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Discrete Orthogonality of Bivariate Polynomials of A_2, C_2 and G_2

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Abstract: We develop discrete orthogonality relations on the finite sets of the generalized Chebyshev nodes related to the root systems A_2, C_2 and G_2. The orthogonality relations are consequences of orthogonality of four types of Weyl orbit functions on the fragments of the dual weight lattices. A uniform recursive construction of the polynomials as well as explicit presentation of all data needed for the discrete orthogonality relations allow practical implementation of the related Fourier methods. The polynomial interpolation method is developed and exemplified.

Keywords: orthogonal polynomial; discrete Fourier transform; Weyl group

1. Introduction

The purpose of this article is to develop uniform explicit discrete orthogonality relations of ten types of bivariate generalized Chebyshev polynomials [1,2]. The discrete orthogonality relations are presented for two families of polynomials corresponding to the Lie algebra A_2, and four, to the algebras C_2 and G_2. Inherent explicit formulas for the bivariate polynomial interpolation are deduced and exemplified.

Orthogonal polynomials of two variables have been studied by many authors during the last several decades—see e.g., [3–7] and the references therein. Polynomials of more than one variable that are orthogonal on a finite set of discrete points are considered in [8–14]. The four types of the current generalized Chebyshev polynomials corresponding to root systems of Weyl groups are induced by the four types of the Weyl orbit functions. The symmetric $C-$ and antisymmetric $S-$functions are inherent for root systems of all types, the hybrid S^s- and S^l-functions are defined for B_n, C_n, F_4 and G_2 only [2]. For the lowest case of root system A_1, the classical Chebyshev polynomials of the first and the second kind are obtained from the corresponding $C-$ and $S-$functions, respectively. The four types of polynomials in [13] are built on G_2 symmetry and their discretization is inherent to G_2 lattice. The A_2 polynomials of [9,11,14,15] and the G_2 polynomials of [13], together with their discretizations, can be translated into our cases by a substitution of variables.

The property that distinguishes our method of discretization of the polynomials is its uniformity. The same chain of construction steps is followed for the polynomials with underlying root system of any type and rank. Our limitation to bivariate polynomials is of a practical kind: the polynomials are presented here in a ready-to-use form for anyone who may have some use for them. Also the uniformity of our treatment of A_2, C_2 and G_2 polynomials becomes obvious. There are two fundamental means of calculation of the polynomials. The recurrence relations construction [13,15] is summarized for bivariate cases in this paper whereas the generating function method is developed recently in [16,17]. If the problem should be regarded as discretization of known polynomials of two continuous variables, then very few such polynomials can be discretized by the method developed in this paper. The polynomials of two continuous variables that are amenable to our discretization are those that are the 'closest' to the properties of finite dimensional irreducible representations of compact simple Lie groups of rank two, namely the groups of types A_2, C_2, and G_2.

Due to their proximity to the simple Lie groups, many of the powerful properties of these groups, that depend on the rank and type of the given group, can be translated to properties of the polynomials. Discretization of characters is a property, which underlies the discrete orthogonality of several infinite families of polynomials for each of the groups of rank two. The discrete orthogonality relations of the polynomials are induced by the orthogonality of the underlying Weyl orbit functions. Several discretizations of the Weyl orbit functions are currently known [18–22]. Dual weight lattice discretization of the orbit functions, used in this paper, is presented in all details in [19,21]. The fragments of the dual weight lattices are transformed to the discrete set of points (not a lattice fragment), where the polynomials are discretely orthogonal. See examples in Figures 1–5.

A uniform description of the polynomials and their orthogonality on discrete sets of points require a precise set up of the details of the theory that is often found in the literature in diverse variants. In Section 2 we first introduce the four bases in the real Euclidean space spanned by the roots of the three groups. There are four bases needed for our consideration of C_2 and G_2. These are the bases of roots, coroots, weights and coweights. There are only two bases for A_2, the bases of roots and coroots coincide, as well as the bases of weights and coweights. Section 3 contains the explicit forms of the A_2, C_2 and G_2 orbit functions, which are needed for the construction of the polynomials, together with a description of their discrete orthogonality. In Section 4, up to four kinds of polynomials are described and the recursion relations, necessary for their explicit calculation, are detailed. The sets of points, the corresponding sets of weights and the resulting discrete orthogonality relations for the polynomials are presented in Section 5. The application of the discrete orthogonality relations to polynomial interpolation is also shown. Section 6 contains various concluding remarks and follow-up questions.

2. Weyl Groups and Corresponding Domains

2.1. Roots, Coroots, Weights, Coweights

It is practical for the uniformity of our construction for all the three cases to use four bases in $\mathbb{R}^2$:

- the $\alpha-$basis of simple roots α_1, α_2,
- the $\alpha^\vee-$basis of coroots $\alpha_1^\vee$, $\alpha_2^\vee$,
- the $\omega-$basis of the fundamental weights ω_1, ω_2,
- the $\omega^\vee-$basis of coweights $\omega_1^\vee$, $\omega_2^\vee$.

The ordered pair of simple roots $\Delta = (\alpha_1, \alpha_2)$ of a simple Lie algebra of rank two consists of two vectors spanning $\mathbb{R}^2$, the real $2-$dimensional Euclidean space [23,24]. The roots of Δ form a basis of $\mathbb{R}^2$ which satisfy certain specific conditions that are different for A_2, C_2, and G_2, namely their relative lengths and angles between them.

For the cases of A_2, C_2 and G_2, the standard convention $\langle \alpha, \alpha \rangle = 2$ is used for the squared length of the long roots. Then for short roots of C_2 we have $\langle \alpha_1, \alpha_1 \rangle = 1$ and for G_2 the squared length of the short root is $\langle \alpha_2, \alpha_2 \rangle = 2/3$. The angle between the long root and the short root is $3\pi/4$ for C_2 and $5\pi/6$ for G_2. The angle between the two (long) roots of A_2 is $2\pi/3$. The coroots $\alpha_1^\vee$, $\alpha_2^\vee$ are defined as $\alpha_i^\vee = 2\alpha_i / \langle \alpha_i, \alpha_i \rangle$, $i = 1, 2$. In addition to the $\alpha-$basis of simple roots, we define the $\omega-$basis by

$$\langle \alpha_i^\vee, \omega_j \rangle = \left\langle \alpha_i, \omega_j^\vee \right\rangle = \delta_{ij}, \qquad i, j \in \{1, 2\},$$

where the $\omega^\vee-$basis, is given by $\omega_i^\vee = 2\omega_i / \langle \alpha_i, \alpha_i \rangle$, $i = 1, 2$.

The root lattice Q and the coroot lattice $Q^\vee$ are the sets of all integer linear combinations of simple roots and coroots, respectively,

$$Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2, \qquad Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee.$$

Similarly the weight lattice P and the coweight lattice $P^\vee$ are given as

$$P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \qquad P^\vee = \mathbb{Z}\omega_1^\vee + \mathbb{Z}\omega_2^\vee.$$

Important subsets of the weight lattice P are the cone of dominant weights P^+, and the cone of strictly dominant weights P^{++},

$$P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2, \qquad P^{++} = \mathbb{Z}^{>0}\omega_1 + \mathbb{Z}^{>0}\omega_2.$$

2.2. Weyl Group and Affine Weyl Group

The reflection r_α, $\alpha \in \Delta$, which fixes the hyperplane orthogonal to α and passes through the origin of $\mathbb{R}^2$, can be explicitly written as

$$r_\alpha x = x - \langle \alpha, x \rangle \alpha^\vee \,, \qquad x \in \mathbb{R}^2 \,.$$

Given a simple Lie algebra with the set of simple roots Δ, the associated Weyl group W is a finite group generated by reflections $r_i \equiv r_{\alpha_i}$, $i = 1, 2$. Acting on the simple roots Δ, the resulting system of vectors $W\Delta$ is the root system that contains the highest root $\xi \in W\Delta$. The highest roots are linear combinations of simple roots with positive integer coefficients, $\xi = m_1\alpha_1 + m_2\alpha_2$. For all three cases they are given as follows:

$$A_2 : \; \xi = \alpha_1 + \alpha_2 \,, \qquad C_2 : \; \xi = 2\alpha_1 + \alpha_2 \,, \qquad G_2 : \; \xi = 2\alpha_1 + 3\alpha_2 \,. \tag{1}$$

The affine reflection r_0 with respect to the highest root ξ is given by

$$r_0 x = r_\xi x + \frac{2\xi}{\langle \xi, \xi \rangle} \,, \qquad r_\xi x = x - \frac{2\langle x, \xi \rangle}{\langle \xi, \xi \rangle}\xi \,, \qquad x \in \mathbb{R}^2 \,. \tag{2}$$

By adding the affine reflection r_0 to the set of generators $\{r_1, r_2\}$ one obtains the affine Weyl group W^{aff}. The group W^{aff} consists of transformations of $\mathbb{R}^2$ from W and of shifts by vectors from the coroot lattice $Q^\vee$. In fact it holds that $W^{\mathrm{aff}} = Q^\vee \rtimes W$. The fundamental domain F of the action of W^{aff} on $\mathbb{R}^2$ is a triangle with vertices $\left\{0, \frac{\omega_1^\vee}{m_1}, \frac{\omega_2^\vee}{m_2}\right\}$, where m_1, m_2 are the coefficients of the highest root ξ in $\alpha-$basis (2). The subset $F^s \subset F$, which contains points of F that are not stabilized by any reflection corresponding to a short root, is called the short fundamental domain F^s. The subset $F^l \subset F$, which contains points of F that are not stabilized by any reflection corresponding to a long root or by r_0, is called the long fundamental domain F^l.

The hyperplane orthogonal to the highest root and containing the origin of R^2, divides the roots in the root system $W\Delta$ into two subsets: the set of positive roots, which contains all simple roots, and the set of negative roots. The half-sum of the positive roots is the vector denoted ϱ. For all three cases it is given by

$$\varrho = \omega_1 + \omega_2. \tag{3}$$

The vectors ϱ^s, ϱ^l which are halves of the sums of the positive short or long roots, respectively, are given for C_2 and G_2 as follows,

$$\begin{aligned} C_2: \quad & \varrho^s = \omega_1, \qquad \varrho^l = \omega_2, \\ G_2: \quad & \varrho^s = \omega_2, \qquad \varrho^l = \omega_1. \end{aligned} \tag{4}$$

2.3. Dual Affine Weyl Group

The set of coroots $\Delta^\vee = (\alpha_1^\vee, \alpha_2^\vee)$, viewed as a set of simple roots by its own right, also generates the identical Weyl group W. The system of vectors $W\Delta^\vee$ is a root system and contains the highest dual root $\eta \in W\Delta^\vee$. These highest roots, which can again be expressed as combinations of simple roots with natural coefficients $\eta = m_1^\vee \alpha_1 + m_2^\vee \alpha_2$, are given for all three cases as follows:

$$A_2: \ \eta = \alpha_1^\vee + \alpha_2^\vee, \qquad C_2: \ \eta = \alpha_1^\vee + 2\alpha_2^\vee, \qquad G_2: \ \eta = 3\alpha_1^\vee + 2\alpha_2^\vee. \tag{5}$$

The dual affine reflection $r_0^\vee$ with respect to the highest dual root is given by

$$r_0^\vee x = r_\eta x + \frac{2\eta}{\langle \eta, \eta \rangle}, \qquad r_\eta x = x - \frac{2\langle x, \eta \rangle}{\langle \eta, \eta \rangle}\eta, \qquad x \in \mathbb{R}^2.$$

By adding the dual affine reflection $r_0^\vee$ to the set of generators $\{r_1, r_2\}$ one obtains the dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$, see [19]. The dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ consists of transformations of $\mathbb{R}^2$ from W and of shifts by vectors from the root lattice Q; it holds that $\widehat{W}^{\mathrm{aff}} = Q \rtimes W$.

The dual fundamental domain $F^\vee$ of the action of $\widehat{W}^{\mathrm{aff}}$ on $\mathbb{R}^2$ is a triangle with vertices $\left\{0, \frac{\omega_1}{m_1^\vee}, \frac{\omega_2}{m_2^\vee}\right\}$, where $m_1^\vee, m_2^\vee$ are the coefficients (5) of the highest dual root η in $\alpha^\vee-$basis, $\eta = m_1^\vee \alpha_1^\vee + m_2^\vee \alpha_2^\vee$. The subset $F^{s\vee} \subset F^\vee$, which contains points of $F^\vee$ that are not stabilized by any reflection corresponding to a short root or by $r_0^\vee$, is called the dual short fundamental domain $F^{s\vee}$. The subset $F^{l\vee} \subset F^\vee$, which contains points of $F^\vee$ that are not stabilized by any reflection corresponding to a long root, is called the dual long fundamental domain $F^{l\vee}$.

3. Orbit Functions and Corresponding Characters

3.1. Orbit Functions and Characters of Two Variables

Each of the four types of special functions, which correspond to the Weyl groups, induces a family of orthogonal polynomials. For the three types of orbit functions which are not symmetric, there is a symmetric character function. These character functions then generate the family of orthogonal polynomials. The functions in the family of symmetric $C-$functions $\Phi_\lambda : \mathbb{R}^2 \to \mathbb{C}$ are parametrized by $\lambda \in P^+$ and given explicitly [25] as

$$\Phi_\lambda(x) = \sum_{w \in W} e^{2\pi i \langle w\lambda, x \rangle}, \qquad x \in \mathbb{R}^2, \quad \lambda \in P^+. \tag{6}$$

The functions in the family of antisymmetric orbit $S-$functions [26] are labeled by $\lambda + \varrho$ and have the explicit form

$$\varphi_{\lambda+\varrho}(x) = \sum_{w \in W} (\det w) e^{2\pi i \langle w(\lambda+\varrho), x \rangle}, \qquad x \in \mathbb{R}^2, \quad \lambda \in P^+. \tag{7}$$

The corresponding symmetric character functions χ_λ are given by the Weyl character formula,

$$\chi_\lambda(x) = \frac{\varphi_{\lambda+\varrho}(x)}{\varphi_\varrho(x)}, \quad \lambda \in P^+.$$

Two types of sign homomorphisms [2] determine two additional families of functions for the algebras C_2 and G_2. The short sign homomorphism $\sigma^s : W \to \{\pm 1\}$ is defined by its values on the generators r_α, $\alpha \in \Delta$ of W. To the r_α of the short simple root α is assigned -1; to the r_α of the long simple root α is assigned 1. The long sign homomorphism σ^l is given conversely, assigning the value -1 to the r_α of the long simple root α. The short homomorphism induces a family of S^s-functions, which are labeled by $\lambda + \varrho^s$ and are of the explicit form

$$\varphi^s_{\lambda+\varrho^s}(x) = \sum_{w \in W} \sigma^s(w) e^{2\pi i \langle w(\lambda+\varrho^s), x \rangle}, \qquad x \in \mathbb{R}^2, \quad \lambda \in P^+.$$

The corresponding symmetric short character functions χ^s_λ are given as

$$\chi^s_\lambda(x) = \frac{\varphi^s_{\lambda+\varrho^s}(x)}{\varphi^s_{\varrho^s}(x)}, \qquad \lambda \in P^+.$$

The long homomorphism induces a family of S^l-functions, which are labeled by $\lambda + \varrho^l$ and are of the explicit form

$$\varphi^l_{\lambda+\varrho^l}(x) = \sum_{w \in W} \sigma^l(w) e^{2\pi i \langle w(\lambda+\varrho^l), x \rangle}, \qquad x \in \mathbb{R}^2, \quad \lambda \in P^+.$$

The corresponding symmetric long character functions χ^l_λ are given as

$$\chi^l_\lambda(x) = \frac{\varphi^l_{\lambda+\varrho^l}(x)}{\varphi^l_{\varrho^l}(x)}, \quad \lambda \in P^+.$$

For $\lambda = a\omega_1 + b\omega_2$ and for $x = x_1\alpha_1^\vee + x_2\alpha_2^\vee$, the inner product is equal to:

$$\langle \lambda, x \rangle = \langle a\omega_1 + b\omega_2, x_1\alpha_1^\vee + x_2\alpha_2^\vee \rangle = ax_1 + bx_2. \tag{8}$$

3.2. Orbit Functions of A_2

In this section, we work out the $C-$function (6) and the $S-$functions (7) of A_2 relative to the $\alpha^\vee-$basis (x_1, x_2) and to the $\omega-$basis (a, b). Since the functions $\Phi_\lambda(x)$ and $\varphi_\lambda(x)$ differ only by signs of certain terms, we write them together, understanding that the upper signs refer to $\Phi_\lambda(x)$, while the lower ones belong to $\varphi_\lambda(x)$:

$$\begin{aligned} &\cos(2\pi((a+b)x_1 - ax_2)) &&\pm \cos(2\pi(bx_1 + ax_2)) &&\pm \cos(2\pi((a+b)x_1 - bx_2)) \\ &+ \cos(2\pi(ax_1 + bx_2)) &&\pm \cos(2\pi(-ax_1 + (a+b)x_2)) &&+ \cos(2\pi(-bx_1 + (a+b)x_2)) \\ &- i\,[\sin(2\pi((a+b)x_1 - bx_2)) &&\pm \sin(2\pi(bx_1 + ax_2)) &&\mp \sin(2\pi((a+b)x_1 - bx_2)) \\ &- \sin(2\pi(ax_1 + bx_2)) &&\mp \sin(2\pi(-ax_1 + (a+b)x_2)) &&+ \sin(2\pi(-bx_1 + (a+b)x_2))]\,. \end{aligned} \tag{9}$$

There is another difference between $\Phi_\lambda(x)$ and $\varphi_\lambda(x)$ that is not visible from (9). The weight coordinates a and b have a different range,

$$\Phi_{(a,b)}(x) \;:\quad a, b \in \mathbb{Z}^{\geq 0}; \qquad \varphi_{(a,b)}(x) \;:\quad a, b \in \mathbb{N}\,.$$

3.3. Orbit Functions of C_2

Here we present the $C-$, $S-$, S^s-, and S^l-functions of C_2. The coordinates (x_1, x_2) are given relative to the $\alpha^\vee-$basis, and the weight (a, b) in $\omega-$basis of C_2. The $C-$ and $S-$functions of C_2 differ only by signs of several terms. The upper sign belongs to $\Phi_{(a,b)}(x_1,x_2)$, while the lower sign is from $\varphi_{(a,b)}(x_1,x_2)$:

$$\begin{aligned}2[&\cos(2\pi((a+2b)x_1+(-a-b)x_2)) \pm \cos(2\pi((a+2b)x_1-bx_2)) \\ &\pm\cos(2\pi(-ax_1+(a+b)x_2)) \qquad +\cos(2\pi(ax_1+bx_2)]\,.\end{aligned}$$

Range of the weight coordinates differ for $\Phi_{(a,b)}(x_1,x_2)$ and for $\varphi_{(a,b)}(x_1,x_2)$. It is due to the presence of $\varrho=(1,1)$ in (a,b) of $\varphi_{(a,b)}(x_1,x_2)$ and to its absence in $\Phi_{(a,b)}(x_1,x_2)$.

The S^s-, and S^l-functions of C_2 differ only by signs of several terms. The upper sign belongs to $\varphi^s_{(a,b)}(x_1,x_2)$, while the lower sign is from $\varphi^l_{(a,b)}(x_1,x_2)$:

$$\begin{aligned}-2[&\cos(2\pi((a+2b)x_1+(-a-b)x_2)) \mp \cos(2\pi((a+2b)x_1-bx_2)) \\ &\pm\cos(2\pi(-ax_1+(a+b)x_2)) \qquad -\cos(2\pi(ax_1+bx_2)].\end{aligned}$$

Ranges of (a,b) in $\varphi^s_{(a,b)}(x_1,x_2)$ and in $\varphi^l_{(a,b)}(x_1,x_2)$ differ because the first contains $\varrho^s=(1,0)$ while the second contains $\varrho^l=(0,1)$. One has

$$\varphi^s_{(a,b)}(x_1,x_2)\ :\quad a\in\mathbb{N},\ b\in\mathbb{Z}^{\geq 0};\qquad \varphi^l_{(a,b)}(x_1,x_2)\ :\quad a\in\mathbb{Z}^{\geq 0},\ b\in\mathbb{N}\,.$$

3.4. Orbit Functions of G_2

The $C-$, $S-$, S^s-, and S^l-functions of G_2 are presented here. The coordinates (x_1,x_2) are given relative to the $\alpha^\vee-$basis, and the weight (a,b) in $\omega-$basis of G_2. As in the previous cases, the $C-$ and $S-$functions of G_2 differ only by signs of several terms. The upper sign belongs to $\Phi_{(a,b)}(x_1,x_2)$, while the lower sign is from $\varphi_{(a,b)}(x_1,x_2)$:

$$\begin{aligned}2[&\cos(2\pi((a+b)x_1+(-3a-2b)x_2)) \quad +\cos(2\pi(-ax_1-bx_2)) \\ &+\cos(2\pi((2a+b)x_1+(-3a-b)x_2)) \ \ \pm\cos(2\pi(ax_1+(-3a-b)x_2)) \\ &\pm\cos(2\pi((2a+b)x_1+(-3a-2b)x_2)) \pm\cos(2\pi((a+b)x_1-bx_2))].\end{aligned}$$

Also in this case, the ranges of admissible values of a and b are different for the two functions. Namely

$$\Phi_{(a,b)}(x_1,x_2)\ :\quad a,b\in\mathbb{Z}^{\geq 0};\qquad \varphi_{(a,b)}(x_1,x_2)\ :\quad a,b\in\mathbb{N}\,.$$

The S^s-, and S^l-functions of G_2 also differ only by signs of several terms. The upper sign belongs to $\varphi^s_{(a,b)}(x_1,x_2)$, while the lower one is from $\varphi^l(x_1,x_2)$:

$$\begin{aligned}2i[&\sin(2\pi((a+b)x_1+(-3a-2b)x_2)) \quad -\sin(2\pi(-ax_1-bx_2)) \\ &\pm\sin(2\pi((2a+b)x_1+(-3a-2b)x_2)) \mp\sin(2\pi(ax_1+(-3a-b)x_2)) \\ &-\sin(2\pi((2a+b)x_1+(-3a-b)x_2)) \ \ \mp\sin(2\pi((a+b)x_1-bx_2))].\end{aligned}$$

As in the C_2 case, the ranges of (a,b) in $\varphi^s_{(a,b)}(x_1,x_2)$ and in $\varphi^l_{(a,b)}(x_1,x_2)$ differ because the first contains $\varrho^s=(0,1)$ while the second contains $\varrho^l=(1,0)$. One has

$$\varphi^s_{(a,b)}(x_1,x_2)\ :\quad a\in\mathbb{Z}^{\geq 0},\ b\in\mathbb{N};\qquad \varphi^l_{(a,b)}(x_1,x_2)\ :\quad a\in\mathbb{N},\ b\in\mathbb{Z}^{\geq 0}\,.$$

3.5. Discrete Orthogonality of Orbit Functions

Discrete orthogonality of the orbit functions represents a starting point for the discrete orthogonality of the corresponding polynomials. First, we choose some arbitrary natural number $M\in\mathbb{N}$ which controls the density of the grids appearing in the discrete orthogonality relations [19].

Please note that the discrete calculus of the orbit functions is performed over the finite quotient group $\frac{1}{M}P^\vee/Q^\vee$ and the finite complement set of weights is taken as the quotient group P/MQ. The four finite sets, on which the discrete calculus of the four types of orbit functions is restricted, are denoted F_M, $\widetilde{F}_M$, F^s_M and F^l_M and defined as intersections of $\frac{1}{M}P^\vee/Q^\vee$ with the corresponding subsets of the fundamental domain:

$$\begin{aligned} F_M &= \frac{1}{M}P^\vee/Q^\vee \cap F\,, \qquad & \widetilde{F}_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^\circ\,, \\ F^s_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^s\,, \qquad & F^l_M &= \frac{1}{M}P^\vee/Q^\vee \cap F^l\,. \end{aligned} \tag{10}$$

The point sets F_M and $\widetilde{F}_M$ are depicted for the A_2 case in [27], the four point sets are F_M, $\widetilde{F}_M$, F^s_M and F^l_M are illustrated for the C_2 case in [19,21] and for the G_2 case in [18,28].

The four complementary finite sets of weights, which label the four types of orbit functions, are denoted Λ_M, $\widetilde{\Lambda}_M$, Λ^s_M and Λ^l_M and defined as intersections of P/MQ with the corresponding subsets of the magnified dual fundamental domain:

$$\begin{aligned} \Lambda_M &= P/MQ \cap MF^\vee\,, \qquad & \widetilde{\Lambda}_M &= P/MQ \cap MF^{\vee\circ}\,, \\ \Lambda^s_M &= P/MQ \cap MF^{s\vee}\,, \qquad & \Lambda^l_M &= P/MQ \cap MF^{l\vee}\,. \end{aligned} \tag{11}$$

The sizes of orbits and stabilizers on the maximal torus are also needed for the formulation of discrete orthogonality. For $x \in \mathbb{R}^2/Q^\vee$, we denote the orbit of the group W and its size by

$$Wx = \left\{ wx \in \mathbb{R}^2/Q^\vee \mid w \in W \right\}, \qquad \varepsilon(x) = |Wx|, \tag{12}$$

and for any $\lambda \in P/MQ$, we denote the stabilizer $\mathrm{Stab}^\vee(\lambda)$ of λ in W and its size by

$$\mathrm{Stab}^\vee(\lambda) = \{ w \in W \mid w\lambda = \lambda \}\,, \qquad h(\lambda) = |\mathrm{Stab}^\vee(\lambda)|. \tag{13}$$

Lastly, we recall the well known numbers of the elements of the Weyl group $|W|$ and the determinant of the Cartan matrix c that appear in discrete orthogonality,

$$A_2 \;:\quad |W| = 6,\ c = 3; \qquad C_2 \;:\quad |W| = 8,\ c = 2; \qquad G_2 \;:\quad |W| = 12,\ c = 1\,.$$

Following [19,21], the discrete orthogonality of the four types of orbit functions can be summarized as follows

$$\sum_{x \in F_M} \varepsilon(x)\Phi_\lambda(x)\overline{\Phi_{\lambda'}(x)} = c\,|W|\,M^2 h(\lambda)\delta_{\lambda,\lambda'}, \qquad \lambda,\lambda' \in \Lambda_M, \tag{14}$$

$$\sum_{x \in \widetilde{F}_M} \varphi_\lambda(x)\overline{\varphi_{\lambda'}(x)} = cM^2\delta_{\lambda,\lambda'}, \qquad \lambda,\lambda' \in \widetilde{\Lambda}_M, \tag{15}$$

$$\sum_{x \in F^s_M} \varepsilon(x)\varphi^s_\lambda(x)\overline{\varphi^s_{\lambda'}(x)} = c\,|W|\,M^2 h(\lambda)\delta_{\lambda,\lambda'}, \qquad \lambda,\lambda' \in \Lambda^s_M, \tag{16}$$

$$\sum_{x \in F^l_M} \varepsilon(x)\varphi^l_\lambda(x)\overline{\varphi^l_{\lambda'}(x)} = c\,|W|\,M^2 h(\lambda)\delta_{\lambda,\lambda'}, \qquad \lambda,\lambda' \in \Lambda^l_M. \tag{17}$$

Here the overline stands for complex conjugation.

4. Orthogonal Polynomials

4.1. Four Types of Orthogonal Polynomials of Two Variables

The families of orthogonal polynomials are built over four types of functions, $C-$, $S-$, S^s-, and S^l-functions, and the three character functions χ_λ, χ^s_λ and χ^l_λ. The polynomial variables X_1, X_2 are the lowest character functions χ_{ω_1}, χ_{ω_2} for the cases C_2, G_2, and are taken as $\mathrm{Re}\,\chi_{\omega_1}$, $\mathrm{Im}\,\chi_{\omega_1}$ for the case A_2, where the character values are complex. Two families of orthogonal polynomials $\mathbb{T}_\lambda$ and $\mathbb{U}_\lambda$, parametrized by $\lambda \in P^+$ and defined for all cases as two variable generalizations of Chebyshev polynomials of the first and second kind, are induced by the relations

$$\mathbb{T}_\lambda(X_1(x), X_2(x)) = \Phi_\lambda(x), \qquad \mathbb{U}_\lambda(X_1(x), X_2(x)) = \chi_\lambda(x), \qquad x \in \mathbb{R}^2.$$

Two additional families of orthogonal polynomials $\mathbb{U}^s_\lambda$ and $\mathbb{U}^l_\lambda$, parametrized by $\lambda \in P^+$ and defined for C_2 and G_2, are induced by the relations

$$\mathbb{U}^s_\lambda(X_1(x), X_2(x)) = \chi^s_\lambda(x), \qquad \mathbb{U}^l_\lambda(X_1(x), X_2(x)) = \chi^l_\lambda(x), \qquad x \in \mathbb{R}^2.$$

In the following sections, we present the explicit forms of the polynomial variables and recursion relations for the construction of the polynomials. These relations are needed for the construction of point sets on which the discrete orthogonality is defined, subsequently. A crucial property is the dependence of polynomial variables $X_1(x)$ and $X_2(x)$ on the variables $x = x_1\alpha_1^\vee + x_2\alpha_2^\vee$. It allows us to transform the lattice points in $\mathbb{R}^2$ to the discrete points in the space of the polynomials, where they do not form a lattice fragment. Then, we carry over the discrete orthogonality of the orbit functions to the discrete orthogonality of the polynomials.

4.2. Orthogonal Polynomials of A_2

The polynomial variables $X_1 = \mathrm{Re}\,\chi_{\omega_1}$ and $X_2 = \mathrm{Im}\,\chi_{\omega_1}$ can be written down as follows, using the explicit form of $C-$functions:

$$\begin{aligned} X_1(x_1,x_2) &= \tfrac{1}{4}(\Phi_{(1,0)}(x_1,x_2) + \Phi_{(0,1)}(x_1,x_2)) \\ &= \cos(2\pi x_1) + \cos(2\pi(x_1 - x_2)) + \cos(2\pi x_2), \end{aligned} \tag{18}$$

$$\begin{aligned} X_2(x_1,x_2) &= \tfrac{1}{4i}(\Phi_{(1,0)}(x_1,x_2) - \Phi_{(0,1)}(x_1,x_2)) \\ &= \sin(2\pi x_1) - \sin(2\pi(x_1 - x_2)) - \sin(2\pi x_2). \end{aligned} \tag{19}$$

4.2.1. Recurrence Relations for $\mathbb{T}-$polynomials of A_2.

Generic recursion relations for $\mathbb{T}-$polynomials of A_2 are for $k, l \geq 1$ of the following form:

$$\begin{aligned} \mathbb{T}_{(k+1,l)} &= (X_1 + iX_2)\mathbb{T}_{(k,l)} - \mathbb{T}_{(k-1,l+1)} - \mathbb{T}_{(k,l-1)}\,, \\ \mathbb{T}_{(k,l+1)} &= (X_1 - iX_2)\mathbb{T}_{(k,l)} - \mathbb{T}_{(k+1,l-1)} - \mathbb{T}_{(k-1,l)}\,, \\ \mathbb{T}_{(k+1,0)} &= (X_1 + iX_2)\mathbb{T}_{(k,0)} - 2\mathbb{T}_{(k-1,1)}\,, \\ \mathbb{T}_{(0,l+1)} &= (X_1 - iX_2)\mathbb{T}_{(0,l)} - 2\mathbb{T}_{(1,l-1)}\,. \end{aligned} \tag{20}$$

The lowest $\mathbb{T}-$polynomials, that are needed to solve the recursion relations (20), are presented in Table 1.

Table 1. The lowest irreducible $\mathbb{T}-$polynomials of A_2.

(k,l)	$\mathbb{T}_{(k,l)}(X_1,X_2)=\overline{\mathbb{T}_{(l,k)}(X_1,X_2)}$
$(0,0)$	6
$(1,0)$	$2X_1+2iX_2$
$(1,1)$	$X_1^2+X_2^2-3$
$(2,0)$	$2X_1^2-2X_2^2+4iX_1X_2-4X_1+4iX_2$
$(3,0)$	$2X_1^3-2iX_2^3-6X_1X_2^2+6iX_1^2X_2-6X_1^2-6X_2^2+6$
$(2,1)$	$X_1^3+iX_2^3+X_1X_2^2+iX_1^2X_2-2X_1^2+2X_2^2+4iX_1X_2-X_1-iX_2$

4.2.2. Recurrence Relations for $\mathbb{U}-$polynomials of A_2.

Generic recursion relations for $\mathbb{U}-$polynomials of A_2 are for $k,l\geq 1$ of the following form:

$$\begin{aligned}\mathbb{U}_{(k+1,l)}&=(X_1+iX_2)\mathbb{U}_{(k,l)}-\mathbb{U}_{(k-1,l+1)}-\mathbb{U}_{(k,l-1)},\\ \mathbb{U}_{(k,l+1)}&=(X_1-iX_2)\mathbb{U}_{(k,l)}-\mathbb{U}_{(k+1,l-1)}-\mathbb{U}_{(k-1,l)},\\ \mathbb{U}_{(k+1,0)}&=(X_1+iX_2)\mathbb{U}_{(k,0)}-\mathbb{U}_{(k-1,1)},\\ \mathbb{U}_{(0,l+1)}&=(X_1-iX_2)\mathbb{U}_{(0,l)}-\mathbb{U}_{(1,l-1)}\,.\end{aligned}\tag{21}$$

The lowest $\mathbb{U}-$polynomials, needed in (21), are presented in Table 2.

Table 2. The irreducible $\mathbb{U}-$polynomials of A_2 of degree $k+l\leq 2$.

(k,l)	$\mathbb{U}_{(k,l)}(X_1,X_2)=\overline{\mathbb{U}_{(l,k)}(X_1,X_2)}$
$(0,0)$	1
$(1,0)$	X_1+iX_2
$(1,1)$	$X_1^2+X_2^2-1$
$(2,0)$	$X_1^2-X_2^2+2iX_1X_2-X_1+iX_2$

4.3. Orthogonal Polynomials of C_2

In this section, we present in coordinates (X_1,X_2) the recursion relations and the lowest orthogonal polynomials generated by the Weyl group of C_2. Having the scalar product (8), we can write explicit expressions for the orbit functions in X_1 and X_2:

$$\begin{aligned}X_1(x_1,x_2)&=\tfrac{1}{2}\Phi_{(1,0)}(x_1,x_2)\qquad=2\cos(2\pi x_1)+2\cos(2\pi(x_1-x_2))\,,\\ X_2(x_1,x_2)&=\tfrac{1}{2}\Phi_{(0,1)}(x_1,x_2)+1=2\cos(2\pi x_2)+2\cos(2\pi(2x_1-x_2))+1\,.\end{aligned}\tag{22}$$

4.3.1. Recurrence Relations for $\mathbb{T}-$polynomials of C_2.

Generic recursion relations for $\mathbb{T}-$polynomials of C_2 are of the form

$$\begin{aligned}\mathbb{T}_{(k+1,l)}&=X_1\mathbb{T}_{(k,l)}-\mathbb{T}_{(k-1,l)}-\mathbb{T}_{(k+1,l-1)}-\mathbb{T}_{(k-1,l+1)},\\ \mathbb{T}_{(k+1,0)}&=X_1\mathbb{T}_{(k,0)}-\mathbb{T}_{(k-1,0)}-2\mathbb{T}_{(k-1,1)},\\ \mathbb{T}_{(0,l+1)}&=X_2\mathbb{T}_{(0,l)}-\mathbb{T}_{(0,l-1)}-2\mathbb{T}_{(2,l-1)}-\mathbb{T}_{(0,l)},\\ \mathbb{T}_{(1,l+1)}&=X_2\mathbb{T}_{(1,l)}-2\mathbb{T}_{(1,l)}-\mathbb{T}_{(1,l-1)}-\mathbb{T}_{(3,l-1)},\end{aligned}\tag{23}$$

for $k\geq 1,\ l\geq 1$ and

$$\mathbb{T}_{(k,l+1)}=X_2\mathbb{T}_{(k,l)}-\mathbb{T}_{(k,l-1)}-\mathbb{T}_{(k+2,l-1)}-\mathbb{T}_{(k-2,l+1)}-\mathbb{T}_{(k,l)},\tag{24}$$

for $k\geq 2,\ l\geq 1$. The lowest $\mathbb{T}-$polynomials, necessary to solve all the recursions above, are presented in Table 3.

Table 3. The irreducible $\mathbb{T}-$polynomials of C_2.

(k,l)	$\mathbb{T}_{(k,l)}(X_1,X_2)$
$(0,0)$	8
$(1,0)$	$2X_1$
$(0,1)$	$2X_2-2$
$(1,1)$	$X_1X_2-3X_1$

4.3.2. Recurrence Relations for $\mathbb{U}-$polynomials of C_2.

Generic recursion relations for $\mathbb{U}-$polynomials of C_2 are of the form

$$\begin{aligned}
\mathbb{U}_{(k+1,l)} &= X_1\mathbb{U}_{(k,l)} - \mathbb{U}_{(k-1,l)} - \mathbb{U}_{(k+1,l-1)} - \mathbb{U}_{(k1,l+1)},\\
\mathbb{U}_{(k+1,0)} &= X_1\mathbb{U}_{(k,0)} - \mathbb{U}_{(k-1,0)} - \mathbb{U}_{(k-1,1)},\\
\mathbb{U}_{(0,l+1)} &= X_2\mathbb{U}_{(0,l)} - \mathbb{U}_{(0,l-1)} - \mathbb{U}_{(2,l-1)},\\
\mathbb{U}_{(1,l+1)} &= X_2\mathbb{U}_{(1,l)} - \mathbb{U}_{(1,l-1)} - \mathbb{U}_{(3,l-1)} - \mathbb{U}_{(1,l)},
\end{aligned} \tag{25}$$

for $k \geq 1,\ l \geq 1$ and

$$\mathbb{U}_{(k,l+1)} = X_2\mathbb{U}_{(k,l)} - \mathbb{U}_{(k,l-1)} - \mathbb{U}_{(k+2,l-1)} - \mathbb{U}_{(k-2,l+1)} - \mathbb{U}_{(k,l)}, \tag{26}$$

for $k \geq 2,\ l \geq 1$. The lowest $\mathbb{U}-$polynomials are presented in Table 4.

Table 4. The irreducible $\mathbb{U}-$polynomials of C_2 degree $k+l \leq 2$.

(k,l)	$\mathbb{U}_{(k,l)}(X_1,X_2)$
$(0,0)$	1
$(0,1)$	X_2
$(1,0)$	X_1
$(1,1)$	$X_1X_2-X_1$

4.3.3. Recurrence Relations for $\mathbb{U}^l-$polynomials of C_2.

Generic recursion relations for $\mathbb{U}^l-$polynomials of C_2 are of the form

$$\mathbb{U}^l_{(k+1,l)} = X_1\mathbb{U}^l_{(k,l)} - \mathbb{U}^l_{(k-1,l)} - \mathbb{U}^l_{(k+1,l-1)} - \mathbb{U}^l_{(k-1,l+1)},$$

for $k \geq 1,\ l \geq 1$ and

$$\begin{aligned}
\mathbb{U}^l_{(k,l+1)} &= X_2\mathbb{U}^l_{(k,l)} - \mathbb{U}^l_{(k,l-1)} - \mathbb{U}^l_{(k+2,l-1)} - \mathbb{U}^l_{(k-2,l+1)} - \mathbb{U}^l_{(k,l)},\\
\mathbb{U}^l_{(k+1,0)} &= X_1\mathbb{U}^l_{(k,0)} - \mathbb{U}^l_{(k-1,1)} - \mathbb{U}^l_{(k-1,0)},\\
\mathbb{U}^l_{(0,l+1)} &= X_2\mathbb{U}^l_{(0,l)} - \mathbb{U}^l_{(0,l-1)} - 2\mathbb{U}^l_{(2,l-1)} - \mathbb{U}^l_{(0,l)},\\
\mathbb{U}^l_{(1,l+1)} &= X_2\mathbb{U}^l_{(1,l)} - 2\mathbb{U}^l_{(1,l)} - \mathbb{U}^l_{(1,l-1)} - \mathbb{U}^l_{(3,l-1)},
\end{aligned}$$

for $k \geq 2,\ l \geq 1$. The lowest $\mathbb{U}^l-$polynomials necessary to solve all above recursions are presented in Table 5.

Table 5. The irreducible $\mathbb{U}^l-$polynomials of C_2.

(k,l)	$\mathbb{U}^l_{(k,l)}(X_1,X_2)$
$(0,0)$	1
$(1,0)$	$\frac{1}{2}X_1$
$(0,1)$	X_2-1
$(1,1)$	$\frac{1}{2}X_1X_2-X_1$

4.3.4. Recurrence Relations for $\mathbb{U}^s-$polynomials of C_2.

Generic recursion relations for $\mathbb{U}^s-$polynomials of C_2 are of the form

$$\begin{aligned}
\mathbb{U}^s_{(k+1,l)} =& X_1\mathbb{U}^s_{(k,l)} - \mathbb{U}^s_{(k-1,l)} - \mathbb{U}^s_{(k+1,l-1)} - \mathbb{U}^s_{(k-1,l+1)},\\
\mathbb{U}^s_{(k+1,0)} =& X_1\mathbb{U}^s_{(k,0)} - \mathbb{U}^s_{(k-1,0)} - 2\mathbb{U}^s_{(k-1,1)},\\
\mathbb{U}^s_{(0,l+1)} =& X_2\mathbb{U}^s_{(0,l)} - \mathbb{U}^s_{(0,l-1)} - \mathbb{U}^s_{(2,l-1)},\\
\mathbb{U}^s_{(1,l+1)} =& X_2\mathbb{U}^s_{(1,l)} - \mathbb{U}^s_{(1,l)} - \mathbb{U}^s_{(1,l-1)} - \mathbb{U}^s_{(3,l-1)},
\end{aligned}$$

for $k\geq 1,\ l\geq 1$ and

$$\mathbb{U}^s_{(k,l+1)} = X_2\mathbb{U}^s_{(k,l)} - \mathbb{U}^s_{(k,l-1)} - \mathbb{U}^s_{(k+2,l-1)} - \mathbb{U}^s_{(k-2,l+1)} - \mathbb{U}^s_{(k,l)},$$

for $k\geq 2,\ l\geq 1$. The lowest $\mathbb{U}^s-$polynomials necessary to solve all above recursions are presented in Table 6.

Table 6. The irreducible $\mathbb{U}^s-$polynomials of C_2.

(k,l)	$\mathbb{U}^s_{(k,l)}(X_1,X_2)$
$(0,0)$	1
$(1,0)$	X_1
$(0,1)$	$\frac{1}{2}X_2+\frac{1}{2}$
$(1,1)$	$\frac{1}{2}X_1X_2-\frac{1}{2}X_1$

4.4. Orthogonal Polynomials of G_2

In this section, we present recursion relations and the lowest orthogonal polynomials generated from the group G_2 in new coordinates (X_1,X_2). Using the scalar product (8), we can write the orbit functions in $\alpha^\vee-$basis in an explicit form:

$$\begin{aligned}
X_1(x_1,x_2) =& \tfrac{1}{2}\Phi_{(1,0)}(x_1,x_2) + \tfrac{1}{2}\Phi_{(0,1)}(x_1,x_2) + 2\\
=& 2 + 2\cos(2\pi x_1) + 2\cos(2\pi(x_1-3x_2)) + 2\cos(2\pi(2x_1-3x_2))\\
&+ 2\cos(2\pi(x_1-2x_2)) + 2\cos(2\pi(x_1-x_2)) + 2\cos(2\pi x_2), \qquad (27)\\
X_2(x_1,x_2) =& \tfrac{1}{2}\Phi_{(0,1)}(x_1,x_2) + 1\\
=& 1 + 2\cos(2\pi(x_1-2x_2)) + 2\cos(2\pi(x_1-x_2)) + 2\cos(2\pi x_2)\,. \qquad (28)
\end{aligned}$$

4.4.1. Recurrence Relations for $\mathbb{T}-$polynomials of G_2.

Generic recursion relations for $\mathbb{T}-$polynomials of G_2 are of the form

$$\begin{aligned}
\mathbb{T}_{(k+1,l)} =& X_1\mathbb{T}_{(k,l)} - \mathbb{T}_{(k-1,l)} - \mathbb{T}_{(k+1,l-3)} - \mathbb{T}_{(k-1,l+3)} - \mathbb{T}_{(k+2,l-3)} - \mathbb{T}_{(k-2,l+3)}\\
&- \mathbb{T}_{(k,l-1)} - \mathbb{T}_{(k+1,l-1)} - \mathbb{T}_{(k-1,l+1)} - \mathbb{T}_{(k,l+1)} - \mathbb{T}_{(k-1,l+2)} - \mathbb{T}_{(k+1,l-2)} - 2\mathbb{T}_{(k,l)},
\end{aligned}$$

for $k \geq 2,\ l \geq 3$,

$$\mathbb{T}_{(k,l+1)} = X_2\mathbb{T}_{(k,l)} - \mathbb{T}_{(k,l-1)} - \mathbb{T}_{(k+1,l-1)} - \mathbb{T}_{(k-1,l+1)} - \mathbb{T}_{(k+1,l-2)} - \mathbb{T}_{(k-1,l+2)} - \mathbb{T}_{(k,l)},$$

for $k \geq 1,\ l \geq 2$ and

$$\begin{aligned}
\mathbb{T}_{(k+1,0)} &= X_1\mathbb{T}_{(k,0)} - \mathbb{T}_{(k-1,0)} - 2\mathbb{T}_{(k,0)} - 2\mathbb{T}_{(k-1,1)} - 2\mathbb{T}_{(k,1)} - 2\mathbb{T}_{(k-1,2)} - 2\mathbb{T}_{(k-1,3)} - 2\mathbb{T}_{(k-2,3)},\\
\mathbb{T}_{(k+1,1)} &= X_1\mathbb{T}_{(k,1)} - \mathbb{T}_{(k-1,1)} - \mathbb{T}_{(k-2,4)} - \mathbb{T}_{(k-1,4)} - 2\mathbb{T}_{(k,2)} - 2\mathbb{T}_{(k-1,2)} - \mathbb{T}_{(k-1,3)}\\
&\quad - \mathbb{T}_{(k+1,0)} - 3\mathbb{T}_{(k,1)} - \mathbb{T}_{(k,0)},\\
\mathbb{T}_{(k+1,2)} &= X_1\mathbb{T}_{(k,2)} - \mathbb{T}_{(k+1,0)} - \mathbb{T}_{(k-1,5)} - \mathbb{T}_{(k-2,5)} - \mathbb{T}_{(k-1,4)} - \mathbb{T}_{(k-1,3)} - \mathbb{T}_{(k,3)} - \mathbb{T}_{(k-1,2)}\\
&\quad - 2\mathbb{T}_{(k,2)} - 2\mathbb{T}_{(k,1)} - 2\mathbb{T}_{(k+1,1)},\\
\mathbb{T}_{(0,l+1)} &= X_2\mathbb{T}_{(0,l)} - \mathbb{T}_{(0,l)} - \mathbb{T}_{(0,l-1)} - 2\mathbb{T}_{(1,l-1)} - 2\mathbb{T}_{(1,l-2)},
\end{aligned}$$

for $k \geq 2,\ l \geq 2$. The lowest $\mathbb{T}-$polynomials necessary to solve all above recursions are presented in Table 7.

Table 7. The irreducible $\mathbb{T}-$polynomials of G_2.

(k,l)	$\mathbb{T}_{(k,l)}(X_1,X_2)$
$(0,0)$	12
$(0,1)$	$2X_2-2$
$(1,0)$	$2X_1-2X_2-2$
$(0,2)$	$2X_2^2-4X_1-4X_2-2$
$(1,1)$	$X_1X_2-3X_2^2+3X_1+2X_2+5$
$(2,0)$	$2X_1^2-4X_2^3+8X_1X_2+2X_2^2+4X_1+8X_2-2$
$(1,2)$	$X_1X_2^2-2X_1^2-X_2^3-X_1X_2+4X_2^2-2X_1-X_2-2$
$(2,1)$	$X_1^2X_2-2X_2^4+3X_1X_2^2+X_1^2+4X_2^3-2X_1X_2+2X_2^2-3X_1-6X_2-2$
$(2,2)$	$X_1^2X_2^2-2X_2^5-2X_1^3+8X_1X_2^3-10X_1^2X_2+3X_2^4-9X_1^2+4X_2^3-16X_1X_2-8X_2^2-4X_1+2X_2+5$

4.4.2. Recurrence Relations for $\mathbb{U}-$polynomials of G_2.

Generic recursion relations for $\mathbb{U}-$polynomials of G_2 are of the form

$$\begin{aligned}
\mathbb{U}_{(k+1,l)} &= X_1\mathbb{U}_{(k,l)} - \mathbb{U}_{(k-1,l)} - \mathbb{U}_{(k+1,l-3)} - \mathbb{U}_{(k-1,l+3)} - \mathbb{U}_{(k+2,l-3)} - \mathbb{U}_{(k-2,l+3)}\\
&\quad - \mathbb{U}_{(k,l-1)} - \mathbb{U}_{(k+1,l-1)} - \mathbb{U}_{(k-1,l+1)} - \mathbb{U}_{(k,l+1)} - \mathbb{U}_{(k-1,l+2)} - \mathbb{U}_{(k+1,l-2)} - 2\mathbb{U}_{(k,l)},
\end{aligned}$$

for $k \geq 2,\ l \geq 3$

$$\mathbb{U}_{(k,l+1)} = X_2\mathbb{U}_{(k,l)} - \mathbb{U}_{(k,l-1)} - \mathbb{U}_{(k+1,l-1)} - \mathbb{U}_{(k-1,l+1)} - \mathbb{U}_{(k+1,l-2)} - \mathbb{U}_{(k-1,l+2)} - \mathbb{U}_{(k,l)},$$

for $k \geq 1,\ l \geq 2$

$$\begin{aligned}
\mathbb{U}_{(k+1,0)} &= X_1\mathbb{U}_{(k,0)} - \mathbb{U}_{(k-1,0)} - \mathbb{U}_{(k-2,3)} - \mathbb{U}_{(k-1,3)} - \mathbb{U}_{(k-1,2)} - \mathbb{U}_{(k,0)},\\
\mathbb{U}_{(k+1,1)} &= X_1\mathbb{U}_{(k,1)} - \mathbb{U}_{(k-1,2)} - \mathbb{U}_{(k-1,1)} - \mathbb{U}_{(k-1,4)} - \mathbb{U}_{(k-1,3)} - \mathbb{U}_{(k,2)} - \mathbb{U}_{(k-2,4)} - 2\mathbb{U}_{(k,1)},\\
\mathbb{U}_{(k+1,2)} &= X_1\mathbb{U}_{(k,2)} - \mathbb{U}_{(k-1,1)} - \mathbb{U}_{(k-1,0)} - \mathbb{U}_{(k,1)} - \mathbb{U}_{(k-1,2)} - \mathbb{U}_{(k-1,5)} - \mathbb{U}_{(k-1,4)} - \mathbb{U}_{(k,3)}\\
&\quad - \mathbb{U}_{(k-2,5)} - \mathbb{U}_{(k-1,3)} - 2\mathbb{U}_{(k,2)},\\
\mathbb{U}_{(0,l+1)} &= X_2\mathbb{U}_{(0,l)} - \mathbb{U}_{(1,l-1)} - \mathbb{U}_{(0,l)} - \mathbb{U}_{(1,l-2)} - \mathbb{U}_{(1,l-1)},
\end{aligned}$$

for $k \geq 2,\ l \geq 2$. The lowest $\mathbb{U}-$polynomials necessary to solve all above recursions are presented in Table 8.

Table 8. The irreducible $\mathbb{U}-$polynomials of G_2.

(k,l)	$\mathbb{U}_{(k,l)}(X_1,X_2)$
$(0,0)$	1
$(0,1)$	X_2
$(1,0)$	X_1
$(0,2)$	$X_2^2-X_1-X_2-1$
$(1,1)$	$X_1X_2-X_2^2+X_1+1$
$(2,0)$	$X_1+X_1^2+2X_2+2X_1X_2-X_2^3$
$(1,2)$	$X_1X_2^2-X_1-X_1^2+X_2+X_2^2-X_2^3$
$(2,1)$	$X_1^2X_2+X_1^2-X_2+2X_2^2+X_1X_2^2+X_2^3-X_2^4-1$
$(2,2)$	$X_1^2X_2^2+-2X_1^2-X_1^3-X_2-4X_1X_2-2X_1^2X_2-4X_2^2-X_1X_2^2+2X_2^3+2X_1X_2^3+2X_2^4-X_2^5+1$

4.4.3. Recurrence Relations for $\mathbb{U}^l-$polynomials of G_2.

Generic recursion relations for $\mathbb{U}^l-$polynomials of G_2 are of the form

$$\begin{aligned}\mathbb{U}^l_{(k+1,l)} =& X_1\mathbb{U}^l_{(k,l)}-\mathbb{U}^l_{(k-1,l)}-\mathbb{U}^l_{(k+1,l-3)}-\mathbb{U}^l_{(k-1,l+3)}-\mathbb{U}^l_{(k+2,l-3)}-\mathbb{U}^l_{(k-2,l+3)}\\ &-\mathbb{U}^l_{(k,l-1)}-\mathbb{U}^l_{(k+1,l-1)}-\mathbb{U}^l_{(k-1,l+1)}-\mathbb{U}^l_{(k,l+1)}-\mathbb{U}^l_{(k-1,l+2)}-\mathbb{U}^l_{(k+1,l-2)}-2\mathbb{U}^l_{(k,l)},\end{aligned}$$

for $k\geq 2,\ l\geq 3$,

$$\mathbb{U}^l_{(k,l+1)} = X_2\mathbb{U}^l_{(k,l)}-\mathbb{U}^l_{(k,l-1)}-\mathbb{U}^l_{(k+1,l-1)}-\mathbb{U}^l_{(k-1,l+1)}-\mathbb{U}^l_{(k+1,l-2)}-\mathbb{U}^l_{(k-1,l+2)}-\mathbb{U}^l_{(k,l)},$$

for $k\geq 1,\ l\geq 2$ and

$$\begin{aligned}\mathbb{U}^l_{(k+1,0)} =& X_1\mathbb{U}^l_{(k,0)}-\mathbb{U}^l_{(k-1,0)}-2\mathbb{U}^l_{(k,0)}-2\mathbb{U}^l_{(k-1,1)}-2\mathbb{U}^l_{(k,1)}-2\mathbb{U}^l_{(k-1,2)}-2\mathbb{U}^l_{(k-1,3)}-2\mathbb{U}^l_{(k-2,3)},\\ \mathbb{U}^l_{(k+1,1)} =& X_1\mathbb{U}^l_{(k,1)}-\mathbb{U}^l_{(k-1,1)}-\mathbb{U}^l_{(k-2,4)}-\mathbb{U}^l_{(k-1,4)}-2\mathbb{U}^l_{(k,2)}-2\mathbb{U}^l_{(k-1,2)}-\mathbb{U}^l_{(k-1,3)}\\ &-\mathbb{U}^l_{(k+1,0)}-3\mathbb{U}^l_{(k,1)}-\mathbb{U}^l_{(k,0)},\\ \mathbb{U}^l_{(k+1,2)} =& X_1\mathbb{U}^l_{(k,2)}-\mathbb{U}^l_{(k+1,0)}-\mathbb{U}^l_{(k-1,5)}-\mathbb{U}^l_{(k-2,5)}-\mathbb{U}^l_{(k-1,4)}-\mathbb{U}^l_{(k-1,3)}-\mathbb{U}^l_{(k,3)}-\mathbb{U}^l_{(k-1,2)}\\ &-2\mathbb{U}^l_{(k,2)}-2\mathbb{U}^l_{(k,1)}-2\mathbb{U}^l_{(k+1,1)},\\ \mathbb{U}^l_{(0,l+1)} =& X_2\mathbb{U}^l_{(0,l)}-\mathbb{U}^l_{(1,l-1)}-\mathbb{U}^l_{(0,l)}-\mathbb{U}^l_{(1,l-2)}-\mathbb{U}^l_{(0,l-1)},\end{aligned}$$

for $k\geq 2,\ l\geq 2$. The lowest $\mathbb{U}^l-$polynomials necessary to solve all above recursions are presented in Table 9.

Table 9. The irreducible $\mathbb{U}^l-$polynomials of G_2.

(k,l)	$\mathbb{U}^l_{(k,l)}(X_1,X_2)$
$(0,0)$	1
$(0,1)$	$\frac{1}{2}X_2-\frac{1}{2}$
$(1,0)$	X_1-X_2+1
$(0,2)$	$\frac{1}{2}X_2^2-X_1-\frac{1}{2}X_2-1$
$(1,1)$	$\frac{1}{2}X_1X_2-X_2^2+\frac{1}{2}X_1+X_2+1$
$(2,0)$	$X_1^2-X_2^3+X_1X_2+X_2^2+X_1+2X_2-1$
$(1,2)$	$\frac{1}{2}X_1X_2^2-X_1^2-\frac{1}{2}X_2^3+\frac{3}{2}X_2^2-\frac{3}{2}X_1-\frac{1}{2}X_2-\frac{1}{2}$
$(2,1)$	$\frac{1}{2}X_1^2X_2-\frac{1}{2}X_2^4+\frac{1}{2}X_1^2+\frac{3}{2}X_2^3-\frac{1}{2}X_1X_2+\frac{1}{2}X_1-2X_2$
$(2,2)$	$\frac{1}{2}X_1^2X_2^2-\frac{1}{2}X_2^5-X_1^3+\frac{3}{2}X_1X_2^3-2X_1^2X_2+X_2^4+\frac{1}{2}X_1X_2^2-\frac{5}{2}X_1^2+\frac{1}{2}X_2^3-4X_1X_2-\frac{3}{2}X_2^2-X_1+\frac{1}{2}$

4.4.4. Recurrence Relations for $\mathbb{U}^s-$polynomials of G_2.

Generic recursion relations for $\mathbb{U}^s-$polynomials of G_2 are of the form

$$\begin{aligned}\mathbb{U}^s_{(k+1,l)} =& X_1\mathbb{U}^s_{(k,l)} - \mathbb{U}^s_{(k-1,l)} - \mathbb{U}^s_{(k+1,l-3)} - \mathbb{U}^s_{(k-1,l+3)} - \mathbb{U}^s_{(k+2,l-3)} - \mathbb{U}^s_{(k-2,l+3)} \\ & - \mathbb{U}^s_{(k,l-1)} - \mathbb{U}^s_{(k+1,l-1)} - \mathbb{U}^s_{(k-1,l+1)} - \mathbb{U}^s_{(k,l+1)} - \mathbb{U}^s_{(k-1,l+2)} - \mathbb{U}^s_{(k+1,l-2)} - 2\mathbb{U}^s_{(k,l)},\end{aligned}$$

for $k \geq 2,\ l \geq 3$,

$$\mathbb{U}^s_{(k,l+1)} = X_2\mathbb{U}^s_{(k,l)} - \mathbb{U}^s_{(k,l-1)} - \mathbb{U}^s_{(k+1,l-1)} - \mathbb{U}^s_{(k-1,l+1)} - \mathbb{U}^s_{(k+1,l-2)} - \mathbb{U}^s_{(k-1,l+2)} - \mathbb{U}^s_{(k,l)},$$

for $k \geq 1,\ l \geq 2$ and

$$\begin{aligned}\mathbb{U}^s_{(k+1,0)} =& X_1\mathbb{U}^s_{(k,0)} - \mathbb{U}^s_{(k-1,0)} - \mathbb{U}^s_{(k-2,3)} - \mathbb{U}^s_{(k-1,3)} - \mathbb{U}^s_{(k-1,2)} - \mathbb{U}^s_{(k,0)}, \\ \mathbb{U}^s_{(k+1,1)} =& X_1\mathbb{U}^s_{(k,1)} - \mathbb{U}^s_{(k-1,4)} - \mathbb{U}^s_{(k-2,4)} - \mathbb{U}^s_{(k-1,3)} - \mathbb{U}^s_{(k-1,2)} - \mathbb{U}^s_{(k,2)} - \mathbb{U}^s_{(k-1,1)} - 2\mathbb{U}^s_{(k,1)}, \\ \mathbb{U}^s_{(k+1,2)} =& X_1\mathbb{U}^s_{(k,2)} - \mathbb{U}^s_{(k,3)} - \mathbb{U}^s_{(k+1,1)} - \mathbb{U}^s_{(k-1,5)} - \mathbb{U}^s_{(k-1,4)} - \mathbb{U}^s_{(k-2,5)} - \mathbb{U}^s_{(k+1,0)} - \mathbb{U}^s_{(k-1,3)} \\ & - \mathbb{U}^s_{(k-1,2)} - \mathbb{U}^s_{(k,1)} - 2\mathbb{U}^s_{(k,2)}, \\ \mathbb{U}^s_{(0,l+1)} =& X_2\mathbb{U}^s_{(0,l)} - \mathbb{U}^s_{(0,l)} - \mathbb{U}^s_{(0,l-1)} - 2\mathbb{U}^s_{(1,l-1)} - 2\mathbb{U}^s_{(1,l-2)},\end{aligned}$$

for $k \geq 2,\ l \geq 2$. The lowest $\mathbb{U}^s-$polynomials necessary to solve all above recursions are presented in Table 10.

Table 10. The irreducible $\mathbb{U}^s-$polynomials of G_2.

(k,l)	$\mathbb{U}^s_{(k,l)}(X_1,X_2)$
$(0,0)$	1
$(0,1)$	X_2+1
$(1,0)$	$\frac{1}{2}X_1+\frac{1}{2}X_2-\frac{1}{2}$
$(0,2)$	$X_2^2-X_1-X_2-1$
$(1,1)$	$\frac{1}{2}X_1X_2-\frac{1}{2}X_2-\frac{1}{2}X_2^2+X_1$
$(2,0)$	$\frac{1}{2}X_1^2-X_2^3+\frac{5}{2}X_1X_2+\frac{3}{2}X_1+X_2$
$(1,2)$	$\frac{1}{2}X_1X_2^2-\frac{1}{2}X_1^2-\frac{1}{2}X_2^3+X_2+\frac{1}{2}$
$(2,1)$	$\frac{1}{2}X_1^2X_2-X_2^4+2X_1X_2^2+\frac{1}{2}X_1^2+\frac{1}{2}X_2^3+X_1X_2+\frac{3}{2}X_2^2-X_1-\frac{1}{2}X_2-\frac{1}{2}$
$(2,2)$	$\frac{1}{2}X_1^2X_2^2-X_2^5-\frac{1}{2}X_1^3+3X_1X_2^3-\frac{5}{2}X_1^2X_2+\frac{3}{2}X_2^4-X_1X_2^2-2X_1^2+2X_2^3-5X_1X_2-3X_2^2-\frac{1}{2}X_1-\frac{1}{2}X_2+1$

5. Discrete Orthogonality

5.1. Sets of Points

The discrete orthogonality of the four types of polynomials is performed on the corresponding subsets (10) of the finite set F_M. These sets have to be transformed by a transformation corresponding to the polynomial variables X_1 and X_2. Firstly, let us introduce an index set I_M of triplets which labels the points of F_M,

$$I_M = \left\{ [u_0, u_1, u_2] \in (\mathbb{Z}^{\geq 0})^3 \mid u_0 + m_1u_1 + m_2u_2 = M \right\},$$

with m_1, m_2 being the coefficients of the highest root ξ. Then the representative points of the set F_M can be expressed as [19]

$$F_M = \left\{ \frac{u_1}{M}\omega_1^\vee + \frac{u_2}{M}\omega_2^\vee \,\middle|\, [u_0, u_1, u_2] \in I_M \right\}.$$

Similarly defined are index sets $\widetilde{I}_M$, I^s_M and I^l_M, which label the points of $\widetilde{F}_M$, F^s_M and F^l_M, respectively. The form of these index sets can be for all cases deduced from [19,21].

For each label $j = [u_0, u_1, u_2] \in I_M$, determining a point $x_j = \frac{u_1}{M}\omega_1^\vee + \frac{u_2}{M}\omega_2^\vee \in F_M$, we define a transformed point Z_j by

$$Z_j = (X_1(x_j), X_2(x_j)) \in \mathbb{R}^2. \tag{29}$$

We also assign to each $j \in I_M$ the size of the orbit ε_j of $x_j \in F_M$, defined by (12),

$$\varepsilon_j = \varepsilon(x_j).$$

For all cases, the coefficients ε_j are listed in Table 11.

Table 11. Orders of orbits ε_j, $j \in I_M$ for the cases A_2, C_2 and G_2. It is assumed that $u_0, u_1, u_2 \neq 0$.

$j \in I_M$	A_2	C_2	G_2
$[u_0, u_1, u_2]$	6	8	12
$[0, u_1, u_2]$	3	4	6
$[u_0, 0, u_2]$	3	4	6
$[u_0, u_1, 0]$	3	4	6
$[0, 0, u_2]$	1	1	2
$[0, u_1, 0]$	1	2	3
$[u_0, 0, 0]$	1	1	1

All transformed points are then collected in the sets $\mathfrak{F}_M$, $\widetilde{\mathfrak{F}}_M$, $\mathfrak{F}_M^s$ and $\mathfrak{F}_M^l$,

$$\begin{aligned} \mathfrak{F}_M &= \{Z_j \mid j \in I_M\}, \quad \widetilde{\mathfrak{F}}_M = \left\{Z_j \mid j \in \widetilde{I}_M\right\}, \\ \mathfrak{F}_M^s &= \{Z_j \mid j \in I_M^s\}, \quad \mathfrak{F}_M^l = \left\{Z_j \mid j \in I_M^l\right\}. \end{aligned}$$

The numbers of generalized Chebyshev nodes in each of the sets coincide with the corresponding numbers of points in the sets F_M, $\widetilde{F}_M$, F_M^s and F_M^l from [19,21]. The explicit counting formulas are derived in Theorem 3.3, Proposition 3.5 in [19] and Theorem 5.2 in [21].

Please note that applying the transform (29) to any $x \in F$ one obtains the transformed domain $\mathfrak{F} \subset \mathbb{R}^2$ of F,

$$\mathfrak{F} = \left\{(X_1(x), X_2(x)) \in \mathbb{R}^2 \mid x \in F\right\},$$

and indeed $\mathfrak{F}_M \subset \mathfrak{F}$.

5.2. General Orthogonality Relations

The general orthogonality relations of bivariate polynomials generalize the standard discrete weighted orthogonality relations of the Chebyshev polynomials from [29]. The halving of the first and last boundary terms in the discrete sums in [29] is generalized to bivariate cases by using the triplets from L_M and the corresponding orbit coefficients ε_j. Before the orthogonality relations can be made explicit, it is necessary to reformulate the discrete orthogonality relations of orbit functions to the corresponding polynomials. Let us define the following three polynomials $J(X_1, X_2)$, $J^s(X_1, X_2)$ and $J^l(X_1, X_2)$ by the relations

$$\begin{aligned} &J(X_1(x), X_2(x)) = \left|\varphi_\varrho(x)\right|^2, \quad J^s(X_1(x), X_2(x)) = |\varphi^s_{\varrho^s}(x)|^2, \\ &J^l(X_1(x), X_2(x)) = |\varphi^l_{\varrho^l}(x)|^2, \quad x \in \mathbb{R}^2. \end{aligned}$$

Since the three functions $\left|\varphi_\varrho(x)\right|^2$, $|\varphi^s_{\varrho^s}(x)|^2$, $|\varphi^l_{\varrho^l}(x)|^2$ are $W-$invariant sums of exponentials, these polynomials in terms of X_1 and X_2 indeed exist [1]. The calculation procedure for the weight polynomials takes advantage of the product-to-sum decompositions of all four types of orbit functions [21,25]. Each function $\left|\varphi_\varrho(x)\right|^2$ is decomposed as a product $\varphi_\varrho(x)\varphi_{-\varrho}(x)$ into a sum of $C-$functions (6) that are subsequently expressed in the form of the $\mathbb{T}-$polynomials.

Using the resulting polynomials J, J^s and J^l as weight functions to cancel out the denominators of characters in discrete orthogonality relations (14)–(17), we obtain new discrete orthogonality relations for the polynomials,

$$\sum_{j\in I_M} \varepsilon_j \mathbb{T}_\lambda(Z_j)\overline{\mathbb{T}_{\lambda'}(Z_j)} = c\,|W|\,M^2 h(\lambda)\delta_{\lambda,\lambda'}, \quad \lambda,\lambda'\in\Lambda_M, \tag{30}$$

$$\sum_{j\in \widetilde{I}_M} J(Z_j)\mathbb{U}_\lambda(Z_j)\overline{\mathbb{U}_{\lambda'}(Z_j)} = cM^2\delta_{\lambda,\lambda'}, \quad \lambda,\lambda'\in\widetilde{\Lambda}_M-\varrho, \tag{31}$$

$$\sum_{j\in I^s_M} \varepsilon_j J^s(Z_j)\mathbb{U}^s_\lambda(Z_j)\overline{\mathbb{U}^s_{\lambda'}(Z_j)} = c\,|W|\,M^2 h(\lambda+\varrho^s)\delta_{\lambda,\lambda'}, \quad \lambda,\lambda'\in\Lambda^s_M-\varrho^s, \tag{32}$$

$$\sum_{j\in I^l_M} \varepsilon_j J^l(Z_j)\mathbb{U}^l_\lambda(Z_j)\overline{\mathbb{U}^l_{\lambda'}(Z_j)} = c\,|W|\,M^2 h(\lambda+\varrho^l)\delta_{\lambda,\lambda'}, \quad \lambda,\lambda'\in\Lambda^l_M-\varrho^l. \tag{33}$$

5.3. Sets of Weights

Let us introduce an index set L_M of triplets which labels the points of Λ_M,

$$L_M = \left\{[t_0,t_1,t_2]\in(\mathbb{Z}^{\geq 0})^3 \mid t_0+m_1^\vee t_1+m_2^\vee t_2 = M\right\},$$

with $m_1^\vee$, $m_2^\vee$ being the coefficients of the highest dual root η. Then the representative points of the set Λ_M can be expressed as [19],

$$\Lambda_M = \{t_1\omega_1+t_2\omega_2 \mid [t_0,t_1,t_2]\in L_M\}.$$

Similarly are defined subsets $\widetilde{L}_M$, L^s_M and L^l_M of L_M, which label the points of $\widetilde{\Lambda}_M-\varrho$, $\Lambda^s_M-\varrho^s$ and $\Lambda^l_M-\varrho^l$, respectively. Explicit form of these index sets can be deduced from [19,21] and they are detailed in the next section for all cases. For each point $\lambda_k = t_1\omega_1+t_2\omega_2\in\Lambda_M$, labeled by $k=[t_0,t_1,t_2]\in L_M$, we assign the size of the stabilizer h_k of λ_k given by (13),

$$h_k = h(\lambda_k).$$

For all cases, the coefficients h_k are listed in Table 12.

Table 12. Orders of stabilizers h_k, $k\in L_M$ for the cases A_2, C_2 and G_2. It is assumed that $t_0,t_1,t_2\neq 0$.

$k\in L_M$	A_2	C_2	G_2
$[t_0,t_1,t_2]$	1	1	1
$[0,t_1,t_2]$	2	2	2
$[t_0,0,t_2]$	2	2	2
$[t_0,t_1,0]$	2	2	2
$[0,0,t_2]$	6	4	4
$[0,t_1,0]$	6	8	6
$[t_0,0,0]$	6	8	12

5.4. Discrete Orthogonality of A_2 Polynomials

The sets of labels of points I_M and $\widetilde{I}_M$ are given explicitly as

$$I_M = \left\{[u_0,u_1,u_2]\in(\mathbb{Z}^{\geq 0})^3 \mid u_0+u_1+u_2 = M\right\},$$
$$\widetilde{I}_M = \left\{[u_0,u_1,u_2]\in\mathbb{N}^3 \mid u_0+u_1+u_2 = M\right\},$$

and the sets of labels of weights L_M and $\widetilde{L}_M$ are given as

$$L_M = \left\{ [t_0, t_1, t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + t_1 + t_2 = M \right\},$$
$$\widetilde{L}_M = \left\{ [t_0, t_1, t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + t_1 + t_2 = M - 3 \right\}.$$

The coordinates of the points Z_j in the sets $\mathfrak{F}_M$ and $\widetilde{\mathfrak{F}}_M$ are obtained by using the functions (18), (19) into substitution (29). The sets $\mathfrak{F}_M$ and $\widetilde{\mathfrak{F}}_M$ are for $M = 6$ and $M = 12$ depicted in Figure 1.

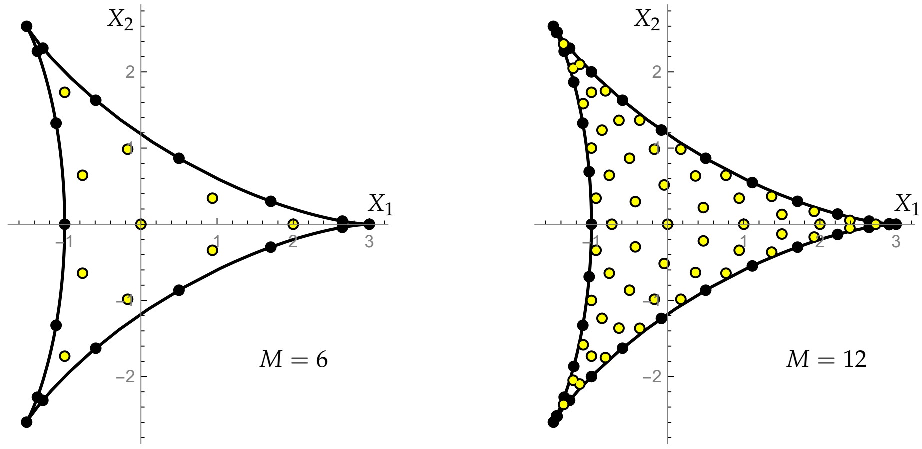


Figure 1. The generalized Chebyshev nodes of the A_2 polynomials for $M = 6$ and $M = 12$. The points from the set $\mathfrak{F}_M$ are shown as the larger black dots. The points from the set $\widetilde{\mathfrak{F}}_M$ are shown as the smaller yellow dots.

Since the function $\varphi_{(1,1)}(x)$ is real-valued, the function $\left|\varphi_\varrho(x)\right|^2$ is decomposed into the sum of $C-$functions as

$$\left|\varphi_{(1,1)}(x)\right|^2 = \varphi_{(1,1)}(x)\varphi_{(1,1)}(x) = -\Phi_{(0,0)}(x) + 2\Phi_{(1,1)}(x) + \Phi_{(2,2)}(x) - \Phi_{(3,0)}(x) - \Phi_{(0,3)}(x).$$

Therefore, the polynomial weight function $J(X_1, X_2)$ has the following form,

$$\begin{aligned} J(X_1, X_2) &= -\mathbb{T}_{(0,0)}(X_1, X_2) + 2\mathbb{T}_{(1,1)}(X_1, X_2) + \mathbb{T}_{(2,2)}(X_1, X_2) - \mathbb{T}_{(3,0)}(X_1, X_2) - \mathbb{T}_{(0,3)}(X_1, X_2) \\ &= X_1^4 + X_2^4 + 2X_1^2X_2^2 - 8X_1^3 + 24X_1X_2^2 + 18X_1^2 + 18X_2^2 - 27. \end{aligned}$$

The resulting orthogonality relations for the two types of polynomials are thus given as

$$\begin{aligned} \sum_{j \in I_M} \varepsilon_j \mathbb{T}_{(t_1,t_2)}(Z_j)\overline{\mathbb{T}_{(t_1',t_2')}(Z_j)} &= 18M^2 h_{[t_0,t_1,t_2]}\delta_{t_1t_1'}\delta_{t_2t_2'}, \quad [t_0, t_1, t_2], [t_0', t_1', t_2'] \in L_M, \\ \sum_{j \in \widetilde{I}_M} J(Z_j)\mathbb{U}_{(t_1,t_2)}(Z_j)\overline{\mathbb{U}_{(t_1',t_2')}(Z_j)} &= 3M^2\delta_{t_1t_1'}\delta_{t_2t_2'}, \quad [t_0, t_1, t_2], [t_0', t_1', t_2'] \in \widetilde{L}_M. \end{aligned}$$

5.5. Discrete Orthogonality of C_2 Polynomials

The sets of labels of points I_M and $\widetilde{I}_M$ are given explicitly as

$$I_M = \left\{ [u_0, u_1, u_2] \in (\mathbb{Z}^{\geq 0})^3 \mid u_0 + 2u_1 + u_2 = M \right\},$$
$$\widetilde{I}_M = \left\{ [u_0, u_1, u_2] \in \mathbb{N}^3 \mid u_0 + 2u_1 + u_2 = M \right\},$$

and the sets I^s_M and I^l_M are given as

$$I^s_M = \left\{ [u_0, u_1, u_2] \in (\mathbb{Z}^{\geq 0})^3 \mid u_1 \in \mathbb{N},\, u_0 + 2u_1 + u_2 = M \right\},$$
$$I^l_M = \left\{ [u_0, u_1, u_2] \in (\mathbb{Z}^{\geq 0})^3 \mid u_0, u_2 \in \mathbb{N},\, u_0 + 2u_1 + u_2 = M \right\}.$$

The sets of labels of weights L_M, $\widetilde{L}_M$, L^s_M and L^l_M are given as

$$L_M = \left\{ [t_0, t_1, t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + t_1 + 2t_2 = M \right\},$$
$$\widetilde{L}_M = \left\{ [t_0, t_1, t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + t_1 + 2t_2 = M - 4 \right\},$$
$$L^s_M = \left\{ [t_0, t_1, t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + t_1 + 2t_2 = M - 2 \right\},$$
$$L^l_M = \left\{ [t_0, t_1, t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + t_1 + 2t_2 = M - 2 \right\}.$$

The coordinates of the points Z_j in the sets $\mathfrak{F}_M$, $\widetilde{\mathfrak{F}}_M$, $\mathfrak{F}^s_M$ and $\mathfrak{F}^l_M$ are obtained by using the functions (22) into substitution (29). The sets $\mathfrak{F}_M$, $\widetilde{\mathfrak{F}}_M$, $\mathfrak{F}^s_M$ and $\mathfrak{F}^l_M$ are for $M = 6$ and $M = 12$ depicted in Figures 2 and 3.

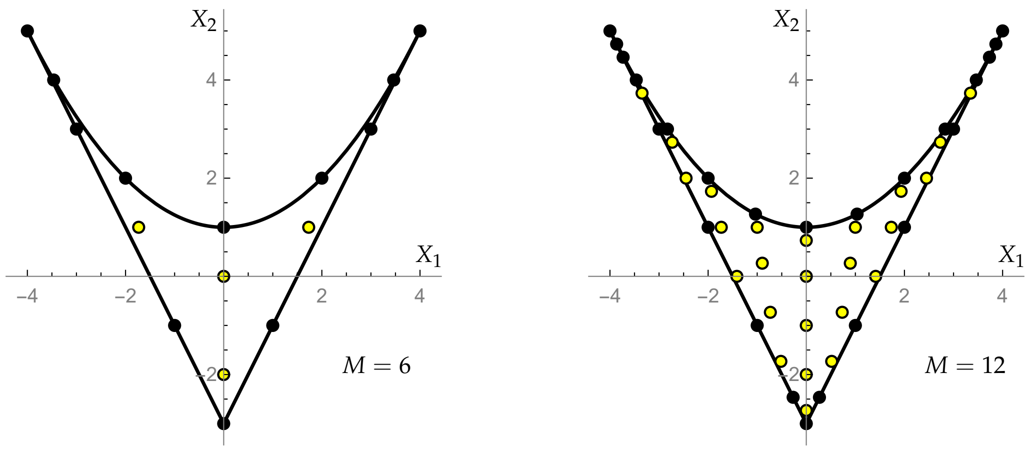


Figure 2. The generalized Chebyshev nodes of the C_2 polynomials for $M = 6$ and $M = 12$. The points from the set $\mathfrak{F}_M$ are shown as the larger black dots. The points from the set $\widetilde{\mathfrak{F}}_M$ are shown as the smaller yellow dots.

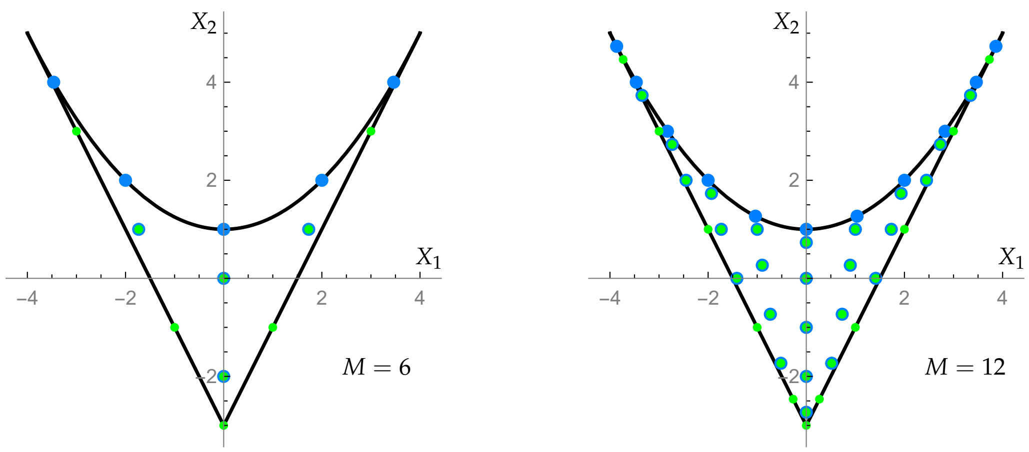


Figure 3. The short and long sets of the generalized Chebyshev nodes of the C_2 polynomials for $M = 6$ and $M = 12$. The points from the set $\mathfrak{F}^s_M$ are depicted as the smaller green dots and the points from the set $\mathfrak{F}^l_M$ are depicted as the larger blue dots.

The polynomial weight functions $J(X_1, X_2)$, $J^s(X_1, X_2)$ and $J^l(X_1, X_2)$ have the following form

$$\begin{aligned}
J(X_1, X_2) =& X_1^2X_2^2 - 4X_1^4 - 4X_2^3 + 22X_1^2X_2 - 20X_2^2 - 7X_1^2 - 12X_2 + 36,\\
J^s(X_1, X_2) =& 4(X_1^2 - 4X_2 + 4),\\
J^l(X_1, X_2) =& 4(X_2^2 - 4X_1^2 + 6X_2 + 9).
\end{aligned}$$

The resulting orthogonality relations for the four types of polynomials are thus given as

$$\begin{aligned}
\sum_{j\in I_M} \varepsilon_j \mathbb{T}_{(t_1,t_2)}(Z_j)\mathbb{T}_{(t_1',t_2')}(Z_j) =& 16M^2 h_{[t_0,t_1,t_2]}\delta_{t_1t_1'}\delta_{t_2t_2'}, \quad [t_0,t_1,t_2],[t_0',t_1',t_2'] \in L_M,\\
\sum_{j\in \widetilde{I}_M} J(Z_j)\mathbb{U}_{(t_1,t_2)}(Z_j)\mathbb{U}_{(t_1',t_2')}(Z_j) =& 2M^2\delta_{t_1t_1'}\delta_{t_2t_2'}, \quad [t_0,t_1,t_2],[t_0',t_1',t_2'] \in \widetilde{L}_M,\\
\sum_{j\in I_M^s} \varepsilon_j J^s(Z_j)\mathbb{U}^s_{(t_1,t_2)}(Z_j)\mathbb{U}^s_{(t_1',t_2')}(Z_j) =& 16M^2 h_{[t_0+1,t_1+1,t_2]}\delta_{t_1t_1'}\delta_{t_2t_2'}, \quad [t_0,t_1,t_2],[t_0',t_1',t_2'] \in L_M^s,\\
\sum_{j\in I_M^l} \varepsilon_j J^l(Z_j)\mathbb{U}^l_{(t_1,t_2)}(Z_j)\mathbb{U}^l_{(t_1',t_2')}(Z_j) =& 16M^2 h_{[t_0,t_1,t_2+1]}\delta_{t_1t_1'}\delta_{t_2t_2'}, \quad [t_0,t_1,t_2],[t_0',t_1',t_2'] \in L_M^l.
\end{aligned}$$

5.6. Discrete Orthogonality of G_2 Polynomials

The sets of labels of points I_M and $\widetilde{I}_M$ are given explicitly as

$$\begin{aligned}
I_M &= \left\{[u_0,u_1,u_2] \in (\mathbb{Z}^{\geq 0})^3 \mid u_0 + 2u_1 + 3u_2 = M\right\},\\
\widetilde{I}_M &= \left\{[u_0,u_1,u_2] \in \mathbb{N}^3 \mid u_0 + 2u_1 + 3u_2 = M\right\},
\end{aligned}$$

and the sets I_M^s and I_M^l are given as

$$\begin{aligned}
I_M^s &= \left\{[u_0,u_1,u_2] \in (\mathbb{Z}^{\geq 0})^3 \mid u_2 \in \mathbb{N},\ u_0 + 2u_1 + 3u_2 = M\right\},\\
I_M^l &= \left\{[u_0,u_1,u_2] \in (\mathbb{Z}^{\geq 0})^3 \mid u_0,u_1 \in \mathbb{N},\ u_0 + 2u_1 + 3u_2 = M\right\}.
\end{aligned}$$

The sets of labels of weights L_M, $\widetilde{L}_M$, L_M^s and L_M^l are given as

$$\begin{aligned}
L_M &= \left\{[t_0,t_1,t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + 3t_1 + 2t_2 = M\right\},\\
\widetilde{L}_M &= \left\{[t_0,t_1,t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + 3t_1 + 2t_2 = M - 6\right\},\\
L_M^s &= \left\{[t_0,t_1,t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + 3t_1 + 2t_2 = M - 3\right\},\\
L_M^l &= \left\{[t_0,t_1,t_2] \in (\mathbb{Z}^{\geq 0})^3 \mid t_0 + 3t_1 + 2t_2 = M - 3\right\}.
\end{aligned}$$

The coordinates of the points Z_j in the sets $\mathfrak{F}_M$, $\widetilde{\mathfrak{F}}_M$, $\mathfrak{F}_M^s$ and $\mathfrak{F}_M^l$ are obtained by using the functions (27), (28) into substitution (29). The sets $\mathfrak{F}_M$, $\widetilde{\mathfrak{F}}_M$, $\mathfrak{F}_M^s$ and $\mathfrak{F}_M^l$ are for $M = 10$ and $M = 20$ depicted Figures 4 and 5.

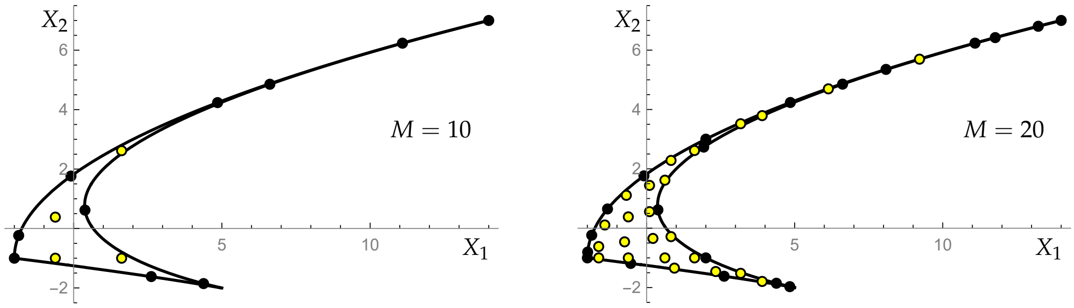


Figure 4. The generalized Chebyshev nodes of the G_2 polynomials for $M = 10$ and $M = 20$. The points from the set $\mathfrak{F}_M$ are shown as the larger black dots. The points from the set $\widetilde{\mathfrak{F}}_M$ are shown as the smaller yellow dots.

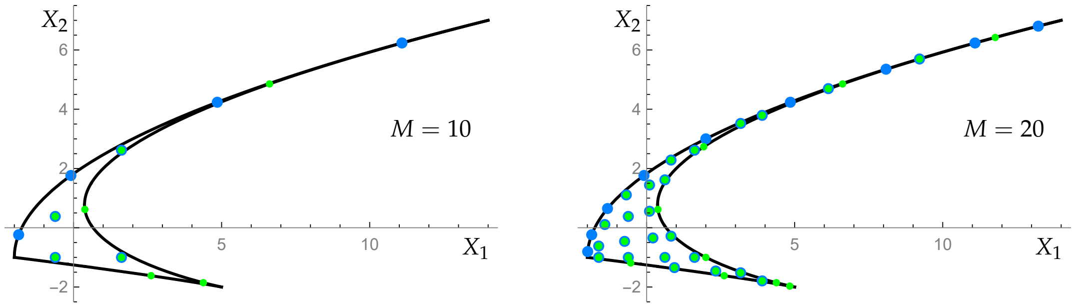


Figure 5. The short and long sets of the generalized Chebyshev nodes of the G_2 polynomials for $M = 10$ and $M = 20$. The points from the set $\mathfrak{F}^s_M$ are depicted as the smaller green dots and the points from the set $\mathfrak{F}^l_M$ are depicted as the larger blue dots.

The polynomial weight functions $J(X_1, X_2)$, $J^l(X_1, X_2)$ and $J^s(X_1, X_2)$ have the following form

$$\begin{aligned}
J(X_1, X_2) =& X_1^2X_2^2 - 4X_1^3 - 4X_2^5 + 26X_1X_2^3 - 38X_1^2X_2 - 7X_2^4 + 26X_1X_2^2 - 47X_1^2 + 32X_2^3 \\
& - 58X_1X_2 - 10X_2^2 - 42X_1 - 28X_2 + 49, \\
J^l(X_1, X_2) =& 28 - 40X_1 - 4X_1^2 - 8X_2 - 40X_1X_2 - 4X_2^2 + 16X_2^3, \\
J^s(X_1, X_2) =& 28 + 16X_1 - 8X_2 - 4X_2^2.
\end{aligned}$$

The resulting orthogonality relations for the four types of polynomials are thus given as

$$\begin{aligned}
\sum_{j\in I_M} \varepsilon_j \mathbb{T}_{(t_1,t_2)}(Z_j)\mathbb{T}_{(t'_1,t'_2)}(Z_j) =& 12M^2 h_{[t_0,t_1,t_2]}\delta_{t_1t'_1}\delta_{t_2t'_2}, \quad [t_0,t_1,t_2],[t'_0,t'_1,t'_2] \in L_M, \\
\sum_{j\in \widetilde{I}_M} J(Z_j)\mathbb{U}_{(t_1,t_2)}(Z_j)\mathbb{U}_{(t'_1,t'_2)}(Z_j) =& M^2\delta_{t_1t'_1}\delta_{t_2t'_2}, \quad [t_0,t_1,t_2],[t'_0,t'_1,t'_2] \in \widetilde{L}_M, \\
\sum_{j\in I^s_M} \varepsilon_j J^s(Z_j)\mathbb{U}^s_{(t_1,t_2)}(Z_j)\mathbb{U}^s_{(t'_1,t'_2)}(Z_j) =& 12M^2 h_{[t_0+1,t_1,t_2+1]}\delta_{t_1t'_1}\delta_{t_2t'_2}, \quad [t_0,t_1,t_2],[t'_0,t'_1,t'_2] \in L^s_M, \\
\sum_{j\in I^l_M} \varepsilon_j J^l(Z_j)\mathbb{U}^l_{(t_1,t_2)}(Z_j)\mathbb{U}^l_{(t'_1,t'_2)}(Z_j) =& 12M^2 h_{[t_0,t_1+1,t_2]}\delta_{t_1t'_1}\delta_{t_2t'_2}, \quad [t_0,t_1,t_2],[t'_0,t'_1,t'_2] \in L^l_M.
\end{aligned}$$

5.7. Polynomial Interpolation

The polynomial interpolation in numerical analysis is the interpolation of a given data set by a polynomial: given some data on points $\mathfrak{F}_M$ inside the fundamental region $\mathfrak{F}$, find a polynomial which goes exactly through this data. The method for the polynomial interpolation follows via standard Fourier methods from the discrete orthogonality relations of polynomials—see also [19].

Using $\mathbb{T}-$polynomials of A_2 we choose the following model function on $\mathfrak{F}$

$$f(X_1, X_2) = e^{-\frac{X_1^2+X_2^2}{2\sigma^2}},$$

where $\sigma = \sqrt{0.15}$. The interpolating polynomial $I_M[f]$ of f has the following form

$$I_M[f](Z) = \sum_{[t_0,t_1,t_2]\in L_M} a_{[t_0,t_1,t_2]} \overline{\mathbb{T}_{(t_1,t_2)}(Z)},$$

where $Z = (X_1, X_2) \in \mathfrak{F}$ and the coefficients $a_{[t_0,t_1,t_2]} \in \mathbb{C}$ with $[t_0, t_1, t_2] \in L_M$ are given by

$$a_{[t_0,t_1,t_2]} = \frac{1}{18M^2 h_{[t_0,t_1,t_2]}} \sum_{j\in I_M} \varepsilon_j \mathbb{T}_{(t_1,t_2)}(Z_j) f(Z_j).$$

The coefficients ε_j and $h_{[t_0,t_1,t_2]}$ are taken from Tables 11 and 12, respectively, Z_j's are defined via Equation (29).

We present the explicit form of the interpolating polynomials for $M = 3$, 6. The interpolating polynomials are computed numerically and the coefficients $a_{[t_0,t_1,t_2]}$, for which $|a_{[t_0,t_1,t_2]}| < 10^{-7}$, are neglected. For $M = 3$ is the resulting interpolating polynomial $I_3[f]$ of the following form

$$I_3[f](X_1, X_2) = 1 - 0.555525X_1^2 + 0.148138X_1^3 - 0.555525X_2^2 - 0.444414X_1X_2^2$$

and the form of $I_6[f]$ is

$$\begin{aligned} I_6[f](X_1, X_2) = &1 - 1.72233X_1^2 + 0.450768X_1^3 + 0.732702X_1^4 - 0.415715X_1^5 + 0.060357X_1^6 \\ &- 1.72233X_2^2 - 1.3523X_1X_2^2 + 1.4654X_1^2X_2^2 + 0.831431X_1^3X_2^2 - 0.44988X_1^4X_2^2 \\ &+ 0.732702X_2^4 + 1.24715X_1X_2^4 + 0.601705X_1^2X_2^4 - 0.009749X_2^6. \end{aligned}$$

The contour plot of function f is shown in Figure 6 and the interpolating polynomials $I_6[f]$, $I_9[f]$ and $I_{12}[f]$ are plotted in Figure 7. One observes that for higher M, the interpolating polynomial $I_M[f](X_1, X_2)$ is getting closer to the model function $f(X_1, X_2)$. We calculate also the integral error of the interpolation, values of the integral errors are shown in Table 13.

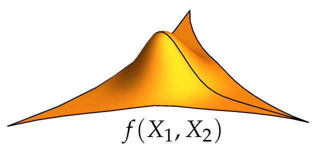


Figure 6. The model function f plotted over the domain $\mathfrak{F}$ of A_2.

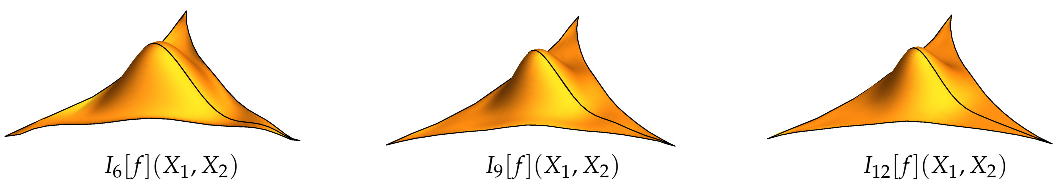


Figure 7. The contour plots of the interpolating polynomials $I_6[f]$, $I_9[f]$ and $I_{12}[f]$ plotted over the domain $\mathfrak{F}$ of A_2.

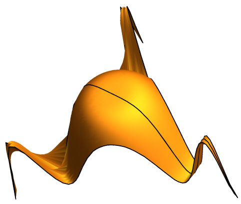
Table 13. Values of the integral errors of the polynomial interpolations of the functions f and g.

M	**Number of Points**	$\frac{1}{\pi^2}\int_{\mathfrak{F}(A_2)} \lvert f - I_M[f]\rvert^2 J^{-1/2}$	$\frac{1}{\pi^2}\int_{\mathfrak{F}(A_2)} \lvert g - I_M[g]\rvert^2 J^{-1/2}$
3	10	0.0146447	0.062784
6	28	0.0013806	0.015796
9	55	0.0002049	0.004072
12	91	0.0000716	0.000906

Consider another model function g on $\mathfrak{F}$ given by

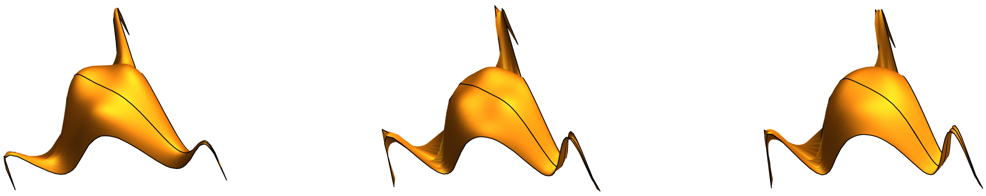
$$g(X_1, X_2) = \cos\left(\pi \frac{X_1^2 + X_2^2}{3}\right).$$

The contour plot of function g is shown in Figure 8 and the interpolating polynomials $I_6[g]$, $I_9[g]$ and $I_{12}[g]$ are plotted in Figure 9. The values of the integral errors are shown in Table 13.



$g(X_1, X_2)$

Figure 8. The model function g plotted over the domain $\mathfrak{F}$ of A_2.



$I_6[g](X_1, X_2)$ $\quad$ $I_9[g](X_1, X_2)$ $\quad$ $I_{12}[g](X_1, X_2)$

Figure 9. The contour plots of the interpolating polynomials $I_6[g]$, $I_9[g]$ and $I_{12}[g]$ plotted over the domain $\mathfrak{F}$ of A_2.

Example 1. *We present an example of the orthogonality for* $\mathbb{U}-$*polynomials for the case* A_2 *and* $M = 4$. *Then we have the following points in Kac coordinates,*

$$\widetilde{I}_4 = \{[1,1,2],[1,2,1],[2,1,1]\}, \qquad \widetilde{L}_4 = \{[1,0,0],[0,1,0],[0,0,1]\}.$$

For each $j \in \widetilde{I}_4$*, the transform* (29) *converts the points* $x_j \in \widetilde{F}_4(A_2)$ *in* $\omega^\vee-$*basis to points* $Z_j \in \widetilde{\mathfrak{F}}_4(A_2)$ *as*

$$\left(\tfrac{1}{4}, \tfrac{1}{2}\right) \to \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right), \quad \left(\tfrac{1}{2}, \tfrac{1}{4}\right) \to \left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right), \quad \left(\tfrac{1}{4}, \tfrac{1}{4}\right) \to (1,0).$$

The values of $\mathbb{U}-$*polynomials are given by*

$$\begin{aligned}
&\mathbb{U}_{(0,0)}(1,0) = 1, && \mathbb{U}_{(0,0)}\left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right) = 1, && \mathbb{U}_{(0,0)}\left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right) = 1,\\
&\mathbb{U}_{(1,0)}(1,0) = 1, && \mathbb{U}_{(1,0)}\left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right) = -\tfrac{1+i\sqrt{3}}{2}, && \mathbb{U}_{(1,0)}\left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right) = -\tfrac{1-i\sqrt{3}}{2},\\
&\mathbb{U}_{(0,1)}(1,0) = 1, && \mathbb{U}_{(0,1)}\left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right) = -\tfrac{1-i\sqrt{3}}{2}, && \mathbb{U}_{(0,1)}\left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right) = -\tfrac{1+i\sqrt{3}}{2}.
\end{aligned}$$

The values of the Jacobian J *at points* $Z_j \in \widetilde{\mathfrak{F}}_4(A_2)$ *are the same for all points and equal* $J(Z_j) = 16$.

We show the orthogonality only for the weight $[1,0,0] \in \widetilde{L}_4$. *The other relations are anologon,*

$$\begin{aligned}
&J(1,0)\mathbb{U}_{(0,0)}(1,0)\overline{\mathbb{U}_{(0,0)}}(1,0) + J(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\mathbb{U}_{(0,0)}(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\overline{\mathbb{U}_{(0,0)}}(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\\
&+ J(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2})\mathbb{U}_{(0,0)}(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2})\overline{\mathbb{U}_{(0,0)}}(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2}) = 48 = 3\cdot 4^2,\\
&J(1,0)\mathbb{U}_{(0,0)}(1,0)\overline{\mathbb{U}_{(1,0)}}(1,0) + J(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\mathbb{U}_{(0,0)}(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\overline{\mathbb{U}_{(1,0)}}(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\\
&+ J(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2})\mathbb{U}_{(0,0)}(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2})\overline{\mathbb{U}_{(1,0)}}(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2}) = 0,\\
&J(1,0)\mathbb{U}_{(0,0)}(1,0)\overline{\mathbb{U}_{(0,1)}}(1,0) + J(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\mathbb{U}_{(0,0)}(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\overline{\mathbb{U}_{(0,1)}}(-\tfrac{1}{2},\tfrac{\sqrt{3}}{2})\\
&+ J(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2})\mathbb{U}_{(0,0)}(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2})\overline{\mathbb{U}_{(0,1)}}(-\tfrac{3}{2},-\tfrac{3\sqrt{3}}{2}) = 0.
\end{aligned}$$

Example 2. *For case G_2 we check the orthogonality of $\mathbb{U}^l$ polynomials and $M=8$. Then it holds that*

$$I_8^l = \{[6,1,0],[4,2,0],[3,1,1],[2,3,0],[1,2,1]\},$$

$$L_8^l = \{[5,0,0],[3,0,1],[2,1,0],[1,0,2],[0,1,1]\}.$$

For each $j \in I_8^l$, the transform (29) converts the points $x_j \in F_8^l(G_2)$ in $\omega^\vee-$basis to points $Z_j \in \mathfrak{F}_8^l(G_2)$ as

$$\begin{aligned}
&\left(\tfrac{1}{8},0\right) \to \left(4(1+\sqrt{2}),3+2\sqrt{2}\right), \quad &&\left(\tfrac{1}{4},0\right) \to (2,3), \quad &&\left(\tfrac{1}{8},\tfrac{1}{8}\right) \to (0,1),\\
&\left(\tfrac{3}{8},0\right) \to \left(4(1-\sqrt{2}),3-2\sqrt{2}\right), \quad &&\left(\tfrac{1}{4},\tfrac{1}{8}\right) \to (0,-1).
\end{aligned}$$

The values of $\mathbb{U}-$polynomials are given by

$$\begin{aligned}
\mathbb{U}_{(0,0)}\left(4(1+\sqrt{2}),3+2\sqrt{2}\right) &= 1, &\mathbb{U}_{(0,0)}(2,3) &= 1, &\mathbb{U}_{(0,0)}(0,1) &= 1,\\
\mathbb{U}_{(0,0)}\left(4(1-\sqrt{2}),3-2\sqrt{2}\right) &= 1, &\mathbb{U}_{(0,0)}(2,3) &= 1,\\
\mathbb{U}_{(0,1)}\left(4(1+\sqrt{2}),3+2\sqrt{2}\right) &= 1+\sqrt{2}, &\mathbb{U}_{(0,1)}(2,3) &= 1, &\mathbb{U}_{(0,1)}(0,1) &= 0,\\
\mathbb{U}_{(0,1)}\left(4(1-\sqrt{2}),3-2\sqrt{2}\right) &= 1-\sqrt{2}, &\mathbb{U}_{(0,1)}(2,3) &= -1,\\
\mathbb{U}_{(1,0)}\left(4(1+\sqrt{2}),3+2\sqrt{2}\right) &= 2+2\sqrt{2}, &\mathbb{U}_{(1,0)}(2,3) &= 0, &\mathbb{U}_{(1,0)}(0,1) &= 0,\\
\mathbb{U}_{(1,0)}\left(4(1-\sqrt{2}),3-2\sqrt{2}\right) &= 2-2\sqrt{2}, &\mathbb{U}_{(1,0)}(2,3) &= 2,\\
\mathbb{U}_{(0,2)}\left(4(1+\sqrt{2}),3+2\sqrt{2}\right) &= 2+\sqrt{2}, &\mathbb{U}_{(0,2)}(2,3) &= 0, &\mathbb{U}_{(0,2)}(0,1) &= -1,\\
\mathbb{U}_{(0,2)}\left(4(1-\sqrt{2}),3-2\sqrt{2}\right) &= 2-\sqrt{2}, &\mathbb{U}_{(0,2)}(2,3) &= 0,\\
\mathbb{U}_{(1,1)}\left(4(1+\sqrt{2}),3+2\sqrt{2}\right) &= 3+2\sqrt{2}, &\mathbb{U}_{(1,1)}(2,3) &= -1, &\mathbb{U}_{(1,1)}(0,1) &= 1,\\
\mathbb{U}_{(1,1)}\left(4(1-\sqrt{2}),3-2\sqrt{2}\right) &= 3-2\sqrt{2}, &\mathbb{U}_{(1,1)}(2,3) &= -1.
\end{aligned}$$

The values of the Jacobian J^l at points $Z_j \in \mathfrak{F}_8^l(G_2)$ are

$$\begin{aligned}
J^l\left(4(1+\sqrt{2}),3+2\sqrt{2}\right) &= 48-32\sqrt{2}, &J^l(2,3) &= 64, &J^l(0,1) &= 32,\\
J^l\left(4(1-\sqrt{2}),3-2\sqrt{2}\right) &= 48+32\sqrt{2}, &J^l(0,-1) &= 16.
\end{aligned}$$

Similarly as in Example 1 we show the orthogonality for one weight $[1,0,2] \in L_8^l$,

$$12J^l(0,-1)\mathbb{U}^l_{(0,2)}(0,-1)\overline{\mathbb{U}^l_{(0,2)}}(0,-1) + 12J^l(0,1)\mathbb{U}^l_{(0,2)}(0,1)\overline{\mathbb{U}^l_{(0,2)}}(0,1)$$

$$+ 6J^l(4-4\sqrt{2},3-2\sqrt{2})\mathbb{U}^l_{(0,2)}(4-4\sqrt{2},3-2\sqrt{2})\overline{\mathbb{U}^l_{(0,2)}}(4-4\sqrt{2},3-2\sqrt{2})$$
$$+ 6J^l(4+4\sqrt{2},3+2\sqrt{2})\mathbb{U}^l_{(0,2)}(4+4\sqrt{2},3+2\sqrt{2})\overline{\mathbb{U}^l_{(0,2)}}(4+4\sqrt{2},3+2\sqrt{2})$$
$$+ 6J^l(2,3)\mathbb{U}^l_{(0,2)}(2,3)\overline{\mathbb{U}^l_{(0,2)}}(2,3) = 768 = 12 \cdot 8^2 \cdot 1.$$

6. Concluding Remarks

- The orthogonality of our polynomials is expressed as the sum of values of the polynomials over the points of the domain. Crucial for the orthogonality are the weight functions J, J^s, J^l that are determined here. The weight functions are different for polynomials of each type and for each of the three groups A_2, C_2, G_2. Let us emphasize that the set of points in the domain for any value of $M \in \mathbb{N}$ is not a fragment of a lattice. It would be interesting to characterize such sets of isolated points without reference to the orthogonality of orbit functions on lattices.
- Our approach begins with the sums of exponential functions ('orbit functions') determined by the orbits of the Weyl groups of A_2, C_2, G_2 on the corresponding weight lattice and their refinements. Discrete orthogonality of orbit functions [19,21] on finite fragments of 2D lattices of any density is the input that makes our work possible. Such functions serve us as a departure point for a description of the polynomials of two discrete variables that are orthogonal on a set of isolated points inside of a finite domain of a complex space. Moreover, their orthogonality is maintained when the finite domain is populated by ever increasing set of points. The density is determined by our choice of any positive integer M. The greater M, the greater number of points is in the domain.
- The classical product-to-sum formula for the Chebyshev polynomials [7] of the first kind $T_n(x)$, which is for $m,n \geq 0$ of the form $2T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$ has a straightforward generalization to the $\mathbb{T}-$polynomials. This 2D product-to-sum formula is readily derived from the product-to-sum formulas of $C-$orbit functions, see e.g. [25], and has the following form
$$\mathbb{T}_\lambda(X_1,X_2)\cdot\mathbb{T}_{\lambda'}(X_1,X_2) = \sum_{w\in W}\mathbb{T}_{|\lambda+w\lambda'|}(X_1,X_2),$$
where $|\lambda + w\lambda'|$ denotes the unique point of $W(\lambda+w\lambda')-$orbit which lies in P^+. Even though the existence of more complicated formulas, which would generalize product-to-sum formulas for the Chebyshev polynomials of the second kind $U_n(x)$ to 2D relations for $\mathbb{U}-$, $\mathbb{U}^s-$ and $\mathbb{U}^l-$polynomials, may be expected, their explicit form deserves further study.
- The example of interpolation demonstrates promising behavior of the polynomial interpolation f_M. Visual inspection and the integral error estimates of the interpolations of the model function f lead to the conclusion that the interpolation error is small once the minimal distances of the points $\mathfrak{F}_M$ become smaller than the variance σ^2 of the model function. The existence of general conditions for the convergence of the functional series of polynomials $\{f_M\}_{M=1}^\infty$ poses an open problem. The aliasing problem of the classical Chebyshev polynomials is generalized via the action of the magnified dual affine Weyl group $MQ \rtimes W$. Two polynomials $\mathbb{T}_\lambda$ and $\mathbb{T}_{\lambda'}$ are identical on the nodes $\mathfrak{F}_M$ if there exists an element of the magnified dual affine Weyl group $w \in MQ \rtimes W$ such that $\lambda = w\lambda'$. Possible generalization of the current Weyl group multivariate approach to wavelet analysis [30–32] poses open problem.
- Another practical aspect of the discrete orthogonality relations is the possibility of deriving inherent cubature formulas [2,33]. For successful computer implementation of these numerical integration formulas the data contained in the presented discrete orthogonality relations, it is necessary to include the weight functions. The data specifying the groups A_2, C_2, G_2 are different. However, our approach is uniform to the extent that it can be directly applied to study the polynomials of higher rank groups, their discrete orthogonality, polynomial interpolation properties, cubature formulas and other practical aspects.

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