lambda}(1,-1,\xi) = \mathcal{NS}_*^{\lambda}(\xi)$ and is represented by

$$\mathcal{NS}_*^{\lambda}(\xi) = \left\{ f : f(z) \in \Omega \text{ and } \Re\left(e^{-i\lambda}\frac{f(-z) - f(z)}{2zf'(z)}\right) > \xi\cos\lambda \ \ (z \in \mathbb{E}) \right\}.$$

For detail of the related topics, see the work of Al-Amiri and Mocanu [3], Rosihan and Ravichandran [4], Aouf and Hossen [5], Arif [6], Goyal and Prajapat [7], Joshi and Srivastava [8], Liu and Srivastava [9], Raina and Srivastava [10], Sun et al. [11], Shi et al. [12] and Owa et al. [13], see also [14–16].

For simplicity and ignoring the repetition, we state here the constraints on each parameter as $0 < \xi < 1$, $-1 \leq t < s \leq 1$ with $s \neq 0 \neq t$, $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$ and $p \in \mathbb{N}$.

We need to mention the following lemmas which will use in the main results.

**Lemma 1.** *"Let $H \subset \mathbb{C}$ and let $\Phi : \mathbb{C}^2 \times \mathbb{E}^* \to \mathbb{C}$ be a mapping satisfying $\Phi(ia, b : z) \notin H$ for $a, b \in \mathbb{R}$ such that $b \leq -n\frac{1+a^2}{2}$. If $p(z) = 1 + c_nz^n + \cdots$ is regular in $\mathbb{E}^*$ and $\Phi(p(z), zp'(z) : z) \in H \ \forall\, z \in \mathbb{E}^*$, then $\Re(p(z)) > 0$."*

**Lemma 2.** *"Let $p(z) = 1 + c_1z + \cdots$ be regular in $\mathbb{E}^*$ and $\eta$ be regular and starlike univalent in $\mathbb{E}^*$ with $\eta(0) = 0$. If $zp'(z) \prec \eta(z)$, then*

$$p(z) \prec 1 + \int_0^z \frac{\eta(t)}{t}dt.$$

*This result is the best possible."*

## 2. Sufficiency Criterion for the Family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$

In this section, we investigate the sufficiency criterion for any meromorphic $p$-valent functions belonging to the introduced family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ :

Now, we obtain the necessary and sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ as follows:

**Theorem 1.** *Let the function $f(z)$ be the member of the family $\Omega_p$. Then*

$$f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi) \Leftrightarrow \left| \frac{e^{i\lambda}}{\mathcal{G}(z)} - \frac{1}{2\xi\cos\lambda} \right| < \frac{1}{2\xi\cos\lambda}, \tag{2}$$

*where*

$$\mathcal{G}(z) = \frac{p\,s^pt^p}{(s^p - t^p)}\frac{f(sz) - f(tz)}{zf'(z)}. \tag{3}$$

**Proof.** Suppose that inequality (2) holds. Then, we have

$$\begin{aligned}
\left|\frac{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}(z)}{2\xi\cos\lambda e^{-i\lambda}\mathcal{G}(z)}\right| &< \frac{1}{2\xi\cos\lambda} \\
&\Leftrightarrow \left|\frac{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}(z)}{2\xi\cos\lambda e^{-i\lambda}\mathcal{G}(z)}\right|^2 < \frac{1}{4\xi^2\cos^2\lambda} \\
&\Leftrightarrow \left(2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}(z)\right)\left(\overline{2\xi\cos\lambda - e^{-i\lambda}\mathcal{G}(z)}\right) < \left(e^{i\lambda}\overline{\mathcal{G}(z)}\right)e^{-i\lambda}\mathcal{G}(z) \\
&\Leftrightarrow 4\xi^2\cos^2\lambda - 2\xi\cos\lambda\left(e^{i\lambda}\overline{\mathcal{G}(z)} + e^{-i\lambda}\mathcal{G}(z)\right) < 0 \\
&\Leftrightarrow 2\xi\cos\lambda - 2\Re\left(e^{-i\lambda}\mathcal{G}(z)\right) < 0 \\
&\Leftrightarrow \Re\left(e^{-i\lambda}\mathcal{G}(z)\right) > \xi\cos\lambda,
\end{aligned}$$

and hence the result follows. □

Next, we investigate the sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ in the following theorem:

**Theorem 2.** *If $f(z)$ belongs to the family $\Omega_p$ of meromorphic p-valent functions and obeying*

$$\sum_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}s^p t^p - \frac{n\beta\cos\lambda}{p}e^{i\lambda}\right)\right||a_n| < \frac{1}{2}\left(1 - \left|1 - 2\beta\cos\lambda e^{i\lambda}\right|\right), \tag{4}$$

*then $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$.*

**Proof.** To prove the required result we only need to show that

$$\left|\frac{2e^{i\lambda}\xi\cos\lambda z f'(z)/p - \frac{s^p t^p}{(t^p - s^p)}\left(f(tz) - f(sz)\right)}{\frac{s^p t^p}{(t^p - s^p)}\left(f(tz) - f(sz)\right)}\right| < 1. \tag{5}$$

Now consider the left hand side of (5), we get

$$\begin{aligned}
LHS &= \left|\frac{2e^{i\lambda}\xi\cos\lambda z f'(z)/p - \frac{s^p t^p}{(t^p - s^p)}\left(f(tz) - f(sz)\right)}{\frac{s^p t^p}{(t^p - s^p)}\left(f(tz) - f(sz)\right)}\right| \\
&= \left|\frac{\left(2e^{i\lambda}\xi\cos\lambda - 1\right) + \sum\limits_{n=p+1}^{\infty}\left(\frac{s^n - t^n}{s^p - t^p}s^p t^p - \frac{2n\xi\cos\lambda}{p}e^{i\lambda}\right)a_n z^{n+p}}{1 + \sum\limits_{n=p+1}^{\infty}\left(\frac{s^n - t^n}{s^p - t^p}\right)s^p t^p a_n z^{n+p}}\right| \\
&\leq \frac{\left|2e^{i\lambda}\xi\cos\lambda - 1\right| + \sum\limits_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}s^p t^p - 2\beta\cos\lambda e^{i\lambda}\frac{n}{p}\right)\right||a_n||z^{n+p}|}{1 - \sum\limits_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}\right)s^p t^p\right||a_n||z^{n+p}|} \\
&\leq \frac{\left|2e^{i\lambda}\xi\cos\lambda - 1\right| + \sum\limits_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}s^p t^p - 2\beta\cos\lambda e^{i\lambda}\frac{n}{p}\right)\right||a_n|}{1 - \sum\limits_{n=p+1}^{\infty}\left|\left(\frac{s^n - t^n}{s^p - t^p}\right)s^p t^p\right||a_n|}.
\end{aligned}$$

By virtue of inequality (4), we at once get the desired result. □

Also, we obtain another sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ by using Lemma 1, in the following theorem:

**Theorem 3.** *If $f(z) \in \Omega_p$ satisfies*

$$\Re\left\{e^{-i\lambda}\left(\alpha z\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}+1\right)\mathcal{G}(z)\right\} > \beta\cos\lambda - \frac{n}{2}\left((1-\beta)\alpha\cos\lambda\right),$$

*then $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, where $\mathcal{G}(z)$ is defined in Equation (3).*

**Proof.** Let we choose the function $q(z)$ by

$$q(z) = \frac{e^{-i\lambda}\mathcal{G}(z) - \beta\cos\lambda + i\sin\lambda}{(1-\beta)\cos\lambda}, \tag{6}$$

then Equation (6) shows that $q(z)$ is holomorphic in $\mathbb{E}$ and also normalized by $q(0) = 1$.

From Equation (6), we can easily obtain that

$$e^{-i\lambda}\mathcal{G}(z)\left(1+\alpha z\frac{\mathcal{G}'(z)}{\mathcal{G}(z)}\right) = \Phi\left(q(z), zq'(z), z\right),$$

where

$$\Phi\left(q(z), zq'(z), z\right) = \left[(1-\beta)\alpha zq'(z) + (1-\beta)q(z) + \beta\right]\cos\lambda - i\sin\lambda.$$

Now for all $a, b \in \mathbb{R}$ satisfying $2y \leq -n\left(1+a^2\right)$, we have

$$\begin{aligned} \Re\left\{\Phi(ia,b,z)\right\} &\leq \beta\cos\lambda - \frac{n}{2}\left(1+a^2\right)(1-\beta)\alpha\cos\lambda \\ &\leq \beta\cos\lambda - \frac{n}{2}(1-\beta)\alpha\cos\lambda. \end{aligned}$$

Now, let us define a set as

$$H = \left\{\zeta : \Re(\zeta) > \beta\cos\lambda - \frac{n}{2}\left((1-\beta)\alpha\cos\lambda\right)\right\},$$

then, we see that $\Phi(ia,b,z) \notin H$ and $\Phi\left(q(z), zq'(z), z\right) \in H$. Therefore, by using Lemma 1, we obtain that $\Re(q(z)) > 0$.
□

Further, in the next theorem, we obtain the sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ by using Lemma 2.

**Theorem 4.** *If $f(z)$ is a member of the family $\Omega_p$ of meromorphic p-valent functions and satisfies*

$$\left|\frac{e^{i\lambda}}{\mathcal{G}(z)}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right)\right| < \frac{1}{\beta\cos\lambda} - 1, \tag{7}$$

*then $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, where $\mathcal{G}(z)$ is given by Equation (3).*

**Proof.** In order to prove the required result, we need to define the following function

$$q(z)\cos\lambda = e^{-i\lambda}\mathcal{G}(z) + i\sin\lambda,$$

then, Equation (6) shows that th function $q(z)$ is holomorphic in $\mathbb{E}$ and also normalized by $q(0) = 1$.

Now, by routine computations, we get

$$\frac{zq'(z)}{q(z) - i\tan\lambda} = \frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}.$$

Now, let us consider $z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)'$ and then by using inequality (7), we have

$$\left|z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)'\right| = \left|\frac{e^{i\lambda}}{\mathcal{G}(z)}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right)\right| < \frac{1}{\beta\cos\lambda} - 1,$$

therefore

$$z\left(\frac{1}{q(z)\cos\lambda - i\sin\lambda}\right)' \prec \frac{(1-\beta\cos\lambda)z}{\beta\cos\lambda}.$$

Using Lemma 2, we have

$$\frac{1}{(q(z) - i\tan\lambda)\cos\lambda} \prec 1 + \frac{(1-\beta\cos\lambda)}{\beta\cos\lambda}z,$$

equivalently

$$(q(z) - i\tan\lambda)\cos\lambda \prec \frac{\beta\cos\lambda}{\beta\cos\lambda + (1-\beta\cos\lambda)z} = H(z)\,(say). \tag{8}$$

After simplifications, we get

$$1 + \Re\left(\frac{zH''(z)}{H'(z)}\right) = 2\beta\cos\lambda - 1 > 0, \;\; for\ \frac{1}{2} < \beta < 1.$$

The region $H(\mathbb{E})$ shows that it is symmetric about the real axis and also $H(z)$ is convex. Hence

$$\Re\ (\mathcal{G}(z)) \geq H(1) > 0,$$

or

$$\Re\ (q(z)\cos\lambda - i\sin\lambda) > \beta\cos\lambda,$$

or

$$\Re\ \left(e^{-i\lambda}\mathcal{G}(z)\right) > \beta\cos\lambda,\ for\ \frac{1}{2} < \beta < 1.$$

□

Finally, we investigate the sufficient condition for the p-valent function $f$ to be in the family $\mathcal{NS}_p^{\lambda}(s,t,\xi)$ in the following theorem:

**Theorem 5.** *If $f(z) \in \Omega_p$ satisfies*

$$\left|\left(\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right)'\right| \leq \eta\,|z|^{\gamma},\ for\ \ 0 < \eta \leq \gamma + 1, \tag{9}$$

*then $f(z) \in \mathcal{NS}_p^{\lambda}(s,t,\xi)$, where $\mathcal{G}(z)$ is defined in Equation (3).*

**Proof.** Let us put

$$G(z) = z\left(\frac{2\beta\cos\lambda e^{i\lambda}}{\mathcal{G}(z)} - 1\right).$$

Then $G(0) = 0$ and $G(z)$ is analytic in $\mathbb{E}$. Using inequality (9), we can write

$$\left| \left( \frac{G(z)}{z} \right)' \right| = \left| \left( \frac{2\beta \cos \lambda e^{i\lambda}}{\mathcal{G}(z)} - 1 \right)' \right| \leq \eta \, |z|^{\gamma}.$$

Now,

$$\left| \left( \frac{G(z)}{z} \right) \right| = \left| \int_0^z \left( \frac{G(t)}{t} \right)' dt \right| \leq \int_0^{|z|} \left| \left( \frac{G(t)}{t} \right)' \right| dt \leq \int_0^{|z|} \eta \, |t|^{\gamma} \, dt = \frac{\eta \, |z|^{\gamma+1}}{\gamma+1} < 1,$$

and this implies that

$$\left| \frac{2\beta \cos \lambda e^{i\lambda}}{\mathcal{G}(z)} - 1 \right| < 1.$$

Now by using Theorem 1, we get the result which we needed. □

## 3. Conclusions

In our results, a new subfamily of meromorphic $p$-valent (multivalent) functions were introduced. Further, various sufficient conditions for meromorphic $p$-valent functions belonging to these subfamilies were obtained and investigated.

**Author Contributions:** Conceptualization, H.M.S. and M.A.; Formal analysis, H.M.S. and S.M.; Funding acquisition, S.M. and G.S.; Investigation, E.S.A.A. and S.M.; Methodology, M.A. and F.G.; Supervision, H.M.S. and M.A.; Validation, M.A. and S.M.; Visualization, G.S. and E.S.A.A.; Writing original draft, M.A., S.M. and F.G.; Writing review and editing, M.A., F.G. and S.M.

**Funding:** This research received no external funding.

**Acknowledgments:** The authors would like to thank the reviewers of this paper for their valuable comments on the earlier version of the paper. They would also like to acknowledge Salim ur Rehman, the Vice Chancellor, Sarhad University of Science & I.T, for providing excellent research environment and his financial support.

**Conflicts of Interest:** All the authors declare that they have no conflict of interest.

## References

1. Srivastava, H.M.; Yang, D.-G.; Xu, N.-E. Some subclasses of meromorphically multivalent functions associated with a linear operator. *Appl. Math. Comput.* **2008**, *195*, 11–23. [CrossRef]
2. Wang, Z.-G.; Jiang, Y.-P.; Srivastava, H.M. Some subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function. *Comput. Math. Appl.* **2009**, *57*, 571–586. [CrossRef]
3. Al-Amiri, H.; Mocanu, P.T. Some simple criteria of starlikeness and convexity for meromorphic functions. *Mathematica (Cluj)* **1995**, *37*, 11–21.
4. Ali, R.M.; Ravichandran, V. Classes of meromorphic $\alpha$-convex functions. *Taiwanese J. Math.* **2010**, *14*, 1479–1490. [CrossRef]
5. Aouf, M.K.; Hossen, H.M. New criteria for meromorphic $p$-valent starlike functions. *Tsukuba J. Math.* **1993**, *17*, 481–486. [CrossRef]
6. Arif, M. On certain sufficiency criteria for $p$-valent meromorphic spiralike functions. In *Abstract and Applied Analysis*; Hindawi: London, UK, 2012.
7. Goyal, S.P.; Prajapat, J.K. A new class of meromorphic multivalent functions involving certain linear operator. *Tamsui Oxf. J. Math. Sci.* **2009**, *25*, 167–176.
8. Joshi, S.B.; Srivastava, H.M. A certain family of meromorphically multivalent functions. *Comput. Math. Appl.* **1999**, *38*, 201–211. [CrossRef]
9. Liu, J.-L.; Srivastava, H.M. A linear operator and associated families of meromorphically multivalent functions. *J. Math. Anal. Appl.* **2001**, *259*, 566–581. [CrossRef]
10. Raina, R.K.; Srivastava, H.M. A new class of mermorphically multivalent functions with applications of generalized hypergeometric functions. *Math. Comput. Model.* **2006**, *43*, 350–356. [CrossRef]

11. Sun, Y.; Kuang, W.-P.; Wang, Z.-G. On meromorphic starlike functions of reciprocal order $\alpha$. *Bull. Malays. Math. Sci. Soc.* **2012**, *35*, 469–477.
12. Shi, L.; Wang, Z.-G.; Yi, J.-P. A new class of meromorphic functions associated with spirallike functions. *J. Appl. Math.* **2012**, *2012*, 1–12. [CrossRef]
13. Owa, S.; Darwish, H.E.; Aouf, M.A. Meromorphically multivalent functions with positive and fixed second coefficients. *Math. Japon.* **1997**, *46*, 231–236.
14. Arif, M.; Ahmad, B. New subfamily of meromorphic starlike functions in circular domain involving q-differential operator. *Math. Slovaca* **2018**, *68*, 1049–1056. [CrossRef]
15. Arif, M.; Raza, M.; Ahmad, B. A new subclass of meromorphic multivalent close-to-convex functions. *Filomat* **2016**, *30*, 2389–2395. [CrossRef]
16. Arif, M.; Sokół J.; Ayaz, M. Sufficient condition for functions to be in a class of meromorphic multivalent Sakaguchi type spiral-like functions. *Acta Math. Sci.* **2014**, *34*, 1–4. [CrossRef]